# SOME SMALL UNIPOTENT REPRESENTATIONS OF INDEFINITE ORTHOGONAL GROUPS

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## 1. INTRODUCTION

Suppose $G$ is a real reductive Lie group and let $\widehat{G}$ denote the set of equivalence classes of irreducible unitary representations. Let $\widehat{G}^\lambda$ denote the subset with infinitesimal character $\lambda$. When $\lambda$ is large and regular in a suitable sense, $\widehat{G}^\lambda$ has a relatively simple and uniform structure; in particular, each element of $\widehat{G}^\lambda$ fits into a well-defined family of representations which behaves very much like the discrete series. (See [S] for precise statements, as well as a history of results of this nature.) But when $\lambda$ becomes small and singular, the uniform structure of $\widehat{G}^\lambda$ breaks down, and a number of fascinating complications arise. One of the most interesting problems in the subject is to organize these complications in a meaningful way.

One idea in this direction is to begin with a representation $\pi(\lambda) \in \widehat{G}^\lambda$ for $\lambda$ large and regular, place it in the family mentioned above, and then formally let the parameter $\lambda$ degenerate into something small and singular. When this procedure can be made precise, it has a chance of yielding interesting singular unitary representations. One virtue of this construction is that it organizes some singular representation in families (when, for instance, the Langlands parameters of these representations may not look like families at all), and one may hope that the procedure sheds at least some light on the most complicated part of $\widehat{G}$. In particular one may hope to locate unipotent representations this way.

Perhaps the earliest codification of this approach is contained in Wallach's construction of the analytic continuation of the holomorphic discrete series ([W]), and in the intervening years a number of authors have elaborated and extended his ideas in many different directions. Most recently Knapp has undertaken a study of this technique, updated by the advances in the algebraic theory of the last twenty years. We refer the reader to [K2]–[K3]. Those references also contain a careful history of this circle of ideas, and we direct the reader to that discussion, particularly the introduction to [K2]. The current paper is devoted to completing the analysis of the representations defined in [K3]. Roughly speaking these are analytic continuations of the smallest representations in the (nonhomomorphic) discrete series for the indefinite orthogonal group.

We specialize to the context of of [K3], and begin by following the introduction of that paper closely. Fix $m$ and $l$ such that $2 \leq m \leq l/2$ and let $G$ denote the universal cover of the connected component of the identity of the indefinite orthogonal group $\mathrm{SO}_\circ(2m, 2l-2m)$. $G$ is nonlinear and connected. Let $\mathfrak{g}_\circ = \mathfrak{so}(2m, 2l-2m)$ denote the Lie algebra of $G$, and let $G^{\mathbb{C}} = \mathrm{Spin}(2l, \mathbb{C})$. Let

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$K \simeq \mathrm{Spin}(2m) \times \mathrm{Spin}(2l-2m)$ denote a maximal compact subgroup of $G$ corresponding to a Cartan involution $\theta$. Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the corresponding complexified Cartan decomposition. Let $\mathrm{U}(\mathfrak{g})$ denote the enveloping algebra of $\mathfrak{g}$. In the usual coordinates with respect to a compact Cartan subalgebra, the roots may be written as $\pm e_i \pm e_j$ for $1 \leq i < j \leq l$, and with the usual choice of positive roots $e_m - e_{m+1}$ is the unique simple noncompact root. Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be the parabolic for which $\mathfrak{u}$ contains the $e_m - e_{m+1}$ root space, but for which all other positive simple roots contribute to $\mathfrak{l}$. More concretely, $\mathfrak{l}_\circ := \mathfrak{l} \cap \mathfrak{g}_\circ \simeq \mathfrak{u}(m,0) \oplus \mathfrak{so}(0, 2l-2m)$. Write $\delta(\mathfrak{u})$ for one-half the sum of the roots in $\mathfrak{u}$. Let $L$ denote the analytic subgroup of $G$ with Lie algebra $\mathfrak{l}_\circ$ and likewise let $L^{\mathbb{C}}$ denote the analytic subgroup of $G^{\mathbb{C}}$ with lie algebra $\mathfrak{l}$.

Let $\lambda_s = \sum_{k=1}^m (-l + \frac{s}{2}) e_k$. This defines a one-dimensional $(\mathfrak{l}, L \cap K)$-module $\mathbb{C}_{\lambda_s}$ which we extend to the opposite parabolic $\overline{\mathfrak{q}}$ by letting the nilradical act trivially. Consider the generalized Verma module $N(\lambda_s + 2\delta(\mathfrak{u})) = \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\overline{\mathfrak{q}})} \mathbb{C}_{\lambda_s + 2\delta(\mathfrak{u})}$, and write $N'(\lambda_s + 2\delta(\mathfrak{u}))$ for its unique irreducible quotient. Set $S = \dim(\mathfrak{u} \cap \mathfrak{k})$, and let $\Pi_S$ denote the $S$th derived Bernstein functor. Finally define $\pi_s = \Pi_S(N(\lambda_s + 2\delta(\mathfrak{u})))$ and $\pi_s' = \Pi_S(N'(\lambda_s + 2\delta(\mathfrak{u})))$. These are the representations studied in [K3] and the ones that we consider in the present paper.

For orientation, we summarize some basic properties of $\pi_s'$ and $\pi_s$. When $s$ is even, it is not difficult to see that both representations factor from the nonlinear group $G$ to $\mathrm{SO}_\circ(2m, 2l-2m)$; when $s$ is odd they do not. In either case, when $s > 2l - 2m$, $\pi_s' = \pi_s$ is in the discrete series; in fact (using [T1] for instance), it is not difficult to see that in this range $\pi_s' = \pi_s$ has the smallest Gelfand-Kirillov dimension among all representations in the discrete series. When $s \geq m+1$, $\lambda_s$ is in the weakly fair range (in the sense of [KV, Definition 0.52]) which automatically implies that $\pi_s' = \pi_s$ and that these representations are unitary (though possibly reducible); see [KV, Theorem 0.53]. In fact the conclusion of the previous sentence also follows from [EPWW], and that analysis extends to show that $\pi_m' = \pi_m$ is unitary as well. [K3] shows that $\pi_s' \neq \pi_s$ for all $0 \leq s < m$, and $s = 0$ is the last point of unitarity in the sense that $\pi_s$ is nonunitary for $-2l + 4m \leq s < 0$.

The paper [K3] provides much more detailed information about the representations $\pi_s'$ which we now briefly summarize. For each $s$, Knapp specifies an orbit $\mathcal{O}^L(s)$ of $L^{\mathbb{C}}$ on $\mathfrak{u} \cap \mathfrak{p}$[1]. Set $\Lambda_s = \lambda_s + 2\delta(\mathfrak{u} \cap \mathfrak{p})$. Fix the standard positive roots $\Delta^+ = \{e_i \pm e_j \mid i < j\}$, and let $\Delta^+_L$ denote those positive roots contributing to $\mathfrak{l}$ and likewise for $\Delta^+_K$. Let $V^\lambda$ denote the finite-dimensional representation of $K$ with $\Delta^+_K$ dominant weight $\lambda$. Then [K3] proves that the multiplicity of $V^\lambda$ in $\pi_s'$ is zero unless $\lambda - \Lambda_s$ is $\Delta^+_L$ dominant; in this case it matches the multiplicity of the representation of $L^{\mathbb{C}}$ with highest weight $\lambda - \Lambda_s$ in the ring of algebraic functions on the closure of $\mathcal{O}^L(s)$. Moreover [K3] gives a clean explicit formula for this multiplicity. In this way the representations $\pi_s'$ are associated to $\mathcal{O}^L(s)$.

Knapp's treatment of the representations $\pi_s'$ is essentially complete. The only small point remaining is that he does not determine whether or not the representation $\pi_s'$ is irreducible (except of course when $s \geq 2l - 2m - 1$ when it is in the discrete series or limits thereof). The point of this paper is

[1]Up until now, our notation has matched [K3], but the notation $\mathcal{O}^L(s)$ differs; the correspondence with Knapp's notation is given in Section 3 below.

to close that gap and indeed prove that all of the representations $\pi'_s$ for $s \geq 0$ are irreducible. The approach is rooted in understanding how each $\pi'_s$ is associated to a nilpotent $K^{\mathbb{C}}$ orbit on $\mathfrak{p}$.

Our proof of the irreducibility is conceptually very simple. In Section 4.1, we first apply a general result of Vogan to conclude irreducibility for $s \geq m+1$ (i.e. in the weakly fair range). Then in Section 4.2,we compute the associated variety of $\pi'_s$ and find that it is always simply the (irreducible) closure the $K^{\mathbb{C}}$ saturation of $\mathcal{O}^L(s)$. Moreover, we show in Section 4.3 that the multiplicity of this component in the associated cycle is exactly one. In Section 4.4, we then compute the annihilator of $\pi'_s$ for $0 \leq s \leq m+1$ and find that it is always a maximal (and, in fact, weakly unipotent) primitive ideal; this implies that if $\pi'_s$ is reducible, all of its constituents have the same associated variety (Corollary 4.10). Since multiplicities in the associated cycle are suitably additive, if there were at least two irreducible components in $\pi'_s$, the multiplicity in the associated cycle would have to be at least two. But we have seen it is one, and hence the irreducibility follows (see Section 4.5). We summarize our main results.

**Theorem 1.1.** *Let $\nu_s$ denote the infinitesimal character of $\pi'_s$; see (3.5) below for explicit details. Let $\mathcal{O}^K(s) = K^{\mathbb{C}} \cdot \mathcal{O}^L(s)$; this orbit is computed explicitly in terms of the tableau classification in Section 3 below.*

(1) *$\pi'_s$ is irreducible for all $s \geq 0$.*
(2) *The associated variety of $\pi'_s$ is the closure of $\mathcal{O}^K(s)$; in particular it is irreducible.*
(3) *The multiplicity of the irreducible representation $V^\lambda$ of $K$ with $\Delta^+_K$ highest weight $\lambda$ in $\pi'_s$ is zero unless $\lambda - \Lambda_s$ is $\Delta^+_K$-dominant. In this case the multiplicity of $V^\lambda$ in $\pi'_s$ coincides with the multiplicity of the irreducible representation of $K^{\mathbb{C}}$ with highest weight $\lambda - \Lambda_s$ in the ring of algebraic functions on the closure of $\mathcal{O}^K(s)$.*
(4) *The multiplicity of $\mathcal{O}^K(s)$ in the associated cycle of $\pi'_s$ is exactly 1.*
(5) *Suppose $0 \leq s \leq m+1$. Then the annihilator of $\pi'_s$ is the maximal primitive ideal $\mathrm{J}_{\max}(\nu_s)$ at infinitesimal character $\nu_s$. Moreover*
   (a) *If $s$ is even, $\mathrm{J}_{\max}(\nu_s)$ is special unipotent, and thus $\pi'_s$ is special unipotent.*
   (b) *If $s$ is odd, $\mathrm{J}_{\max}(\nu_s)$ is not special unipotent, yet it is weakly unipotent; so $\pi'_s$ is weakly unipotent.*

   *(For the terminology relating to special and weakly primitive ideals, see Section 2.2 below.)*

The virtue of this formulation is that while [K3] shows in what sense $\pi'_s$ is associated to the $L^{\mathbb{C}}$ orbit $\mathcal{O}^L(s)$, this result shows in what sense $\pi'_s$ is associated to the $K^{\mathbb{C}}$ orbit $\mathcal{O}^K(s)$. Theorem 1.1(3) provides an especially tight relationship between the representation and the orbit, and this relationship is of course consonant with the predictions of the orbit method. We also remark that we compute the maximal ideal $\mathrm{J}_{\max}(\nu_s)$ explicitly in terms of the tableau classification of the primitive spectrum in the course of the proof of Lemma 4.7; see also Example 4.8.

While we have stated all of our results for the even indefinite orthogonal groups, there is an entirely parallel (and in fact easier) set of results for the odd orthogonal groups. This is sketched briefly in Section 5, especially Theorem 5.1.

Finally one may ask: for $0 \leq s \leq m+1$, is $\pi'_s$ the only irreducible Harish-Chandra module with maximal primitive ideal $\mathrm{J}_{\max}(\nu_s)$; or, in other words, are there other unipotent representations associated to $\mathcal{O}^K(s)$? The answer is that for $s > 0$, there are indeed other representations. Section 4.6 suggests where to find them.

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## 2. Background on associated cycles and primitive ideals

2.1. **Associated cycles of Harish-Chandra modules.** In this section $G$ may be taken to be an arbitrary real reductive group. (We retain the notational conventions of the introduction.) Let $X$ be a finite length $(\mathfrak{g}, K)$ module, and let $(X^j)$ denote a $K$-invariant good filtration of $X$. Then $\mathrm{gr}(X)$ defines a $K^{\mathbb{C}}$-equivariant sheaf supported on $\mathcal{N}(\mathfrak{p})$. (See [V4] for more details of this construction; here we have used an invariant form to identify $\mathfrak{p}^*$ with $\mathfrak{p}$.) The support of this sheaf is called the associated variety of $X$ and denoted $\mathrm{AV}(X)$; $\mathrm{AV}(X)$ is a (finite) union of the closures of orbits of $K^{\mathbb{C}}$ orbits on $\mathcal{N}(\mathfrak{p})$, the nilpotent cone in $\mathfrak{p}$. Define a subset $\mathrm{av}(X)$ of $K^{\mathbb{C}}$ orbits on $\mathcal{N}(\mathfrak{p})$ by the requirement

$$\mathrm{AV}(X) = \bigcup_{\mathcal{O}^K \in \mathrm{av}(X)} (\mathcal{O}^K)^{\mathrm{cl}}.$$

Here and elsewhere the superscript cl denotes closure. For $\mathcal{O}^K \in \mathrm{av}(X)$, let $m_X(\mathcal{O}^K)$ (or just $m(\mathcal{O}_K)$ if $X$ is clearly fixed) denote the rank of the sheaf $\mathrm{gr}(X)$ along $(\mathcal{O}^K)^{\mathrm{cl}}$. Define the associated cycle of $X$ as

$$\mathcal{AC}(X) = \sum_{\mathcal{O}^K \in \mathrm{av}(X)} m_X(\mathcal{O}^K) \left[ (\mathcal{O}^K)^{\mathrm{cl}} \right].$$

This construction of the $\mathrm{AV}(X)$ and $\mathcal{AC}(X)$ is transparently additive in the following sense. Suppose $0 \to X \to Z \to Y \to 0$ is an exact sequence of Harish-Chandra modules. Then

$$\mathrm{AV}(X) \cup \mathrm{AV}(Y) = \mathrm{AV}(Z). \tag{2.1}$$

Moreover if $\mathcal{O}^K \in \mathrm{av}(Z)$, then

$$m_X(\mathcal{O}^K) + m_Y(\mathcal{O}^K) = m_Z(\mathcal{O}^K). \tag{2.2}$$

The following is a standard interpretation of the multiplicities.

**Proposition 2.1.** *Retain the notation above, and for simplicity suppose $K^{\mathbb{C}}$ is connected. Fix a choice of Cartan $H^{\mathbb{C}}$ for $K^{\mathbb{C}}$ and a choice of positive roots of $\mathfrak{h}$ in $\mathfrak{g}$. Let $\Lambda^+$ denote the corresponding set of dominant weights (so the irreducible representations of $K^{\mathbb{C}}$ are parametrized by $\Lambda^+$). For $\lambda \in \Lambda^+$, let $V^\lambda$ denote the corresponding irreducible representation of $K^{\mathbb{C}}$, and let $c_\lambda(\mathcal{O}^K)$ denote the multiplicity of $V^\lambda$ in the ring of algebraic functions on $(\mathcal{O}^K)^{\mathrm{cl}}$. Similarly let $c_\lambda(X)$ denote the* *multiplicity of $V^\lambda$ in $X$. Finally for each $\lambda$, let $|\lambda|$ denote its length (defined using any fixed invariant bilinear form). If $\mathcal{O}^K \in \mathrm{av}(X)$, then*

$$m_X(\mathcal{O}^K) = \lim_{t\to\infty} \frac{\sum_{|\lambda|<t} c_\lambda(X)\dim(V^\lambda)}{\sum_{|\lambda|<t} c_\lambda(\mathcal{O}^K)\dim(V^\lambda)} \tag{2.3}$$

2.2. **Primitive ideals in $\mathrm{U}(\mathfrak{g})$: generalities.** Next we recall a few facts about primitive ideals in $\mathrm{U}(\mathfrak{g})$ where, for the purposes of this section, $\mathfrak{g}$ may be taken to be any complex semisimple Lie algebra. Recall that a two-sided ideal in $\mathrm{U}(\mathfrak{g})$ is called primitive if it is the annihilator of a simple $\mathrm{U}(\mathfrak{g})$ module. We say a primitive ideal $I = \mathrm{Ann}(X)$ has infinitesimal character $\chi$ if $X$ has infinitesimal character $\chi$; i. e. if $I$ contains the codimension one ideal $Z_\chi$ in the center of $\mathrm{U}(\mathfrak{g})$ corresponding to $\chi$. According to fundamental results of Duflo ([Du]) the set of primitive ideals $\mathrm{Prim}(\mathrm{U}(\mathfrak{g}))_\chi$ with fixed infinitesimal character $\chi$ is finite and, in the inclusion partial order, contains a unique maximal element. We denote this maximal element $\mathrm{J}_{\max}(\chi)$. More intrinsically, $\mathrm{J}_{\max}(\chi)$ is the largest proper ideal in $\mathrm{U}(\mathfrak{g})$ containing $Z_\chi$.

Recall the associated variety $\mathrm{AV}(I)$ of a primitive ideal $I$. According to a result of Borho-Brylinski [BB], $\mathrm{AV}(I)$ is the closure of a single nilpotent coadjoint orbit in $\mathfrak{g}$. (Again we identify $\mathfrak{g}^*$ with $\mathfrak{g}$.) The associated variety of a Harish-Chandra module $X$ is an equidimensional union of irreducible components of $\mathrm{AV}(X) \cap \mathfrak{p}$; see Section 6 of [V4] for more details. In particular,

$$\mathrm{AV}(\mathrm{Ann}(X)) = G^{\mathbb{C}} \cdot \mathrm{AV}(X); \tag{2.4}$$

here $G^{\mathbb{C}}$ may be taken to be the adjoint group of $\mathfrak{g}$ if $G$ is connected.

The following result is well-known, but we set is aside as we have frequent occasion to refer to it below.

**Proposition 2.2** ([BK], Korollar 4.7)**.** *Suppose $I$ and $J$ are primitive ideals such that $I \subset J$. Then either $I = J$ or $\mathrm{AV}(I) \supsetneq \mathrm{AV}(J)$.*

Next we recall the definition of a special unipotent primitive ideal (Definition 12.10 of [V3], for example). Such primitive ideals are parameterized by nilpotent orbits $\mathcal{O}^\vee$ in the dual algebra $\mathfrak{g}^\vee$. Let $\mathfrak{h}^\vee$ denote a Cartan subalgebra of $\mathfrak{g}^\vee$; so $\mathfrak{h}^\vee$ is canonically isomorphic to $\mathfrak{h}^*$ for a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. For an orbit $\mathcal{O}^\vee$, let $h^\vee$ denote the semisimple element of a Jacobsen-Morozov triple for $\mathcal{O}^\vee$ with $h^\vee \in \mathfrak{h}^\vee \simeq \mathfrak{h}^*$. Let $\chi(\mathcal{O}^\vee) = \frac{1}{2}h^\vee$. Since $\mathfrak{h}^\vee \simeq$, we may view $\chi(\mathcal{O}^\vee)$ as an element of of $\mathfrak{h}^*$. Different choices of the Jacobsen-Morozov triple may lead to $W$ translates of $\chi(\mathcal{O}^\vee)$ . Consequently while $\chi(\mathcal{O}^\vee)$ is not independent of the choices involved, the infinitesimal character it defines is. A primitive ideal is said to be *special unipotent* if it is of the form $\mathrm{J}_{\max}\big(\chi(\mathcal{O}^\vee)\big)$ for some nilpotent orbit $\mathcal{O}^\vee$; in this case we write $\mathrm{J}_{\max}(\mathcal{O}^\vee)$ instead of $\mathrm{J}_{\max}\big(\chi(\mathcal{O}^\vee)\big)$. An irreducible Harish-Chandra module whose annihilator is a special unipotent primitive ideal is also called special unipotent.

We recall that an irreducible Harish-Chandra module $X$ is called *weakly unipotent* if its annihilator has real infinitesimal character $\chi$ such that if $Y$ is a Harish-Chandra module with infinitesimal character $\lambda$ that appears as a subquotient of $X \otimes F$ (with $F$ finite-dimensional), then $\lambda$ is (weakly) shorter in length than $\chi$. In this case $\mathrm{Ann}(X)$ is called a weakly unipotent primitive ideal.

According to [BV3, Lemma 5.7], a special unipotent primitive ideal is weakly unipotent. See Chapter 12 of [V3] as well as Chapter 12 of [KV] for more details on these definitions.

Finally we recall the duality map of Spaltenstein as treated in the appendix of [BV3]. Write $\mathcal{N}$ for the set of nilpotent coadjoint orbits in $\mathfrak{g}$ and likewise write $\mathcal{N}^\vee$ for the set such orbits in $\mathfrak{g}^\vee$. The duality of orbits is a map

$$d \,:\, \mathcal{N}^\vee \longrightarrow \mathcal{N}. \tag{2.5}$$

**Proposition 2.3** ([Corollary A3, BV3])**.** *In the notation introduced above,*

$$\mathrm{AV}\left(\mathrm{J}_{\max}(\mathcal{O}^\vee)\right) = d(\mathcal{O}^\vee)^{\mathrm{cl}}.$$

2.3. **Notation for tableau and partitions.** Let $p = (p_1 \geq p_2 \geq \dots)$ be a partition of $n$. We identify $p$ with a Young diagram $D_p$ of size $n$. $D_p$ is a left justified array of boxes arranged so that the $i$th row has $p_i$ boxes. A partition (or Young diagram) is called a domino partition (or a domino Young diagram) if it can be tiled by 1-by-2 and 2-by-1 dominos.

A partition of $2n$ is said to be special (for type D) if the number of odd parts between consecutive even parts or greater than the largest nonzero even part is even.

Suppose $\nu$ is an unordered $n$-tuple of positive real numbers so that the difference of any two entries is an integer. A standard domino $\nu$-tableau is a Young diagram of size $2n$ tiled by dominos labeled with the entries of $\nu$ such that the entries weakly increase across rows and strictly increase down columns, together with the requirement that if two zeros appear as coordinates of $\nu_s$ the tableau must never contain the configuration[2]

$$\begin{array}{|c|}\hline 0 \\ \hline 0 \\ \hline \end{array}\quad . \tag{2.6}$$

2.4. **Nilpotent orbits in $\mathfrak{so}(2l,\mathbb{C})$.** Let $\mathcal{N}$ denote the nilpotent cone in $\mathfrak{g} = \mathfrak{so}(2l,\mathbb{C})$. The Jordan normal form is a complete invariant for orbits of $\mathrm{O}(2l,\mathbb{C})$ on $\mathcal{N}$. Such orbits are parametrized by partitions of $2l$ in which each even part occurs with even multiplicity. Each orbit is a single orbit under $\mathrm{Spin}(2l,\mathbb{C})$ (or $\mathrm{SO}(2l,\mathbb{C})$) unless the orbit is parametrized by a "very even" partition (one in which all parts are even, and each occurs with even multiplicity); in this case they split into two and are parametrized by attaching an additional numeral (1 or 2) to the very even partition.

In terms of this parametrization, the duality map $d$ of Section 2.2 (which in this case map nilpotent orbit for $\mathfrak{so}(2l,\mathbb{C})$ again to nilpotent orbits for $\mathfrak{so}(2l,\mathbb{C})$) may be computed as follows. Given an orbit $\mathcal{O}$ parametrized by the partition $p$, one first passes to the transposed partition $p^{\mathrm{tr}}$. Because $p^{\mathrm{tr}}$ need not have all even rows occurring with even multiplicity, one must "collapse" $p^{\mathrm{tr}}$ by replacing it by the largest partition smaller that $p^{\mathrm{tr}}$ in which all even parts do occur with even multiplicity. Here "largest" and "smaller" refer to the standard partial order on partitions. (See [CMc, Chapter 6] for more details on the collapse procedure.) The resulting partition is well-defined and parametrizes

[2] The conditions about weakly increasing across rows and strictly increasing down columns are of course really $\tau$-invariant conditions on the simple roots $e_i - e_{i+1}$ The perhaps less familiar stipulation on the zero dominos is also a $\tau$-invariant condition, this time on the simple root $e_1 + e_2$.

$d(\mathcal{O})$. (In the case of very even partitions, no collapsing is required, and $d(\mathcal{O})$ is parametrized by $p^{\mathrm{tr}}$, which is also very even, together with the numeral parametrizing $\mathcal{O}$.)

An orbit of $\mathrm{O}(2l,\mathbb{C})$ or $\mathrm{Spin}(2l,\mathbb{C})$ is *special* if the partition parametrizing it is special in the sense of Section 2.3. (The condition that an orbit is special is equivalent to the condition that it be in the image of $d$. This is a general fact.)

The orbits of $K^{\mathbb{C}} = \mathrm{Spin}(2m,\mathbb{C}) \times \mathrm{Spin}(2l-2m,\mathbb{C})$ on $\mathcal{N}(\mathfrak{p})$ coincide with those of $\mathrm{SO}(2m,\mathbb{C}) \times \mathrm{SO}(2l-2m,\mathbb{C})$ and are a little delicate to parametrize explicitly. The orbits of $K^{\mathbb{C}}_{\mathrm{ex}} := \mathrm{O}(2m,\mathbb{C}) \times \mathrm{O}(2l-2m,\mathbb{C})$ are much simpler. (The subscript $ex$ is meant to stand for "extended.") It follows from the first line of the proof of Theorem 9.3.4 in [CMc] (for instance) that the orbits of $K^{\mathbb{C}}_{\mathrm{ex}}$ are parametrized by orthogonal signature $(2m, 2l-2m)$ signed tableau; i.e. equivalence classes of Young diagrams of size $2l$ whose boxes are labeled with $2m$ plus signs and $2l-2m$ minus signs so that signs alternate across rows and so that the number of rows of each fixed even length $2k$ beginning with $+$ coincides with the number of rows of length $2k$ beginning with $-$. (The equivalence relation interchanges rows of equal length.) In addition, we label such a diagram by a string of signed integers $1_+^{n_1^+} 1_-^{n_1^-} 2_+^{n_2^+} 2_-^{n_2^-} \dots$ ; here $n_j^{\epsilon}$ is the number of rows of length $j$ beginning with the sign $\epsilon$.

Most $K^{\mathbb{C}}_{\mathrm{ex}}$ orbits are single orbits under the action of $K^{\mathbb{C}}$ but there are some exceptions. Since the details of those exceptions do no arise in applications for us, we do not discuss them here.

2.5. **Weyl group representations and truncated induction.** We introduce some notation for the Springer correspondence, again in the generality of an arbitrary complex semisimple Lie algebra $\mathfrak{g}$. Suppose $\mathcal{O}$ is a nilpotent coadjoint orbit, and write $\mathrm{sp}(\mathcal{O})$ for the irreducible representation of $W$ attached to the trivial local system on $\mathcal{O}$. The map $\mathcal{O} \mapsto \mathrm{sp}(\mathcal{O})$ is injective, but not surjective usually. Recall that there is a notion of a special Weyl group representation (for instance, [Ca, Section 11.4]). An orbit is called special if $\mathrm{sp}(\mathcal{O})$ is. This condition is made explicit for $\mathfrak{so}(2l,\mathbb{C})$ in the previous section. (We also again remark that the set of special orbits is precisely the image of the duality map $d$ of Section 2.2.)

Let $W'$ be any subgroup of $W$ generated by reflections. Recall the operation of truncated induction $j_{W'}^{W}$ from irreducible representations of $W'$ to those of $W$ (for example, [Ca, Section 11.3]). This operation has the following important property if $\pi' = \mathrm{sp}(\mathcal{O}')$ is a *special* representation of $W'$, then

$$j_{W'}^{W}\left(\mathrm{sp}(\mathcal{O}')\right) = \mathrm{sp}(\mathcal{O}), \tag{2.7}$$

for some nilpotent orbit $\mathcal{O}$. See [Ca, Proposition 11.4.11] for instance.

The operation $j$ is injective and suitably compatible with the operation of taking orbit closures. To make this precise, suppose $\mathcal{O}_1'$ and $\mathcal{O}_2'$ are two special orbits. Write $\mathcal{O}_i$ for the orbits such that $j_{W'}^{W}(\mathrm{sp}(\mathcal{O}_i')) = \mathrm{sp}(\mathcal{O}_i)$ according to (2.7). Then

$$\mathcal{O}_1' \subset (\mathcal{O}_2')^{\mathrm{cl}} \quad \Longrightarrow \quad \mathcal{O}_1 \subset \mathcal{O}_2^{\mathrm{cl}} \tag{2.8}$$

and

$$\mathcal{O}_1 = \mathcal{O}_2 \quad \Longrightarrow \quad \mathcal{O}_1' = \mathcal{O}_2'. \tag{2.9}$$

Again we refer to [Ca, Section 11.4] for these facts.

2.6. **Primitive ideals in $\mathrm{U}(\mathfrak{g})$: classification.** We recall some details of the classification of primitive ideals in $\mathrm{U}(\mathfrak{g})$. The basic references are [BV1]–[BV2] and [G1]–[G4]; see also [Mc2].

Begin with some generalities and assume that $\mathfrak{g}$ is an arbitrary complex semisimple Lie algebra. Fix an infinitesimal character $\lambda$. To each primitive ideal $I$ with infinitesimal character $\lambda$, one may attach a special representation $\pi'_I$ of the integral Weyl group $W'$ of $\lambda$. The map from $I \in \mathrm{Prim}(\mathrm{U}(\mathfrak{g}))_\lambda$ to $\pi'_I$ is surjective onto the set of special representations of $W'$, and the fiber over $\pi'$ has cardinality equal to the dimension of $\pi'$ when $\lambda$ is regular, but strictly smaller if $\lambda$ is singular. Recall that special representation $\pi'$ of $W'$ defines (a not necessarily special) nilpotent orbit $\mathcal{O}$ for $\mathfrak{g}$ by (2.7). This orbit is dense in the associated variety of any $I$ with $\pi'_I = \pi'$.

We make these details more explicit for $\mathfrak{g} = \mathfrak{so}(2l, \mathbb{C})$. In applications below, the integral Weyl group $W'$ will either be all of $W$ or else of type $\mathrm{D}_m \times \mathrm{D}_{l-m}$, and we spell out each case separately.

Suppose $\lambda$ is integral, so $W' = W$, and all coordinates of $\lambda$ are integers. Assume (as we may) that all coordinates are nonnegative. In the terminology of Section 2.3, $\mathrm{Prim}(\mathrm{U}(\mathfrak{g}))_\lambda$ is parametrized by the set of standard domino $\lambda_i$-tableau whose underlying shape is *special*.

Suppose $I$ is a primitive ideal parametrized by such a tableau. The shape of the tableau almost parametrizes a special nilpotent $\mathrm{SO}(2l, \mathbb{C})$ orbit $\mathcal{O}_I$. The modifier "almost" is required because in case the shape is very even, we need also supply a numeral. To do this, we count the number of vertical dominos. The hypothesis that the shape is very even implies that this number is even, so is either congruent to 0 or 2 modulo 4. The extra numeral needed to parametrize $\mathcal{O}_I$ is 1 in the former case, and 2 in the latter. In any event, the orbit $\mathcal{O}_I$ is dense in the associated variety of $I$. Moreover the special representation $\pi_I$ attached to $I$ is $\mathrm{sp}(\mathcal{O}_I)$.

Next suppose the coordinates of $\lambda$ consist of $m$ half-integers and $l - m$ integers, so $W'$ is of type $\mathrm{D}_m \times \mathrm{D}_{l-m}$. Again assume the coordinates of $\lambda$ are nonnegative, and partition $\lambda = \lambda_1 \cup \lambda_2$ into half-integer and integer coordinates respectively. Then $\mathrm{Prim}(\mathrm{U}(\mathfrak{g}))_\lambda$ is parametrized by the set of pairs $(T_1, T_2)$ of domino tableaux such that (in the terminology of Section 2.3) $T_i$ is a standard domino $\lambda$-tableau of special shape. Let $\mathcal{O}_i$ denote the special orbits for $\mathrm{SO}(2m, \mathbb{C})$ and $\mathrm{SO}(2l - 2m, \mathbb{C})$ parametrized by the shape of $T_i$ (and possibly additional numeral as above). Then the special representation of $W'$ corresponding to a primitive ideal $I$ parametrized by $(T_1, T_2)$ is $\pi'_I = \mathrm{sp}(\mathcal{O}_1) \boxtimes \mathrm{sp}(\mathcal{O}_2)$. Using (2.7) write $j_{W'}^{W}(\pi') = \mathrm{sp}(\mathcal{O}_I)$. Then $\mathcal{O}_I$ is dense in the associated variety of $I$.

## 3. Some explicit details

For completeness, we make some of the quantities appearing in Theorem 1.1 a little more explicit. We also prove some elementary results that will be needed later. We maintain the notation of the introduction.

We first recall the structure of $L^{\mathbb{C}}$ orbits on $\mathfrak{u} \cap \mathfrak{p}$. This is treated carefully in [K3, Section 3]. As in [K3, Proposition 1.1] we find it convenient to identify $\mathfrak{u} \cap \mathfrak{p}$ with the space of $m \times 2l - 2m$

complex matrices $M_{m,2l-2m}$. With this identification $(x, y) \in \mathrm{GL}(m, \mathbb{C}) \times \mathrm{O}(2l - 2m, \mathbb{C})$ acts on $M \in M_{m,2l-2m}$ by $xMy$. The orbits of this action are finite in number and are parametrized by two indices $0 \leq q \leq p \leq m$. The corresponding orbit is characterized by the requirement that for each $X' \in \mathcal{O}^L(p, q)$, $X'$ has rank $p$ and $X'(X')^{\mathrm{tr}}$ has rank $q$. This orbit is a single $L^{\mathbb{C}}$ orbit, which we denote $\mathcal{O}^L(p, q)$, except in the special case that $l = 2m$ and $(p, q) = (m, 0)$. In that case the orbit splits into two for $L^{\mathbb{C}}$. Although slightly misleading, we still write $\mathcal{O}^L(m, 0)$ for the union of these two orbits.

Now set $\mathcal{O}_{\mathrm{ex}}(p, q) = \mathcal{O}(2l, \mathbb{C}) \cdot \mathcal{O}^L(p, q)$ and $\mathcal{O}(p, q) = G^{\mathbb{C}} \cdot \mathcal{O}^L(p, q)$. Under the identification of $\mathfrak{u} \cap \mathfrak{p}$ with $M_{m,2l-2m}$, let $X$ be an element of the former space corresponding to $X'$ in the latter. One quickly checks that $X'$ (respectively, $X'(X')^{\mathrm{tr}}$) has rank $m$ if and only if $X$ (respectively $X^2$) has rank $2m$ (respectively $m$). Consequently it is easy to conclude that $X$ has Jordan form $3^q 2^{2p-2q} 1^{2l-4p+q}$. From the discussion of Section 2.4, $\mathcal{O}_{\mathrm{ex}}(p, q)$ is indeed parametrized by $3^q 2^{2p-2q} 1^{2l-4p+q}$.

On the other hand, set $\mathcal{O}^K_{\mathrm{ex}}(p, q) = \mathcal{O}(2l, \mathbb{C}) \cdot \mathcal{O}^L(p, q)$ and $\mathcal{O}^K(p, q) = K^{\mathbb{C}} \cdot \mathcal{O}^L(p, q)$. Since $L^{\mathbb{C}} \subset K^{\mathbb{C}}$, each $\mathcal{O}^K(p, q)$ is a single $K^{\mathbb{C}}$ orbit, save for the one exception mentioned above; $\mathcal{O}^K_{\mathrm{ex}}(p, q)$ is always a single orbit. In terms of the parametrization of Section 2.4, we leave it to the reader to verify that $\mathcal{O}^K_{\mathrm{ex}}(p, q)$ is parametrized by $3^q_+ 2^{p-q}_+ 2^{p-q}_- 1^{2l-4p+q}_-$.

It transpires that structure of the orbit closure $\mathcal{O}^K(p, q)^{\mathrm{cl}}$ essentially reduces to that of $\mathcal{O}^L(p, q)^{\mathrm{cl}}$.

**Lemma 3.1.** *There exists an injective algebraic map*

$$j \ : \ \mathcal{O}^K(p, q)^{\mathrm{cl}} \longrightarrow \mathcal{O}^L(p, q)^{\mathrm{cl}} \times M_{m,2l-2m}.$$

**Proof.** Extending the identification of $\mathfrak{u} \cap \mathfrak{p}$ with $M_{m,2l-2m}$ (as in [K3, Lemma 1.1]), one may identify $\mathfrak{p}$ with $M_{m,2l-2m} \times M_{m,2l-2m}$. We need to make the action of $K^{\mathbb{C}}$ explicit. Since the orbits of this action do not depend on the isogeny class of $K^{\mathbb{C}}$, there is no harm in instead considering the action of $K^{\mathbb{C}}_\bullet = \mathrm{SO}(2m, \mathbb{C}) \times \mathrm{SO}(2l - 2m, \mathbb{C})$. Let $U_K$ denote the analytic subgroup of $K^{\mathbb{C}}_\bullet$ with Lie algebra $\mathfrak{u} \cap \mathfrak{k}$. Since this latter algebra is abelian a typical element of $U_K$ looks like $\mathrm{Id} + u$ with $u \in \mathfrak{u} \cap \mathfrak{k}$. An analogous description of course applies to the opposite group $\overline{U}_K$ corresponding to $\bar{\mathfrak{u}} \cap \mathfrak{p}$. Since $K^{\mathbb{C}}_\bullet = \overline{U}_K L^{\mathbb{C}}_\bullet U_K$, we need only describe the action of each piece. Start with $L^{\mathbb{C}}_\bullet = \mathrm{GL}(m, \mathbb{C}) \times \mathrm{SO}(2l, \mathbb{C})$ and consider $(x, y) \in L^{\mathbb{C}}_\bullet$. Then for $(M_1, M_2) \in M_{m,2l-2m} \times M_{m,2l-2m}$ the action of $L^{\mathbb{C}}_\bullet$ on $\mathfrak{p}$ corresponds to

$$(x, y) \cdot (M_1, M_2) = (xM_1y, xM_2y).$$

The action of $\mathrm{Id} + u \in U_K$ is given by

$$(\mathrm{Id} + u) \cdot (M_1, M_2) = (M_1 + uM_2, M_2);$$

and for $\mathrm{Id} + \bar{u} \in \overline{U}_K$ by

$$(\mathrm{Id} + \bar{u}) \cdot (M_1, M_2) = (M_1, M_2 + \bar{u}M_1). \tag{3.1}$$

Since $\mathfrak{u} \cap \mathfrak{p}$ identifies with the first copy of $M_{m,2l-2m}$, it is easy to see that $U_K$ acts trivially on $\mathfrak{u} \cap \mathfrak{p}$ and, in particular, on $\mathcal{O}^L(p, q)^{\mathrm{cl}}$. Consequently $\mathcal{O}^K(p, q)^{\mathrm{cl}} = \overline{U}_K \cdot \mathcal{O}^L(p, q)^{\mathrm{cl}}$. In fact, from (3.1) it is clear that

$$\mathcal{O}^K(p,q)^{\mathrm{cl}} = \coprod_{M \in \mathcal{O}^L(p,q)^{\mathrm{cl}}} \overline{U}_K \cdot M;$$

here the union is disjoint. Again using (3.1) and the identification of $\mathfrak{p}$ with $M_{m,2l-2m} \times M_{m,2l-2m}$, the disjoint union becomes

$$\mathcal{O}^K(p,q)^{\mathrm{cl}} = \coprod_{M \in \mathcal{O}^L(p,q)^{\mathrm{cl}}} (M, M + \bar{\mathfrak{u}}M); \tag{3.2}$$

where of course $\bar{\mathfrak{u}}M = \{XM \mid X \in \bar{\mathfrak{u}}\}$. Thus (3.2) gives the injection of the lemma. Since every action in sight is algebraic, the map is indeed algebraic. □

For any algebraic variety $Z$, let $\mathbb{C}[Z]$ denote the ring of algebraic functions on $Z$. The inclusion, say $j$, of $\mathcal{O}^L(p,q)^{\mathrm{cl}}$ into $\mathcal{O}^K(p,q)^{\mathrm{cl}}$ gives a surjective restriction map

$$j^* \; : \; \mathbb{C}[\mathcal{O}^K(p,q)^{\mathrm{cl}}] \longrightarrow \mathbb{C}[\mathcal{O}^L(p,q)^{\mathrm{cl}}].$$

By projecting the image of $\mathcal{O}^K(p,q)$ under the inclusion of Lemma 3.1, we obtain an algebraic projection $p$ from $\mathcal{O}^K(p,q)^{\mathrm{cl}}$ onto $\mathcal{O}^L(p,q)^{\mathrm{cl}}$. On the level of functions we obtain an injection

$$p^* \; : \; \mathbb{C}[\mathcal{O}^L(p,q)^{\mathrm{cl}}] \longrightarrow \mathbb{C}[\mathcal{O}^K(p,q)^{\mathrm{cl}}].$$

From the formulas for the action of $\overline{U}_K$ given in the proof of Lemma 3.1, we can give a more concrete description of $p^*$. Each $\overline{U}_K$ orbit on $\mathcal{O}^K(p,q)^{\mathrm{cl}}$ meets $\mathcal{O}^L(p,q)^{\mathrm{cl}}$ in a single point. So given $f \in \mathbb{C}[\mathcal{O}^L(p,q)^{\mathrm{cl}}]$, we can extend it to $\mathcal{O}^K(p,q)^{\mathrm{cl}}$ by declaring that it be constant on $\overline{U}_K$ orbits. It is easy to check that the resulting extension is nothing but $p^*(f)$.

**Proposition 3.2.** *Recall the choices of positive roots $\Delta_K^+$ and $\Delta_L^+$ given in the introduction. The restriction $j^*$ is a bijection from the set of $\Delta_K^+$ lowest weight vectors in $\mathbb{C}[\mathcal{O}^K(s)^{\mathrm{cl}}]$ to the set of $\Delta_L^+$ lowest weight vectors in $\mathbb{C}[\mathcal{O}^L(s)^{\mathrm{cl}}]$. The inverse bijection is the extension map $p^*$.*

*Proof.* Clearly if $f$ is a $\Delta_K^+$ lowest weight vector, then $j^*(f)$ is a $\Delta_L^+$ lowest weight vector. On the other hand, if $f$ is a $\Delta_L^+$ lowest weight vector, then we have seen that $p^*(f)$ is constant on $\overline{U}_K$ orbits, and so indeed is a $\Delta_K^+$ lowest weight vector. The map $p^*$ is injective. Since any $\Delta_K^+$ lowest weight vector is constant on $\overline{U}_K$ orbits, it is also surjective, with inverse $j^*$. □

Only certain orbits $\mathcal{O}^L(p,q)$ will arise for us, and we find it convenient to specialize our notation further. Define

$$\begin{aligned}
&\mathcal{O}^L(s) = \mathcal{O}(m,m) && \text{if } s \geq m\\
&\mathcal{O}^L(s) = \mathcal{O}(m,s) && \text{if } 0 \leq s \leq m \text{ and } (l,s) \neq (2m,0)\\
&\mathcal{O}^L(s) = \mathcal{O}(m-1,0) && \text{if } s = 0 \text{ and } l = 2m.
\end{aligned}$$

Set $\mathcal{O}^K(s) = K^{\mathbb{C}} \cdot \mathcal{O}^L(s)$ and $\mathcal{O}(s) = G^{\mathbb{C}} \cdot \mathcal{O}^L(s)$. Since each $\mathcal{O}(p,q)$ above is a single $L^{\mathbb{C}}$ orbit and since $L^{\mathbb{C}} \subset K^{\mathbb{C}}$, each $\mathcal{O}^K(s)$ is a single $K^{\mathbb{C}}$ orbit. With this notation, it is useful to reformulate the main results of [K3] discussed in the introduction in light of Proposition 3.2. This is Theorem 1.1(3).

**Corollary 3.3.** *Retain the notation of Proposition 3.2 and of the introduction. In particular, recall the $\Delta_K^+$ dominant weight $\Lambda_s$. Fix an irreducible representation $V^\lambda$ of $K$ with highest weight $\lambda$. Then the multiplicity of $V^\lambda$ in $\pi_s'$ is zero unless $\lambda - \Lambda_s$ is $\Delta_K^+$ dominant. In this case the multiplicity of $V^\lambda$ in $\pi_s'$ coincides with the multiplicity of the irreducible representation of $K^{\mathbb{C}}$ with highest weight $\lambda - \Lambda_s$ in $\mathbb{C}[\mathcal{O}^K(s)^{\mathrm{cl}}]$.*

$\square$

In the notation of Section 2.4, set

$$\mathcal{O}^K_{\mathrm{ex}}(s) = K^{\mathbb{C}}_{\mathrm{ex}} \cdot \mathcal{O}^K(s).$$

We find it convenient to record the explicit tableau parameters as follows.

$$\begin{aligned}
&\mathcal{O}^K_{\mathrm{ex}}(s) = 3_+^m 1_-^{2l-3m} && \text{if } s \geq m;\\
&\mathcal{O}^K_{\mathrm{ex}}(s) = 3_+^s 2_+^{m-s} 2_-^{m-s} 1_-^{2l-4m+s} && \text{if } 0 \leq s \leq m \text{ and } (l,s) \neq (2m,0);\\
&\mathcal{O}^K_{\mathrm{ex}}(s) = 2_+^{m-s-1} 2_-^{m-s-1} 1_-^2 1_+^2 && \text{if } s = 0 \text{ and } l = 2m.
\end{aligned}$$

For applications below, we need some details of the closure ordering of the $K^{\mathbb{C}}$ orbits appearing above. From Theorem 9.3.4 of [CMc], one quickly checks that each of the orbits $\mathcal{O}^K_{\mathrm{ex}}(s)$ splits into two $K^{\mathbb{C}}$ orbits: one is $\mathcal{O}^K(s)$, of course; the other is something different. The main point that we want to make here is that the operation of taking the closure of an orbit does not mix the two kinds of orbits. More precisely, we have

$$\mathcal{O}^K(s)^{\mathrm{cl}} \setminus \mathcal{O}^K(s) = \mathcal{O}(s-1)^{\mathrm{cl}} \quad \text{for } 0 \leq s \leq m \text{ unless } l = 2m \text{ and } s = 1. \tag{3.3}$$

If $l = 2m$ and $s = 1$, then the conclusion must be slightly modified. According to Theorem 9.3.4 of [CMc], the $K^{\mathbb{C}}_{\mathrm{ex}}$ orbit $2_+^m 2_-^m$ splits into four $K^{\mathbb{C}}$ orbits, say $\mathcal{O}_1, \ldots, \mathcal{O}_4$. Then with an appropriate labeling scheme,

$$\mathcal{O}^K(1)^{\mathrm{cl}} \setminus \mathcal{O}^K(1) = \mathcal{O}^K(0)^{\mathrm{cl}} \cup \mathcal{O}_1 \cup \mathcal{O}_2; \quad \text{if } l = 2m. \tag{3.4}$$

here we also note that $\mathcal{O}^K(0)$ has strictly smaller dimension than $\mathcal{O}_1$ or $\mathcal{O}_2$. Equations (3.3) and (3.4) are special cases of the main conjecture of [DLS] whose proof has been announced in general [D]. In our setting, the conclusion of Equations (3.3) and (3.4) are relatively straightforward. We omit the details.

The following lemma will be used below. It follows from the discussion of the previous paragraph.

**Lemma 3.4.** *Fix $k$ such that $0 \leq k \leq m$, and let $d$ denote the dimension of $\mathcal{O}^K(k)$. Then $\mathcal{O}^K(k)$ is the unique $K^{\mathbb{C}}$ orbit of dimension $d$ in the closure of $\mathcal{O}^K(m)$.*

$\square$

We now collect some facts about the relevant complex orbits. In terms of the partition classification, we of course have

$$\begin{aligned}
&\mathcal{O}(s) = 3^m 1^{2l-3m} && \text{if } s \geq m;\\
&\mathcal{O}(s) = 3^s 2^{2m-2s} 1^{2l-4m+s} && \text{if } 0 \leq s \leq m \text{ and } (l,s) \neq (2m,0);\\
&\mathcal{O}(s) = 2^{2m-2s-2} 1^4 && \text{if } s = 0 \text{ and } l = 2m.
\end{aligned}$$

(None of the orbits are very even, so we do not need to attach additional numerals to the parametrizing partition.) Note that $\mathcal{O}(s)$ is special if and only if $s$ is even or $s \geq m$.

For $s$ even, define nilpotent $\mathcal{O}^\vee(s) = d(\mathcal{O}(s))$. Explicitly one may compute

$$\begin{aligned}
&\mathcal{O}^\vee(s) = (2l-2m-1), (2m-s+1), (s-1), 1 && \text{if } 2l-2m \neq 2m-s \text{ and } s \neq 0\\
&\mathcal{O}^\vee(s) = (2l-2m-1), (2m+1) && \text{if } 2l-2m \neq 2m \text{ and } s = 0\\
&\mathcal{O}^\vee(s) = (2l-2m+1)(2l-2m-1) && \text{if } (l,s) = (2m,0).
\end{aligned}$$

Since $d^2$ is the identity on special orbits, $d(\mathcal{O}^\vee(s)) = \mathcal{O}(s))$. (When $s$ is odd, $\mathcal{O}(s)$ is not special, so there is no orbit $\mathcal{O}^\vee$ with $d(\mathcal{O}^\vee) = \mathcal{O}$.)

Let $\nu_s$ denote the infinitesimal character of $\pi_s$. Of course $\nu_s$ is defined only up to the action of the Weyl group. Nonetheless we often find it convenient to pick a particular representative in which all coordinates are positive,

$$\nu_s = (\overbrace{0, 1, \ldots, l-m-1}^{l-m}, \overbrace{\left|\frac{s}{2} - m\right|, \left|\frac{s}{2} - m + 1\right|, \ldots, \left|\frac{s}{2} - 1\right|}^{m}). \tag{3.5}$$

(The use of $\nu_s$ to represent both an infinitesimal character and a particular representative causes no confusion in practice.) Notice $\nu_s$ is integral if $s$ is even; otherwise the integral Weyl group is of type $\mathrm{D}_m \times \mathrm{D}_{l-m}$.

**Lemma 3.5.** *Recall the infinitesimal character $\chi(\mathcal{O}^\vee(s))$ attached to $\mathcal{O}^\vee(s)$ (Section 2.2). Then*

$$\nu_s = \chi(\mathcal{O}^\vee(s)) \qquad \text{if } 0 \leq s \leq m+1 \text{ is even.} \tag{3.6}$$

*If $s$ is odd, there is no nilpotent orbit $\mathcal{O}^\vee$ such that $\nu_s = \chi(\mathcal{O}^\vee)$.*

*Proof.* The first assertion follows easily from the tableau parameters for $\mathcal{O}^\vee(s)$ given above, the explicit computation of the semisimple element of the Jacobsen-Morozov triple given, for instance, in [CMc, Section 5.2], and (3.5). As for the second assertion, note that $\chi(\mathcal{O}^\vee)$ for any orbit $\mathcal{O}^\vee$ is integral (for instance, the last paragraph of the proof of [BV3, Proposition A1]). But $\nu_s$ is not integral is $s$ is odd. ☐

**Corollary 3.6.** *If $0 \leq s \leq m+1$ is even, then $\mathrm{J}_{\max}(\nu_s) = \mathrm{J}_{\max}(\mathcal{O}^\vee(s))$ is special unipotent. If $s$ is odd, then $\mathrm{J}_{\max}(\nu_s)$ is not special unipotent.*

*Proof.* This follows from the definitions. ☐

To conclude this section, we record some very specific truncated induction calculations that we will need below.

**Lemma 3.7.** *Let $W' = W(\mathrm{D}_{l'}) \times W(\mathrm{D}_{l''})$ and $W = W(\mathrm{D}_l)$. Identity nilpotent orbits of $\mathfrak{so}(2l,\mathbb{C})$ with partitions of $2l$ in which even parts occur with even multiplicity. (No very even orbits appear in the statements below, so no additional numerals are needed in this parametrization.) Similarly identify nilpotent orbits in $\mathfrak{so}(2l',\mathbb{C})$ and $\mathfrak{so}(2l'',\mathbb{C})$ by appropriate partitions. Recall the notation for the Springer correspondence and truncated induction in Section 2.5. Let $a$, $b$ and $c$ be even integers such that $c \geq a+b$ and $b \neq 0$. Then*

$$j_{W'}^{W}\left(\mathrm{sp}(2^a 1^b) \otimes \mathrm{sp}(1^c)\right) = \mathrm{sp}(3^{a+1}2^{b-2}1^{c-a-b+1}). \tag{3.7}$$

*Sketch.* The truncated induction operation $j_{W'}^{W}$ is easy to compute in terms of the pairs-of-partitions parametrization ([Ca, Section 11.4]),

$$j_{W'}^{W}(\pi_{(p_1,p_2)}, \pi_{(q_1,q_2)}) = \pi_{(\overline{p_1 q_1}, \overline{p_2 q_2})};$$

here $\overline{p_1 q_1}$ is the partition whose columns (when viewed as a Young diagram) are precisely the union of the columns of $p_1$ and $q_1$; and likewise for $\overline{p_2 q_2}$. See [Ca, Proposition 11.4.4]. Thus if we can compute $\mathrm{sp}(\mathcal{O})$ in terms of the pairs-of-partition classification, we can easily make the computation of the lemma. This computation is due to Lusztig (with some elaboration by McGovern) and is given, for instance, on page 80 of [Mc2]. The proposition follows. □

We isolate the exact statement that we will need below.

**Corollary 3.8.** *Let $W' = W(\mathrm{D}_m) \times W(\mathrm{D}_{l-m})$ with $2m \leq l$ and set $W = W(\mathrm{D}_l)$. Let $s$ be an odd integer between $1$ and $m$. Then*

$$\pi' = \mathrm{sp}\left(2^{s-1}1^{2m-2(s-1)}\right) \boxtimes \mathrm{sp}(1^{2l-2m})$$

*is the unique special representation of $W'$ such that*

$$j_{W'}^{W}(\pi') = \mathrm{sp}\left(\mathcal{O}(s)\right).$$

*Proof.* This follow from (3.7) and the injectivity of (2.9). □

## 4. Proof of Theorem 1.1

We proved Theorem 1.1(3) in Corollary 3.3. We prove the remaining parts of the theorem in this section.

### 4.1. Proof of Theorem 1.1(1) for $\mathbf{s \geq m+1}$.

**Lemma 4.1.** *If $s \geq m+1$, then $\pi'_s$ is irreducible.*

*Proof.* We recall some results from [V2], and for the moment assume $G$ is an arbitrary reductive group in Harish-Chandra's class. Suppose $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is a $\theta$-stable parabolic in $\mathfrak{g}$, and $\mathbb{C}_\lambda$ is a one-dimensional $(\mathfrak{l}, L \cap K)$-module in the weakly fair range for $\mathfrak{q}$ (see page 35 in [KV] for a definition). Define $A_{\mathfrak{q}}(\lambda)$ to be the derived functor module defined in Chapter 5 of [KV]. Then the main result of Section 6 of [V2] asserts that if $G^{\mathbb{C}} \cdot (\mathfrak{u} \cap \mathfrak{p}) = G^{\mathbb{C}} \cdot \mathfrak{u}$, then $A_{\mathfrak{q}}(\lambda)$ is irreducible or zero.

Now return to the setting of Theorem 1.1, and recall the $\theta$-stable parabolic $\mathfrak{q}$ and character $\lambda_s$. In the notation of the previous paragraph $\pi_s = A_{\mathfrak{q}}(\lambda_s)$. As remarked in the introduction the condition $s \geq m{+}1$ implies that $\lambda_s$ is in the weakly fair range for $\mathfrak{q}$ and hence $\pi_s = \pi_s'$. Of course [K3] shows that $\pi_s \neq 0$, so the lemma will follow from Vogan's theorem if we can show that $G^{\mathbb{C}}\cdot(\mathfrak{u}\cap\mathfrak{q}) = G^{\mathbb{C}}\cdot\mathfrak{u}$. (In fact, it follows from Section 6 of [V2], that in our context we can get by with a little less: we need only show the $\mathrm{O}(2l,\mathbb{C})$ saturation of $\mathfrak{u}\cap\mathfrak{p}$ coincides with the $\mathrm{O}(2l,\mathbb{C})$ saturation of $\mathfrak{u}$.) This is very easy to check directly. It is clear (from Proposition 3.1 of [K3], for instance) that an element of $\mathcal{O}^L(m)$ is generic in $\mathfrak{u}\cap\mathfrak{p}$. We have seen (in the discussion after (3.5)) that the Jordan form of such an element consists of $m$ blocks of size 3 and $l-m$ blocks of size 1. We conclude that $G^{\mathbb{C}}\cdot(\mathfrak{u}\cap\mathfrak{p})$ is the closure of a $G^{\mathbb{C}}$ orbit consisting of matrices with this Jordan form. On the other hand, Corollary 7.3.4(ii) of [CMc] (for example) shows that $G^{\mathbb{C}}\cdot\mathfrak{u}$ is the closure of a $G^{\mathbb{C}}$ orbit that also consists of matrices with the same Jordan canonical form. Now instead of considering $G^{\mathbb{C}}$ orbits, enlarge them to $\mathrm{O}(2l,\mathbb{C})$ orbits. Since the Jordan form for such orbits is a complete invariant (Section 2.4), the lemma follows from the previous parenthetic sentence. □

**Remark 4.2.** We thus obtain a Blattner-type formula for the $K$ spectrum of $\pi_s'$ whenever $s \geq m+1$. (See [KV, Theorem 8.29] for a precise statement.) Such a formula of course involves cancellation of positive and negative terms, and thus is very different from the one given in [K3].

4.2. **Proof of Theorem 1.1(2).**

**Lemma 4.3.** *If* $s \geq 0$, $\mathrm{AV}(\pi_s') = \mathcal{O}^K(s)^{\mathrm{cl}}$*; in particular,* $\mathrm{AV}(\mathrm{Ann}(\pi_s')) = \mathcal{O}(s)^{\mathrm{cl}}$.

*Proof.* The second assertion follows from the first by (2.4), and so we may concentrate only on the first. We begin by assuming $s \geq m+1$. The key point (again proved in Section 6 of [V4] for instance) is that the associated variety of any weakly fair $A_{\mathfrak{q}}(\lambda)$ module is simply $K^{\mathbb{C}}\cdot(\mathfrak{u}\cap\mathfrak{p})$. From Lemma 1.1 of [K3], we know that $K^{\mathbb{C}}\cdot\mathcal{O}(m,m)$ is dense in $K^{\mathbb{C}}\cdot(\mathfrak{u}\cap\mathfrak{p})$. Unwinding the definition of $\mathcal{O}^K(s)$ given in Section 3, we see that the current lemma thus follows when $s \geq m+1$.

Now assume that $0 \leq s \leq m$. Suppose for the moment that we can establish that $\mathrm{AV}(\pi_s')$ is *contained* in the closure of $\mathcal{O}^K(m)$. Since the Gelfand-Kirillov dimension of $\pi_s'$ is equal to the dimension of $\mathrm{AV}(\pi_s')$ ([V4]), Lemma 3.4 would then imply that the current lemma would follow from the equality

$$\text{GK-dimension of } \pi_s' = \dim(\mathcal{O}^K(s)).$$

But this follows directly from Corollary 3.3. More precisely, by shifting by $\Lambda_s$, we can translate the degree filtration of functions on $\mathcal{O}^K(s)$ into a good $K$ invariant filtration on $\pi_s'$. The growth of this former filtration is transparently polynomial of degree equal to the dimension of $\mathcal{O}^K(s)$; and the growth of the latter filtration is just the GK dimension of $\pi_s'$. So, indeed, the GK-dimension of $\pi_s'$ equals the dimension of $\mathcal{O}^K(s)$.

Thus the current lemma is reduced to establishing the inclusion $\mathrm{AV}(\pi_s') \subset \mathcal{O}^K(s)^{\mathrm{cl}}$. From the definition of $\pi_s'$, we know that $\pi_s'$ is a subquotient of $\pi_s$. It is easy to see that $\pi_s$ is a subquotient of a translation functor applied to $\pi_{s+2r}$ for any $r \geq 0$. (To see this, note that translation functors commute with the derived Bernstein functor; then on the level of generalized Verma modules use a standard argument reproduced, for instance, in proof of Lemma 8.35 of [KV].) In particular, we may take $r$ so large that $s+2r \geq m+1$. The associated variety is not affected by tensoring with finite dimensionals (see [BB, Lemma 4.1] for instance) and it is additive with respect to subquotients (see (2.1)). Since we have already proved that $\mathrm{AV}(\pi_{s+2r}) = \mathcal{O}^K(m)^{\mathrm{cl}}$ for $s+2r \geq m+1$, it follows that $\mathrm{AV}(\pi'_s) \subset \mathcal{O}^K(m)^{\mathrm{cl}}$. So the lemma follows. □

4.3. **Proof of Theorem 1.1(4).** For $s \geq m+1$ (that is, in the weakly fair range) Theorem 1.1(4) follows from the (relatively elementary) asymptotic localization argument given in Proposition 6.3 of [PT]. But this argument cannot be adapted to the range of $s$ between 0 and $m$. Instead we give a uniform proof that works for all $s \geq 0$, but which is based on the (relatively difficult) main results of [K3] in the guise of Corollary 3.3. The main idea of this section, Proposition 4.6, was suggested by David Vogan.

We are going to use Proposition 2.1 to compute the multiplicity in Theorem 1.1(4). The first point is that in many cases the "edges" of the sums in (2.3) do not affect the limit.

**Lemma 4.4.** *Let $G$ be an arbitrary connected real reductive group with maximal compact subgroup $K$. Let $\mathcal{O}^K$ be a nilpotent orbit of $K^{\mathbb{C}}$ on $\mathfrak{p}$. Fix a Cartan $H^{\mathbb{C}}$ for $K^{\mathbb{C}}$ and choose a set of positive roots $\Delta^+(\mathfrak{k},\mathfrak{h})$. Let $\Lambda^+$ denote the corresponding set of dominant weights for $K^{\mathbb{C}}$, and for $\lambda \in \Lambda^+$ let $V^\lambda$ denote the corresponding irreducible representation of $K^{\mathbb{C}}$. Let $c_\lambda(\mathcal{O}^K)$ denote the multiplicity of $V^\lambda$ in the ring of algebraic functions on the closure of $\mathcal{O}^K$.*

*Let $M^{\mathbb{C}}$ denote the Levi subgroup of $G^{\mathbb{C}}$ corresponding the subsystem*

$$\Delta_M = \{\alpha \in \Delta(\mathfrak{k},\mathfrak{h}) \mid \langle \alpha, \lambda\rangle = 0 \textit{ whenever } c_\lambda \neq 0\}.$$

*Let $\Delta_M^+ = \Delta^+ \cap \Delta_M$ and set $\Delta_{\mathfrak{n}} = \Delta^+ \setminus \Delta_M^+$. Suppose there exists $\eta \in \Lambda^+$ such that $\langle \eta, \alpha\rangle > 0$ for all $\alpha \in \Delta_{\mathfrak{n}}$ and, moreover, whenever $c_\lambda(\mathcal{O}^K)$ is nonzero, then*

$$\text{the sequence } c_\lambda(\mathcal{O}^K), c_{\lambda+\eta}(\mathcal{O}^K), c_{\lambda+2\eta}(\mathcal{O}^K), \ldots \text{ is weakly increasing.} \tag{4.1}$$

*For $t>0$, set $\Lambda^+_{\leq t} = \{\lambda \in \Delta^+ \mid |\lambda| < t\}$. Fix $N>0$ and set*

$$S^N_{\leq t} = \{\lambda \in \Lambda^+_{\leq t} \mid \langle \lambda, \alpha\rangle < N \textit{ for some } \alpha \in \Delta_{\mathfrak{n}}\}.$$

*Set $T^N_{\leq t} = \Lambda^+_{\leq t} \setminus S^N_{\leq t}$. Then*

$$\lim_{t\to\infty} \frac{\sum_{\lambda \in S^N_{\leq t}} c_\lambda(\mathcal{O}^K)\left(\prod_{\alpha\in\Delta_{\mathfrak{n}}} \langle \lambda+\rho,\alpha\rangle\right)}{\sum_{\lambda \in T^N_{\leq t}} c_\lambda(\mathcal{O}^K)\left(\prod_{\alpha\in\Delta_{\mathfrak{n}}} \langle \lambda+\rho,\alpha\rangle\right)} = 0. \tag{4.2}$$

**Remark 4.5.** It seems likely that the existence of $\eta$ satisfying the condition in (4.1) is guaranteed under very mild conditions. In any event, for the orbits $\mathcal{O}^K(s)$ appearing in Theorem 1.1, the explicit computation of [K3, Theorem 9.4(c)] shows that such an $\eta$ does indeed exist. In a little more detail, in the usual coordinates, write a dominant weight for $K^{\mathbb{C}} = \mathrm{Spin}(2m,\mathbb{C}) \times \mathrm{Spin}(2l-2m,\mathbb{C})$ as $(a_1,\ldots,a_m; b_1,\ldots,b_{l-m})$ with $a_1 \geq \cdots \geq a_{m-1} \geq |a_m|$ and similarly for the $b_i$'s. In these coordinates, it follows from [K3, Theorem 9.4(c)] that the $\Delta_M$ of Lemma 4.4 is of Type $\mathrm{D}_{l-2m}$ in the last $l-2m$ coordinates,

$$\Delta_M = \{\pm e_i \pm e_j \mid l-2m+1 \leq i < j \leq l\}.$$

In these coordinates, take

$$\eta = \big(m, m-1, \dots, 1; m, m-1, \dots, 1, \overbrace{0, \dots, 0}^{l-2m}\big).$$

Then if $c_\lambda := c_\lambda(\mathcal{O}^K(s)) \neq 0$, the formulation of [K3, Theorem 9.4(c)] in terms of the Littlewood-Richardson rule shows

$$c_\lambda \leq c_{\lambda+\eta} \leq c_{\lambda+2\eta} \leq \cdots .$$

So indeed (4.1) holds, and it is trivial to check that $\langle \eta, \alpha \rangle > 0$ for all $\alpha \in \Delta_{\mathfrak{n}}$.

*Proof of Lemma 4.4.* Set $c_\lambda = c_\lambda(\mathcal{O}^K)$ for simplicity. Fix $\varepsilon$ and $N$. Choose $M$ such that $N/M < \varepsilon$. For each $t > 0$, choose a number $s(t)$ (depending on $t$) as large as possible such that $M\eta + S^N_{\leq s(t)} \subset \Lambda^+_{\leq t}$. (When $t$ is small compared to $M$ this condition cannot be satisfied and in this case we set $s(t) = -1$.) It is not difficult to see that there is a constant $C$ (independent of $t$ but depending on $M$ and hence on $\varepsilon$) such that

$$t - s(t) < C. \tag{4.3}$$

For each $\lambda' \in S^N_{\leq s(t)}$, let $M_S(\lambda')$ be the largest integer such that $\lambda' + M_S(\lambda')\eta \in S^N_{\leq s(t)}$. We may write $S^N_{\leq s(t)}$ as a *disjoint* union of strings of the form

$$A_S = \{\lambda', \lambda' + \eta, \lambda' + 2\eta, \dots, \lambda' + M_S(\lambda')\eta\}.$$

Since $\langle \eta, \alpha \rangle > 0$ for all $\alpha \in \Delta_{\mathfrak{n}}$, each $M_S(\lambda') \leq N$. On the other hand, we may extend the string in $T^N_{\leq t}$,

$$A_T = \{\lambda' + (M_S(\lambda')+1)\eta, \lambda' + (M_S(\lambda')+2)\eta, \dots, \lambda' + M_T(\lambda')\eta\};$$

here $M_T(\lambda')$ is the largest integer such that $\lambda' + M_T(\lambda')\eta \in T^N_{\leq t}$. Again $T^N_{\leq t}$ is a disjoint union of such strings. By the definition of $s(t)$ (and since $\lambda' \in S^N_{\leq s(t)}$), $M_T(\lambda') \geq M$. Thus (4.1) and the hypothesis that $\langle \eta, \alpha \rangle > 0$ for all $\alpha \in \Delta_{\mathfrak{n}}$ immediately implies that the quotient of the sums over the strings is bounded by $\varepsilon$,

$$\frac{\sum_{\lambda \in A_S} c_\lambda \left(\prod_{\alpha \in \Delta_{\mathfrak{n}}} \langle \lambda + \rho, \alpha \rangle\right)}{\sum_{\lambda \in A_T} c_\lambda \left(\prod_{\alpha \in \Delta_{\mathfrak{n}}} \langle \lambda + \rho, \alpha \rangle\right)} \leq \frac{M_S(\lambda')}{M_T(\lambda')} \leq \frac{N}{M} < \varepsilon.$$

Since each region $S^N_{\leq s(t)}$ and $T^N_{\leq t}$ is a disjoint union of such strings, we conclude

$$\frac{\sum_{\lambda \in S^N_{\leq s(t)}} c_\lambda(\mathcal{O}^K) \left(\prod_{\alpha \in \Delta_{\mathfrak{n}}} \langle \lambda + \rho, \alpha \rangle\right)}{\sum_{\lambda \in T^N_{\leq t}} c_\lambda(\mathcal{O}^K) \left(\prod_{\alpha \in \Delta_{\mathfrak{n}}} \langle \lambda + \rho, \alpha \rangle\right)} < \varepsilon. \tag{4.4}$$

Let $d$ denote the dimension of $\mathcal{O}^K$. The standard theory of Gelfand-Kirillov dimension (together with the Weyl character formula) shows that the denominator in (4.4) grows like some constant times $t^k$ for some $k \leq d$. On the other hand, as $t$ goes to infinity, (4.3) implies that the difference

$$\sum_{\lambda \in S^N_{\leq t}} c_\lambda(\mathcal{O}^K) \left( \prod_{\alpha \in \Delta_{\mathfrak{n}}} \langle \lambda + \rho, \alpha \rangle \right) - \sum_{\lambda \in S^N_{\leq s(t)}} c_\lambda(\mathcal{O}^K) \left( \prod_{\alpha \in \Delta_{\mathfrak{n}}} \langle \lambda + \rho, \alpha \rangle \right)$$

can grow at most like a constant times $t^{k-1}$, the derivative of $t^k$. Thus the conclusion of (4.4) indeed implies (4.2). $\square$

**Proposition 4.6.** *Retain the notation and hypothesis of Lemma 4.4. In particular assume the existence of $\eta$ satisfying (4.1). Fix $\Lambda \in \Lambda^+$. Let $H^{\mathbb{C}}_M = H^{\mathbb{C}} \cap M^{\mathbb{C}}$, and let $V^\Lambda_M$ denote the finite-dimensional representation of $M^{\mathbb{C}}$ with highest weight given by the restriction of $\Lambda$ to $\mathfrak{h}_M$. Then*

$$\lim_{t \to \infty} \frac{\sum_{|\lambda|<t} c_\lambda(\mathcal{O}^K) \dim(V^{\lambda+\Lambda})}{\sum_{|\lambda|<t} c_\lambda(\mathcal{O}^K) \dim(V^\lambda)} = \dim(V^\Lambda_M). \tag{4.5}$$

*Proof.* Again for simplicity set $c_\lambda = c_\lambda(\mathcal{O}^K)$. Let $\rho$ denote the half-sum of the elements of $\Delta^+(\mathfrak{k}, \mathfrak{h})$. Then using the Weyl character formula (and canceling out the Weyl denominators), we may write the quotient in (4.5) as

$$\frac{\sum_{|\lambda|<t} c_\lambda \prod_{\alpha \in \Delta^+} \langle \lambda + \Lambda + \rho, \alpha \rangle}{\sum_{|\lambda|<t} c_\lambda \prod_{\alpha \in \Delta^+} \langle \lambda + \rho, \alpha \rangle};$$

or

$$\frac{\prod_{\alpha \in \Delta^+_M} \langle \Lambda + \rho, \alpha \rangle}{\prod_{\alpha \in \Delta^+_M} \langle \rho, \alpha \rangle} \cdot \frac{\sum_{|\lambda|<t} c_\lambda \prod_{\alpha \in \Delta_{\mathfrak{n}}} \langle \lambda + \Lambda + \rho, \alpha \rangle}{\sum_{|\lambda|<t} c_\lambda \prod_{\alpha \in \Delta_{\mathfrak{n}}} \langle \lambda + \rho, \alpha \rangle}. \tag{4.6}$$

The Weyl dimension formula for $M$ implies that (4.6) becomes

$$\dim(V^\Lambda_M) \frac{\sum_{|\lambda|<t} c_\lambda \prod_{\alpha \in \Delta_{\mathfrak{n}}} \langle \lambda + \Lambda + \rho, \alpha \rangle}{\sum_{|\lambda|<t} c_\lambda \prod_{\alpha \in \Delta_{\mathfrak{n}}} \langle \lambda + \rho, \alpha \rangle}. \tag{4.7}$$

From the definition of $T^M_{\leq t}$, it is easy to see that for every $\varepsilon > 0$, there exists an $N$ such that for each $t > 0$,

$$\frac{\sum_{\lambda \in T^N_{\leq t}} c_\lambda \prod_{\alpha \in \Delta_{\mathfrak{n}}} \langle \lambda + \Lambda + \rho, \alpha \rangle \; - \; \sum_{\lambda \in T^N_{\leq t}} c_\lambda \prod_{\alpha \in \Delta_{\mathfrak{n}}} \langle \lambda + \rho, \alpha \rangle}{\sum_{\lambda \in T_{\leq t}} c_\lambda \prod_{\alpha \in \Delta_{\mathfrak{n}}} \langle \lambda + \rho, \alpha \rangle} < \varepsilon. \tag{4.8}$$

whenever the terms in the numerator and denominator are nonzero.

Now consider the limit appearing in the proposition. Using (4.7), and breaking each sum into pieces according to the partition $\Lambda^+_{\leq t} = S^N_{\leq t} \cup T^N_{\leq t}$, we may write the limit in the proposition as

$$\dim(V^\Lambda_M) \lim_{t \to \infty} \frac{\sum_{\lambda \in S^N_{\leq t}} c_\lambda \prod_{\alpha \in \Delta_{\mathfrak{n}}} \langle \lambda + \Lambda + \rho, \alpha \rangle + \sum_{\lambda \in T^N_{\leq t}} c_\lambda \prod_{\alpha \in \Delta_{\mathfrak{n}}} \langle \lambda + \Lambda + \rho, \alpha \rangle}{\sum_{\lambda \in S^N_{\leq t}} c_\lambda \prod_{\alpha \in \Delta_{\mathfrak{n}}} \langle \lambda + \rho, \alpha \rangle + \sum_{\lambda \in T^N_{\leq t}} c_\lambda \prod_{\alpha \in \Delta_{\mathfrak{n}}} \langle \lambda + \rho, \alpha \rangle}. \tag{4.9}$$

An easy application of Lemma 4.4 shows that the first sum in the numerator and the first sum in the denominator become insignificant as $t$ gets very large. Thus (4.8) implies that for $t$ sufficiently large, the expression in (4.9) is arbitrarily close to $\dim(V^\Lambda_M)$. So the proposition follows. $\square$

Now turn to the proof of Theorem 1.1(4), and adopt the notation used there and the coordinates used in Remark 4.5. Let $\Lambda_s$ denote the highest weight of the lowest $K$-type of $\pi'_s$. Explicitly $\Lambda_s = \lambda_s + 2\delta(\mathfrak{u}\cap\mathfrak{p})$, and in coordinates

$$\Lambda_s = (\overbrace{s/2+l-2m,\ldots,s/2+l-2m}^{m};\overbrace{0,\ldots,0}^{l-2m}). \tag{4.10}$$

In the notation of Proposition 2.1, Corollary 3.3 becomes

$$c_\lambda(\pi'_s) = \begin{cases} 0 & \text{if } \lambda-\Lambda_s \text{ is not dominant; and} \\ c_{\lambda-\Lambda_s}(\mathcal{O}(s)) & \text{if } \lambda-\Lambda_s \text{ is dominant.} \end{cases}$$

Using Theorem 1.1(2), Proposition 2.1, Proposition 4.6, and Remark 4.5, it follows immediately that the multiplicity $m_s$ of $\mathcal{O}(s)$ in the associated cycle is the dimension of $V_M^{\Lambda_s}$ again with notation as in Proposition 4.6. If $(l,s)=(2m,0)$, then $\Lambda_s$ is zero, $V_M^{\Lambda_s}$ is trivial, and the theorem follows. On the other hand, if $(l,s)\neq(2m,0)$ then, as we mentioned in Remark 4.5, $\Delta_M$ is of Type $D_{l-2m}$ in the last $l-2m$ coordinates. So (4.10) shows that the restriction of $\Lambda_s$ to $\mathfrak{h}\cap\mathfrak{m}$ is zero and hence $V_M^{\Lambda_s}$ is trivial. Thus Theorem 1.1(4) follows. ☐

### 4.4. **Proof of Theorem 1.1(5).**

**Lemma 4.7.** *Let $\nu_s$ denote the infinitesimal character of $\pi'_s$ and recall the complex nilpotent orbit $\mathcal{O}(s)$ (Section 3). Assume $0\le s\le m+1$. Then*

$$\mathrm{AV}(\mathrm{J}_{\max}(\nu_s)) = \mathcal{O}(s)^{\mathrm{cl}}. \tag{4.11}$$

*In particular $\mathrm{J}_{\max}(\nu_s)$ is the unique primitive ideal with infinitesimal character $\nu_s$ whose associated variety is contained in $\mathcal{O}(s)^{\mathrm{cl}}$.*

**Proof.** The final assertion follows easily from (4.11): if $I$ has infinitesimal character $\nu_s$, then $I\subset \mathrm{J}_{\max}(\nu_s)$ by maximality, and Proposition 2.2 (together with (4.11)) implies that either $I=\mathrm{J}_{\max}(\nu_s)$ or $\mathcal{O}(s)^{\mathrm{cl}}\subsetneq \mathrm{AV}(I)$.

For $s$ even, we give two proofs of (4.11). The first proof is elegant and conceptual (but does not extend to handle the odd case) and the other is purely computational (but does extend). The latter also supplies some useful explicit axillary information.

Suppose $s$ is even and recall the notation $\mathcal{O}^\vee(s)$ of Section 3; so $d(\mathcal{O}^\vee(s))=\mathcal{O}$. Then $\mathrm{J}_{\max}(\nu_s) = \mathrm{J}_{\max}(\mathcal{O}^\vee(s))$ by Lemma 3.5. Proposition 2.3 computes

$$\mathrm{AV}(\mathrm{J}_{\max}(\nu_s)) = d(\mathcal{O}^\vee(s))^{\mathrm{cl}} = \mathcal{O}(s)^{\mathrm{cl}},$$

and the lemma follows.

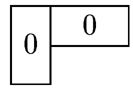
We now give an alternative proof (again assuming $s$ is even) using the tableau classification of primitive ideals. Recall that we know $\mathcal{O}(s)$ has Jordan form

$$3^s 2^{2m-2s} 1^{2l-4m+s} \qquad \text{if } 0\le s\le m \text{ and } (l,s)\neq(2m,0); \tag{4.12}$$

$$2^{2m-2}1^4 \qquad \text{if } s=0 \text{ and } (l,s)=(2m,0). \tag{4.13}$$

We now write down a standard domino $\nu_s$-tableau of this shape. (The reader is invited to consult Example 4.8 below.)

Suppose first $(l,s) \neq (2m,0)$. Then build a standard domino $\nu_s$-tableau by starting with




Next add the tripled coordinates sequentially as vertical dominos, one in each column. Add the smallest doubled coordinate as a single vertical domino in the first column, and then a horizontal domino spanning the second and third columns. Add the remaining doubled coordinates sequentially as vertical dominos, one in each of the first two columns. Finally add the remaining singleton coordinates sequentially as vertical dominos in the first column. When $(l,s) = (2m,0)$, the coordinates of $\nu_s$ consists of a single 0, a single $m$, and each $1,2,\ldots,m-1$ repeated twice. Begin with a single vertical domino labeled 0 in the first column, add the doubled coordinates sequentially as vertical dominos (one in the first column, one in the second), and finally add the terminal $m$ as a vertical domino in the first column.

Call the tableau constructed in the previous paragraph $T_{\max}$. We leave it to the reader to verify

- (†) $T_{\max}$ is the unique standard domino $\nu_s$-tableau of shape given by (4.12)–(4.13).
- (††) There is no standard domino $\nu_s$-tableau $T'$ whose shape is that of a *special* nilpotent orbit in the boundary of $\mathcal{O}(s)^{\mathrm{cl}}$.

Then (†) shows that the primitive ideal $I_{\max}$ corresponding to $T_{\max}$ is the unique primitive ideal with infinitesimal character $\nu_s$ and associated variety $\mathcal{O}(s)^{\mathrm{cl}}$. (††) then implies that $I_{\max} = \mathrm{J}_{\max}(\nu_s)$ (since if $I_{\max} \neq \mathrm{J}_{\max}(\nu_s)$, then there exists $I_{\max} \subsetneq I$ and Proposition 2.2 would imply the existence of a tableau contradicting (††)). Thus these claims prove the lemma.

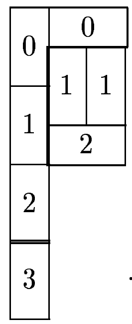
Now suppose $s$ is odd. Recall (from the beginning of Section 3) that the integral Weyl group, say $W_s'$, of $\nu_s$ is of type $\mathrm{D}_m \times \mathrm{D}_{l-m}$. Partition $\nu_s = \nu_s^1 \cup \nu_s^2$ into its half-integer and integer entires. According to Section 2.6, $\mathrm{Prim}(\mathrm{U}(\mathfrak{g}))_{\nu_s}$ is parametrized by pairs $(T_1,T_2)$ with each $T_i$ a standard domino $\nu_s^i$-tableau. If $I$ is parametrized by $(T_1,T_2)$ and has associated variety $\mathcal{O}(s)^{\mathrm{cl}}$, then Corollary 3.8 implies that $T_1$ has shape $2^{s-1}1^{2m-2(s-1)}$ and $T_2$ has shape $1^{2l-2m}$. We write down a pair of standard domino $\nu_s^i$-tableaux of the requisite shapes. (Again the reader is invited to consult Example 4.8 below.) Let $T^2_{\max}$ be the standard domino $\nu_s^2$-tableau consisting of a single column of vertical dominos labeled $0,\ldots,l-m-1$. Let $T^1_{\max}$ be the standard domino $\nu_s^1$-tableau obtained by adding the doubled entries of $\nu_s^1$ entered sequentially as vertical dominos (one in each of the first two columns) and then each of the remaining single entries sequentially as vertical dominos in the first column.

We leave it the reader to verify the following simple combinatorial statements.

- $(*)$ $(T^1_{\max}, T^2_{\max})$ is the unique pair of standard domino $\nu_s^i$-tableau with the respective shapes $2^{s-1}1^{2m-2(s-1)}$ and $1^{2l-2m}$.
- $(**)$ There is no pair of standard domino $\nu_s$-tableau $(T_1',T_2')$ such that $T_1'$ has special shape strictly smaller than the shape of $T^1_{\max}$.

As in the even case, these assertion prove the lemma in the case that $s$ is odd. (Here one must use the conclusion of (2.8).) $\square$

**Example 4.8.** The proof of Lemma 4.7 contains a description of the tableau parameters of $\mathrm{J}_{\max}(\nu_s)$. Here we work out a few examples, Suppose $m = 4$, $l = 8$. Consider first $s = 4$; so according to (3.5), $\nu_4 = (3, 2, 2, 1, 1, 1, 0, 0)$, and the shape of $\mathcal{O}(4)$ is $3^4 1^4$. The procedure described in the proof of Lemma 4.7 produces the domino tableau



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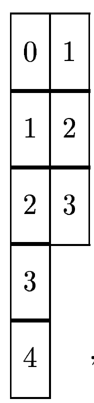
One may quickly check (†), namely that the above configuration is the only standard domino $\nu_s$-tableau of this shape. On the other hand, the special orbits in the boundary of the orbit parametrized by $3^4 1^1$ are parametrized by partition with at most 2 rows of length 3 (and the rest of length 1 or 2). So one quickly deduces that it is impossible to enter the coordinates 0, 0, 1, 1, 1 of $\nu_4$ in such a shape. This is (††) in this example.

Next consider the case $s = 0$ where now

$$\nu_0 = (4, 3, 3, 2, 2, 1, 1, 0),$$

and the shape of $\mathcal{O}(0)$ is $2^6 1^4$. The procedure produces



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It is easy to verify (†) directly. The only special orbit in the boundary of $\mathcal{O}(0)$ have only 1's and 2's in their partition and fewer than $4$ rows of 2's, so there is no way to enter the coordinates in such a shape. This verifies(††).

Now we turn to an odd example. Suppose $m = 5$, $l = 10$, and $s = 5$. Then

$$\nu_5 = (5/2, 3/2, 3/2, 1/2, 1/2, 4, 3, 2, 1, 0).$$

The procedure given in the proof of Lemma 4.7 produces the pair

$$T_1 \quad = \quad \begin{array}{|c|c|} \hline \frac{1}{2} & \frac{1}{2} \\ \hline \frac{3}{2} & \frac{3}{2} \\ \hline \frac{5}{2} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

and

$$T_2 \quad = \quad \begin{array}{|c|} \hline 0 \\ \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}$$

It is trivial to see that this is the unique pair of tableau of this shape with entries in $\nu_5$ as stipulated, so $(*)$ follows. On the other hand, one quickly checks that $T_1$ is the smallest tableau with the given entries. So $(**)$ follows as well.

As another example, let $s = 1$ (with $m = 5$ and $l = 10$ as before). Then

$$\nu_1 = (9/2, 7/2, 5/2, 3/2, 1/2, 4, 3, 2, 1, 0),$$

and the procedure produces the pair consisting of a single column $T_1$ of vertical dominos with the entries $(9/2, 7/2, \ldots, 1/2)$ and a single column $T_2$ (as above) of vertical dominos labeled $(4, 3, 2, 1, 0)$. $(*)$ and $(**)$ again follow easily. $\square$

**Corollary 4.9.** *If* $0 \leq s \leq m$*, then* $\mathrm{J}_{\max}(\nu_s)$ *is weakly unipotent.*

*Proof.* Suppose $X$ is a Harish-Chandra module with annihilator $\mathrm{J}_{\max}(\nu_s)$ and $F$ is finite-dimensional. It is elementary to verify that if $Y$ is an irreducible constituent of $X \otimes F$, then $\mathrm{AV}(\mathrm{Ann}(Y)) \subset \mathcal{O}(s)$; see [BB, Lemma 4.1], for instance. Of course $Y$ has an infinitesimal character that differs from $X$ by an $l$-tuple of integers.

Suppose first that $s$ is odd. The previous paragraph, together with the conclusion of (2.8), implies that the the current corollary amounts to showing that there is no pair of standard domino tableaux $(T_1', T_2')$ such that

(1) $T_1'$ has size $2l$ and shape that is smaller (or the same) as $2^{s-1}1^{2m-2(s-1)}$.
(2) The entries $\nu_1', \ldots, \nu_l'$ of $T_1'$ are positive half-integers.
(3) The shape of $T_2'$ is $1^{2l-2m}$.

(4) The entries $\nu''_1, \ldots, \nu''_{l-m}$ of $T'_2$ are positive integers.
(5) $\sum_i (\nu'_i)^2 + \sum_i (\nu''_i)^2 < \sum_i \nu_i^2$.

A simple combinatorial argument shows this is impossible and the lemma follows. We omit the details.

When $s$ is even, we have already seen $\mathrm{J}_{\max}(\nu_s)$ is special unipotent (Corollary 3.6). So the result of Barbasch-Vogan mentioned in Section 2.2 ([BV3, Lemma 5.7]) implies $\mathrm{J}_{\max}(\nu_s)$ is weakly unipotent. (One can also see this by a combinatorial argument analogous to the one sketched above for $s$ odd.) ☐

**Corollary 4.10.** *Suppose $0 \leq s \leq m$. If $X_s$ is an irreducible factor of $\pi'_s$, then*

$$\mathrm{Ann}(X_s) = \mathrm{Ann}(\pi'_s) = \mathrm{J}_{\max}(\nu_s).$$

*Consequently*

$$\mathrm{AV}(X_s) = \mathcal{O}^K(s)^{\mathrm{cl}}, \tag{4.14}$$

*and, in particular, all such $X_s$ have the same Gelfand-Kirillov dimension.*

*Proof.* Suppose $X_s$ is an irreducible constituent of $\pi'_s$. Then $X_s$ has infinitesimal character $\nu_s$ and

$$\mathrm{AV}(X_s) \subset \mathrm{AV}(\pi'_s) = \mathcal{O}^K(s)^{\mathrm{cl}}; \tag{4.15}$$

the former inclusion follow from (2.1) while the latter equality is Theorem 1.1(2). By (2.4), we thus conclude $\mathrm{AV}(\mathrm{Ann}(X_s)) \subset \mathcal{O}(s)^{\mathrm{cl}}$. So the final assertion of Lemma 4.7 implies $\mathrm{Ann}(X_s) = \mathrm{J}_{\max}(\nu_s)$, proving the first assertion of the current corollary.

To prove (4.14), note that we have just seen in (4.15) that $\mathrm{AV}(X_s) \subset \mathcal{O}^K(s)^{\mathrm{cl}}$. If the inclusion were proper, (2.4) would imply $\mathrm{Ann}(X_s) \subsetneq \mathcal{O}(s)^{\mathrm{cl}}$. But we just saw that $\mathrm{Ann}(X_s) = \mathrm{J}_{\max}(\nu_s)$, so the purported proper inclusion contradicts Lemma 4.7. Thus the inclusion is indeed equality, proving (4.14). The final assertion of the corollary is clear. ☐

4.5. **Proof of Theorem 1.1(1) for $\mathbf{0 \leq s \leq m+1}$.** According to Theorem 1.1(4) (which was proved in Section 4.3), $m_{\mathcal{O}(s)}(\pi'_s) = 1$. The additivity of the associated cycle construction (see (2.2)) together with the conclusion of Corollary 4.10 immediately implies that $\pi'_s$ is irreducible for $0 \leq s \leq m+1$. (Note that for $s = m+1$, the "edge" of the weakly fair range, this argument overlaps with the one given in Section 4.1.) ☐

4.6. **A failed alternative attempt at proving Theorem 1.1(1) for $0 \leq \mathbf{s} \leq \mathbf{m+1}$.** Suppose that one could prove that there is a unique Harish-Chandra module with annihilator $\mathrm{J}_{\max}(\nu_s)$. Then Corollary 4.10 would imply that that $\pi'_s$ is a multiple of a single irreducible representation; but the lowest $K$-type $\Lambda_s$ appears with multiplicity one in $\pi'_s$, so the irreducibility assertion would follow. When $s$ is even, for instance, the number of Harish-Chandra modules with annihilator $\mathrm{J}_{\max}(\nu_s)$ is equal to the multiplicity of the special $W$ representation parametrized by $\mathrm{AV}(\mathrm{J}_{\max}(\nu_s))$ in the full coherent continuation representation at infinitesimal character $\rho$. This latter representation is easy to compute using an old observation of Barbasch-Vogan; see formula (11) in [Mc1] for a very closely related computation. Thus the relevant multiplicity can be explicitly computed. When $(l, s) = (2m, 0)$ it is indeed one (and the irreducibility assertion follows). But outside of this case the multiplicity is strictly greater than 1, and for $s \geq 1$, this approach to the irreducibility assertion fails. One encounters the same kind of failure when considering $s$ odd.

The approach of the previous paragraph evidently fails because there are Harish-Chandra modules other than $\pi'_s$ with weakly unipotent annihilator $\mathrm{J}_{\max}(\nu_s)$. In other words, *the representation $\pi'_s$ is not the only weakly unipotent representation with annihilator* $\mathrm{J}_{\max}(\nu_s)$. Where one might find the missing representations? Since $\mathrm{AV}\big(\mathrm{Ann}(\mathrm{J}_{\max}(\nu_s))\big)$ is not Richardson, it is natural to expect that they arise as constituents of representations constructed exactly as $\pi'_s$ was constructed, except that one relaxes the conditions (essential for [K3]) that $2m \leq l$ and that $\Lambda_s$ be dominant.

When $s$ is even (so that $\pi'_s$ factors to a linear group) and between $0$ and $m+1$, A. Paul has located the representation $\pi'_s$ in the Howe correspondence. Very roughly speaking she obtains $\pi'_s$ as two-step $\theta$ lift from $\mathrm{O}(0, s)$ to $\mathrm{Sp}(2m, \mathbb{R})$ and finally to $\mathrm{O}(2m, 2l - 2m)$. Her results also show where to find the missing unipotent representations: one should find them as the same kind of two-step lift beginning instead with the trivial representation of a noncompact orthogonal group $\mathrm{O}(s_1, s_2)$ with $s_1 + s_2 = s$. We would like to explore this point of view in a future project.

4.7. **Yet another failed alternative attempt at proving Theorem 1.1(1) for $0 \leq \mathbf{s} \leq \mathbf{m} + \mathbf{1}$.** When $s$ is even and $(l, s) \neq (2m, 0)$, then [K1, Theorem 5.1] provides an embedding of $\pi'_s$ into a derived functor module $A_{\mathfrak{q}'}(\lambda')$ induced from a one-dimensional module in the weakly fair range. (Here $\mathfrak{q}'$ is different from the $\mathfrak{q}$ used above.) Thus if one could prove that $A_{\mathfrak{q}'}(\lambda')$ is irreducible, then the irreducibility of $\pi'_s$ would follow. Unfortunately the former module reduces, and this approach fails.

## 5. The case of odd orthogonal groups

Fix, as usual, $0 \leq m \leq l/2$ and let $G$ be the universal cover of $\mathrm{SO}_\circ(2m, 2l - 2m + 1)$. There is a corresponding set of results for this group. The proofs are nearly identical to even case (and often easier), and in this section we outline the explicit details. We set $G^{\mathbb{C}} = \mathrm{Spin}(2l + 1, \mathbb{C})$ and $K^{\mathbb{C}} = \mathrm{Spin}(2m, \mathbb{C}) \times \mathrm{Spin}(2l - 2m + 1, \mathbb{C})$, and adopt analogous notation as used in the previous sections.

Adopt the usual coordinates for the roots of the compact Cartan in $\mathfrak{g}$ and write

$$\Delta = \{\pm e_i \,| 1 \leq i \leq l\} \cup \{\pm e_i \pm e_j \mid 1 \leq i < j \leq l\}.$$

Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be the $\theta$-stable parabolic determined by the unique noncompact simple root $e_m - e_{m+1}$. In these coordinates for $s \in \mathbb{Z}$, set

$$\lambda_s = (-l + (s-1)/2, \ldots, -l + (s-1)/2; 0, \ldots, 0).$$

Let $\Lambda_s = \lambda_s + 2\delta(\mathfrak{u} \cap \mathfrak{p})$, or explicitly

$$\Lambda_s = (l - 2m + 1 + (s-1)/2, \ldots, l - 2m + 1 + (s-1)/2; 0, \ldots, 0).$$

Using notation analogous to that of the introduction, set $S = \dim(\mathfrak{u} \cap \mathfrak{k})$, and let

$$\pi_s = \Pi_s\big(N(\lambda + 2\delta(\mathfrak{u}\cap\mathfrak{p})\big) \qquad \pi_s' = \Pi_s\big(N'(\lambda + 2\delta(\mathfrak{u}\cap\mathfrak{p})\big).$$

As in the even case, these representation as in the discrete series if $s > 2l-1$ (in which case they are again the smallest possible kind of such representations), and $\lambda_s$ is in the weakly fair range if $s \geq m+1$. For more details, see Table 4 of Section 11 of [K3]. In Section 11 (see Table 5) of [K3], Knapp proves that for $s \geq 0$, $\pi_s'$ is associated to a certain $L^{\mathbb{C}}$ orbit $\mathcal{O}^L(s)$ on $\mathfrak{u}\cap\mathfrak{p}$.

**Theorem 5.1.** *Let $\nu_s$ denote the infinitesimal character of $\pi_s'$; see (5.3) below for explicit details. Let $\mathcal{O}^K(s) = K^{\mathbb{C}} \cdot \mathcal{O}^L(s)$; this orbit is computed explicitly in terms of the tableau classification in (5.1) –(5.2).*

(1) *$\pi_s'$ is irreducible for all $s \geq 0$.*
(2) *The associated variety of $\pi_s'$ is the closure of $\mathcal{O}^K(s)$; in particular it is irreducible.*
(3) *The multiplicity of the irreducible representation $V^\lambda$ of $K$ with $\Delta_K^+$ highest weight $\lambda$ in $\pi_s'$ is zero unless $\lambda - \Lambda_s$ is $\Delta_K^+$-dominant. In this case the multiplicity of $V^\lambda$ in $\pi_s'$ coincides with the multiplicity of the irreducible representation of $K$ with highest weight $\lambda - \Lambda_s$ in the ring of algebraic functions on the closure of $\mathcal{O}^K(s)^{\mathrm{cl}}$.*
(4) *The multiplicity of $\mathcal{O}^K(s)$ in the associated cycle of $\pi_s'$ is exactly 1.*
(5) *Suppose $0 \leq s \leq m+1$. Then the annihilator of $\pi_s'$ is the maximal primitive ideal $\mathrm{J}_{\max}(\nu_s)$ at infinitesimal character $\nu_s$. Moreover*
  (a) *If $s$ is odd, $\mathrm{J}_{\max}(\nu_s)$ is special unipotent, and thus $\pi_s'$ is special unipotent.*
  (b) *If $s$ is even, $\mathrm{J}_{\max}(\nu_s)$ is not special unipotent, yet it is weakly unipotent; so $\pi_s'$ is weakly unipotent.*

We now enumerate some explicit details. At the end of the section, we sketch how to use them to trace through the proof of Theorem 1.1 to obtain Theorem 5.1.

Nilpotent orbits for $\mathfrak{so}(2l+1,\mathbb{C})$ are parametrized by partitions of $2l+1$ in which even parts occur with even multiplicity. Such an orbit is special (for type B) if there is an even number of odd parts between consecutive even parts or greater than the largest even part. Nilpotent orbits of $K^{\mathbb{C}}_{\mathrm{ex}} = \mathrm{O}(2m) \times \mathrm{O}(2l-2m+1)$ on $\mathfrak{p}$ are parametrized by orthogonal signature $(2m, 2l-2m+1)$-signed tableau. (The terminology is defined in Section 2.4.) In notation analogous that established in Section 2.4, it is easy to verify that

(5.1) $\qquad \mathcal{O}^K_{\mathrm{ex}}(s)$ is parametrized by $3_+^m 1_-^{2l-3m+1}$ for $s \geq m$; and

(5.2) $\qquad$ by $3_+^s 2_+^{m-s} 2_-^{m-s} 1^{2l-4m+s+1}$ for $0 \leq s \leq m$.

Since we need to use the duality map $d$ of Section 2.2, we need to discuss nilpotent orbits for $\mathfrak{sp}(2l,\mathbb{C})$. Such orbit are parametrized by partitions of $2l$ in which odd parts occur with even multiplicity. In terms of this parametrization, given $s$ *odd* between 1 and $m+1$, set

$$\mathcal{O}^\vee(s) = 2l-2m+1, 2m-s, s.$$

Then

$$d(\mathcal{O}(s)) = \mathcal{O}^\vee(s),$$

and since $\mathcal{O}(s)$ is special $d(\mathcal{O}^\vee(s)) = \mathcal{O}(s)$. For $s$ even, $\mathcal{O}(s)$ is not special, and there is no orbit $\mathcal{O}^\vee$ with $d(\mathcal{O}^\vee)) = \mathcal{O}(s)$.

The infinitesimal character of $\pi_s$ is given by a representative

(5.3) $\quad \nu_s = (|-1+s/2|, |-2+s/2|, \dots, |-m+s/2|; l-m-1/2, l-m-3/2, \dots, 1/2).$

This is integral if $s$ is odd, and has integral Weyl group of type $\mathrm{B}_m \times \mathrm{B}_{l-m}$ if $s$ is even. When $s$ is odd between 1 and $m+1$, then

$$\chi(\mathcal{O}^\vee(s)) = \nu_s,$$

but such an equality cannot hold if $s$ is even. Thus $\mathrm{J}_{\max}(\nu_s)$ is special unipotent if $s$ is odd (and between 1 and $m+1$), but never special unipotent if $s$ is even. (This is the analog of Corollary 3.5.)

If $s$ is odd, primitive ideals in $\mathrm{U}(\mathfrak{g})$ at infinitesimal character $\nu_s$ are parametrized by Young diagrams of size $2l+1$ and special shape where the upper left 1-by-1 box is labeled 0 and and the remaining shape is tiled by dominos labeled with the entries of $\nu_s$ so that the numbers strictly increase down columns and weakly increase across rows. (We continue to call such a configuration a standard domino $\nu_s$-tableau.) If $s$ is even, partition the coordinates of $\nu_s$ into an $m$-tuple of integers $\nu_s^1$ and $l-m$ half-integers $\nu_s^2$. Primitive ideals in $\mathrm{U}(\mathfrak{g})$ at infinitesimal character $\nu_s$ are parametrized by pairs consisting of a standard domino $\nu_s^1$-tableau of special shape (for type B) and a standard domino $\nu_s^2$-tableau of special shape (for type B).

We now write describe the tableau parameters for a primitive ideal at infinitesimal character $\nu_s$ for $0 \le s \le m+1$. First assume $s$ is odd. Begin with the requisite box labeled zero. Add the tripled coordinates of $\nu_s$ as vertical dominos, one in each of the first three columns. Add the smallest doubled coordinate as a single vertical domino in the first column and as a horizontal domino spanning the second and third columns. Add the remaining doubled coordinates as two vertical dominos, one in each of the first two columns. Finally add the singleton coordinates of $\nu_s$ as vertical dominos in the first column. The resulting tableau has shape that coincides with that of $\mathcal{O}(s)$, and hence this orbit is dense in the associated variety of the primitive ideal the tableau parametrizes. If $s$ is odd, build a pair of tableau $(T_1, T_2)$ by letting $T_2$ denote a single column of vertical dominos with the half-integer entries of $\nu_s$. Construct $T_s$ by adding the doubled integer coordinates as vertical dominos to the requisite zero box, one in each of the first two columns, and then add the remaining singleton coordinates as vertical dominos in the first column. A truncated induction calculation along the lines of Corollary 3.8 shows that the primitive ideal parametrized by this pair has associated variety $\mathcal{O}(s)^{\mathrm{cl}}$.

We now sketch how to assemble these details to prove Theorem 5.1. The proof of Theorem 5.1(3) follows as in the proof of Corollary 3.3. The proof of Theorem 5.1(1) for $s \ge m+1$ follows exactly as in Section 4.1 where one must again check the relatively straightforward assertion that $G^{\mathbb{C}} \cdot (\mathfrak{u} \cap \mathfrak{p}) = G^{\mathbb{C}} \cdot \mathfrak{u}$. Theorem 5.1(2) follow from the same argument as in Section 4.2. It rests on Theorem 5.1(3) (already proven) and the fact that the obvious analog of the key observation of Lemma 3.4 is still valid (and is in fact much easier in the odd setting). Theorem 5.1(4) follows as in

Section 4.3. The main point here is that the representation $V_M^{\Lambda_s}$ is still trivial; this follows from the explicit details of [K3, Theorem 11.2(c)]. As for Theorem 5.1(4), a simple combinatorial argument (as in Lemma 4.7) shows that the tableau described above indeed parametrize $\mathrm{J}_{\max}(\nu_s)$ for $0 \leq s \leq m+1$, and that $\mathrm{AV}(\mathrm{J}_{\max}(\nu_s)) = \mathcal{O}(s)^{\mathrm{cl}}$. (This time for the case of $s$ odd, as in the first paragraph of the proof of Lemma 4.7, one could also also deduce the associated variety computation from Proposition 2.3 since when $s$ is odd we mentioned above that $\nu_s = \chi(\mathcal{O}^\vee(s))$ and hence $\mathrm{J}_{\max}(\nu_s)$ is special unipotent.) That $\mathrm{J}_{\max}(\nu_s)$ is weakly unipotent follows from a combinatorial argument as in Corollary 4.9. This proves Theorem 5.1(5). The conclusion of Corollary 4.10 again holds for the same reason, and the argument of Section 4.5 establishes Theorem 5.1(1). This completes our sketch of the proof of the theorem.

We remark that as in Section 4.6, there are other weakly unipotent representations with associated variety $\mathcal{O}^K(s)^{\mathrm{cl}}$ besides $\pi'_s$ for $0 \leq s \leq m+1$. Again Paul's computations identify the $\pi'_s$ (for $s$ odd and between 1 and $m+1$) in the Howe correspondence, and suggest where to find the missing unipotent representations.

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