REGULAR ORBITS OF SYMMETRIC SUBGROUPS ON PARTIAL FLAG VARIETIES

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1. Introduction

Suppose G is a complex connected reductive algebraic group and let θ denote an involutive automorphism of G. Write K for the fixed points of θ, and $\mathcal{P}$ for a variety of parabolic subalgebras of a fixed type in $\mathfrak{g}$, the Lie algebra of G. Then K acts with finitely many orbits on $\mathcal{P}$, and these orbits may be parametrized in a number of ways (e.g. [M], [RS], [BH]). For orientation, if G is a product $G_1 \times G_1$, θ interchanges the two factors, and $\mathcal{P} = \mathcal{B}$ consists of (pairs of) Borel subalgebras, then $K \simeq G_1$ orbits on $\mathcal{B}$ (for the diagonal action) are parametrized by the Weyl group of G_1 according to the usual Bruhat decomposition.

In Section 2, we use the geometry of the moment map for the cotangent bundle $T^*\mathcal{P}$ to give another parametrization of the set $K\backslash\mathcal{P}$ of K orbits on $\mathcal{P}$. In more detail, let $\mathfrak{k}$ denote the Lie algebra of K, write $\mathcal{N}^\theta$ for the cone of nilpotent elements in $(\mathfrak{g}/\mathfrak{k})^*$, and let $\mathcal{N}^\theta_{\mathcal{P}}$ denote the cone of nilpotent elements in

$$(G \cdot (\mathfrak{g}/\mathfrak{p}_\circ)^*) \cap (\mathfrak{g}/\mathfrak{k})^*$$

where $\mathfrak{p}_\circ$ denotes a fixed base-point in $\mathcal{P}$. (Here and elsewhere we implicitly invoke the inclusion of $(\mathfrak{g}/\mathfrak{p}_\circ)^*$ and $(\mathfrak{g}/\mathfrak{k})^*$ into $\mathfrak{g}^*$ and take the intersection there.) The moment map $\mu_{\mathcal{P}}$ from $T^*\mathcal{P}$ to $\mathfrak{g}^*$ maps a point $(\mathfrak{p}, \xi)$ in $T^*\mathcal{P}$ with

$$\xi \in T^*_{\mathfrak{p}}\mathcal{P} \simeq (\mathfrak{g}/\mathfrak{p})^* \tag{1.1}$$

simply to ξ. Consider now the conormal variety for K orbits on $\mathcal{P}$,

$$T^*_K\mathcal{P} = \bigcup_{Q \in K\backslash\mathcal{P}} T^*_Q\mathcal{P},$$

where $T^*_Q\mathcal{P}$ denotes the conormal bundle to the K orbit Q. (In the special case $G = G_1 \times G_1$ and $\mathcal{P} = \mathcal{B}$ mentioned above, the conormal variety is the usual Steinberg variety of triples; see, for instance, the exposition of [DR].) In general we may identify

$$T^*_Q\mathcal{P} = \{(\mathfrak{p}, \xi) \mid \mathfrak{p} \in Q, \xi \in (\mathfrak{g}/(\mathfrak{k} + \mathfrak{p}))^*\}, \tag{1.2}$$

and hence the image of $T^*_K\mathcal{P}$ under $\mu_{\mathcal{P}}$ is simply $\mathcal{N}^\theta_{\mathcal{P}}$. Since $\mu_{\mathcal{P}}$ is G-equivariant, a short argument shows that the image of a particular conormal bundle $T^*_Q\mathcal{P}$ contains a unique dense orbit of K on $\mathcal{N}^\theta_{\mathcal{P}}$. Hence we obtain a map

$$\Phi = \Phi_{\mathcal{P}} \ : \ K\backslash\mathcal{P} \longrightarrow K\backslash\mathcal{N}^\theta_{\mathcal{P}}, \tag{1.3}$$

the latter set being finite by the results of Kostant-Rallis [KR]. Proposition 2.10 below parametrizes the fibers of Φ in terms of certain isotypic components of Springer's Weyl group representations using the partial resolutions of Borho-MacPherson [BM]. This gives the parametrization of $K\backslash\mathcal{P}$ alluded to above (Corollary 2.12).

One is naturally led to ask if the parametrization can be effectively computed. For instance, are the fibers $\Phi^{-1}(\mathcal{O}_K)$ of the map in (1.3) computable in general? In Section 3, we restrict to a seemingly very special case, requiring $\mu_{\mathcal{P}}$ to be birational and $\mathcal{O}_K$ to be as large as possible ("regular" in the terminology of title; see Definition 3.1). In Proposition 3.7 and Remark 3.10 we give an effective algorithm to compute the fibers $\Phi^{-1}(\mathcal{O}_K)$. Perhaps surprisingly this algorithm relies on the Kazhdan-Lusztig-Vogan algorithm [V1] for computing the intersection homology groups (with coefficients) of K orbit closures on the full flag variety.

The setting of Section 3 may appear too restrictive to be of much practical value. But in Section 4 we recall that it is exactly the geometric setting of the Adams-Barbasch-Vogan definition of Arthur packets. (We follow [ABV, Chapter 27], but our exposition is essentially self-contained.) More precisely, since the ground field is $\mathbb{C}$, θ arises as the complexification of a Cartan involution for a real form $G_{\mathbb{R}}$ of G. We show that the algorithm of Remark 3.10 gives an effective means to compute a distinguished constituent of each Arthur packet of integral special unipotent representations for $G_{\mathbb{R}}$. According to the Arthur conjectures, these representations should be unitary. This is a striking prediction (which is still open in general), since the constructions leading to their definition have nothing to do with unitarity.

Finally, in Section 5, we consider a number of examples illustrating some subtleties of the parametrization of Section 2.

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2. PARAMETRIZING $K\backslash\mathcal{P}$

The main result of this section is Corollary 2.12 which gives a parametrization of the K orbits on $\mathcal{P}$. As Propositions 2.8 and 2.13 show, the parametrization is closely related to Springer's Weyl group representations.

We begin with a discussion of the set $K\backslash\mathcal{B}$ of K orbits on $\mathcal{B}$. Basic references for this material are [M] or [RS]. The set $K\backslash\mathcal{B}$ is partially ordered by the inclusion of orbit closures. It is generated by closure relations in codimension one. We will need to distinguish two kinds of such relations. To do so, we fix a base-point $\mathfrak{b}_\circ \in \mathcal{B}$, a decomposition $\mathfrak{b}_\circ = \mathfrak{h}_\circ \oplus \mathfrak{n}_\circ$, and let Δ^+ denote the corresponding set of positive roots. For a simple root α, let $\mathcal{P}_\alpha$ denote the set of parabolic subalgebras of type α, and write π_α for the projection $\mathcal{B} \to \mathcal{P}_\alpha$.

Fix K orbits Q and Q' on $\mathcal{B}$. If K is connected, then Q is irreducible, and hence so is $\pi_\alpha^{-1}(\pi_\alpha(Q))$. Thus $\pi_\alpha^{-1}(\pi_\alpha(Q))$ contains a unique dense K orbit. In general, K need not be connected and Q need not be irreducible. But it is easy to see that the similar reasoning applies to conclude $\pi_\alpha^{-1}(\pi_\alpha(Q))$ always contains a dense K orbit. We write $Q \overset{\alpha}{\to} Q'$ if

$$\dim(Q') = \dim(Q) + 1$$

and

$$Q' \text{ is dense in } \pi_\alpha^{-1}(\pi_\alpha(Q)).$$

This implies that Q is codimension one in the closure of Q'. The relations $Q < Q'$ for $Q \overset{\alpha}{\to} Q'$ do not generate the closure order however. Instead we must also consider a kind of saturation condition. More precisely, whenever a codimension one subdiagram of the form

$$\begin{tikzcd}
 & Q_1 & \\
Q_2 \arrow[ur, "\alpha"] & & Q_3 \\
 & Q_4 \arrow[ul] \arrow[ur, "\alpha"'] &
\end{tikzcd} \tag{2.1}$$

is encountered, we complete it to

(2.2)
$$\begin{tikzcd} & Q_1 & \\ Q_2 \arrow[ur, "\alpha"] & & Q_3 \arrow[ul, dotted] \\ & Q_4 \arrow[ul] \arrow[ur, "\alpha"'] & \end{tikzcd}$$

New edges added in this way are dashed in the diagrams below. Note that this operation must be applied recursively, and thus some of the edges in the original diagram (2.1) may be dashed as the recursion unfolds. Following the terminology of [RS, 5.1], we call the partially ordered set determined by the solid edges the weak closure order.

Now fix a variety of parabolic subalgebras $\mathcal{P}$ of an arbitrary fixed type and write $\pi_{\mathcal{P}}$ for the projection from $\mathcal{B}$ to $\mathcal{P}$. For definiteness fix $\mathfrak{p}_\circ = \mathfrak{l}_\circ \oplus \mathfrak{u}_\circ \in \mathcal{P}$ containing $\mathfrak{b}_\circ$. Then $K\backslash\mathcal{P}$ may be parametrized from a knowledge of the weak closure on $K\backslash\mathcal{B}$ as follows. Consider the relation $Q \sim_{\mathcal{P}} Q'$ if $\pi_{\mathcal{P}}(Q) = \pi_{\mathcal{P}}(Q')$; this is generated by the relations $Q \sim Q'$ if $Q \overset{\alpha}{\to} Q'$ for α simple in $\Delta(\mathfrak{h}_\circ, \mathfrak{l}_\circ)$. Equivalence classes in $K\backslash\mathcal{B}$ clearly are in bijection with $K\backslash\mathcal{P}$. (See also the parametrization of [BH, Section 1], especially Proposition 4.) Fix an equivalence class C and fix a representative $Q \in C$. The same reasoning that shows that $\pi_\alpha^{-1}(\pi_\alpha(Q))$ contains a unique dense K orbit also shows that

$$\pi_{\mathcal{P}}^{-1}(\pi_{\mathcal{P}}(Q))$$

contains a unique dense K orbit $Q_C \in K\backslash\mathcal{B}$. In other words, Q_C is the unique largest dimensional orbit among the elements in C. In fact Q_C is characterized among the elements of C by the condition

$$\dim \pi_\alpha^{-1}(\pi_\alpha(Q_C)) = \dim(Q_C) \tag{2.3}$$

for all α simple in $\Delta(\mathfrak{h}_\circ, \mathfrak{l}_\circ)$. It follows that the full closure order on $K\backslash\mathcal{P}$ is simply the restriction of the full closure on $K\backslash\mathcal{B}$ to the subset of all maximal-dimensional representatives of the form Q_C. By restricting only the weak closure order, we may speak of the weak closure order on $K\backslash\mathcal{P}$.

Here are some elementary properties of the map $\Phi_{\mathcal{P}}$ introduced in (1.3) above.

Proposition 2.4. (1) *Fix $Q \in K\backslash\mathcal{P}$ and suppose $Q' \in K\backslash\mathcal{B}$ is dense in $\pi_{\mathcal{P}}^{-1}(Q)$. Then*

$$\Phi_{\mathcal{B}}(Q') = \Phi_{\mathcal{P}}(Q).$$

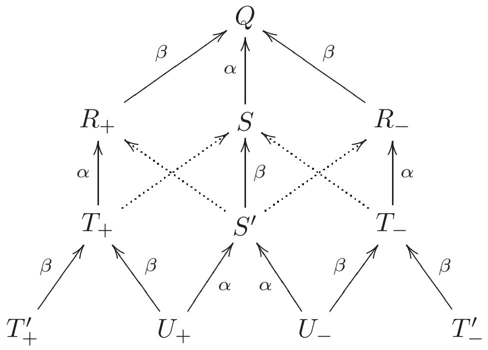
(2) *The map $\Phi_{\mathcal{P}}$ is order reversing from the weak closure order in $K\backslash\mathcal{P}$ to the closure order on $K\backslash\mathcal{N}_{\mathcal{P}}^{\theta}$; that is, if $Q < Q'$ in the weak closure order on $K\backslash\mathcal{P}$, then*

$$\overline{\Phi_{\mathcal{P}}(Q)} \supset \Phi_{\mathcal{P}}(Q').$$

Proof. Part (1) is clear from the definitions. Part (2) reduces to the assertion for $Q \overset{\alpha}{\to} Q'$. In that case, it amounts to a rank one calculation where it is obvious. $\square$

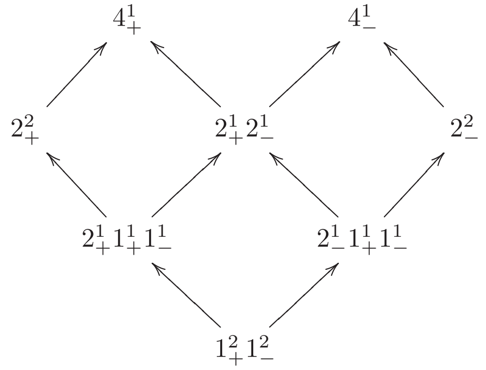
Example 2.5. Proposition 2.4(2) fails for the full closure order on $K\backslash\mathcal{P}$. The first example which exhibits this failure is $G_{\mathbb{R}} = \mathrm{Sp}(4,\mathbb{R})$ and $\mathcal{P} = \mathcal{B}$. Let α denote the short simple root in Δ^+ and β the long one. The closure order for $K\backslash\mathcal{B}$ is as follows. Orbits on the same row of the diagram below all have the same dimension. (The bottom row consists of orbits of dimension one, the next row consists of orbits of dimension two, and so on.) Dashed lines represent relations in the full closure order which are not in the weak order.

(2.6)



Adopt the parametrization of $K\backslash\mathcal{N}^\theta$ given in [CM, Theorem 9.3.5] in terms of signed tableau. Let $(i_1)^{j_1}_{\epsilon_1}(i_2)^{j_2}_{\epsilon_2}\cdots$ denote the tableau with j_k rows of length i_k beginning with sign ϵ_k for each k. Then the closure order on $K\backslash\mathcal{N}^\theta$ is given by

(2.7)



Then $\Phi_{\mathcal{B}}$ maps Q to $1^2_+1^2_-$; $R_\pm$ to $2^1_\pm 1^1_+1^1_-$; S and S' to $2^1_+2^1_-$; $T_\pm$ and $T'_\pm$ to $2^2_\pm$; and $U_\pm$ to $4^1_\pm$. Note that $\Phi_{\mathcal{B}}$ reverses all closure relations *except* the two dashed edges indicating $T_\pm \subset \overline{S}$.

We are now in a position to determine the size of the fiber $\Phi^{-1}_{\mathcal{P}}(\mathcal{O}_K)$ for $\mathcal{O}_K \in K\backslash\mathcal{N}^\theta_{\mathcal{P}}$. For $\xi \in \mathcal{O}_K$, let $A_K(\xi)$ (resp. $A_G(\xi)$) denote the component group of the centralizer in K (resp. G) of ξ. Obviously there is a natural map

$$A_K(\xi) \to A_G(\xi)$$

which we often invoke implicitly. Write $\mathrm{Sp}(\xi)$ for the Springer representation of $W \times A_G(\xi)$ on the top homology of the Springer fiber over ξ (normalized so that $\xi = 0$ gives the sign representations of W). Let

$$\mathrm{Sp}(\xi)^{A_K} = \mathrm{Hom}_{A_K(\xi)}\left(\mathrm{Sp}(\xi), \mathbb{1}\right).$$

Proposition 2.8. *Fix $\xi \in \mathcal{O}_K$. Then*

$$\begin{aligned}\#\Phi^{-1}_{\mathcal{P}}(\mathcal{O}_K) &= \dim \mathrm{Hom}_{W(\mathcal{P})}\left(\mathrm{sgn}, \mathrm{Sp}(\xi)^{A_K}\right)\\ &= \dim \mathrm{Hom}_W(\mathrm{ind}^W_{W(\mathcal{P})}(\mathrm{sgn}), \mathrm{Sp}(\xi)^{A_K}).\end{aligned}$$

Proof. The second equality follows by Frobenius reciprocity. For the first, set

$$S_{\mathcal{P}} = \{Q \in K\backslash\mathcal{B} \mid Q \text{ is dense in } \pi^{-1}_{\mathcal{P}}(\pi_{\mathcal{P}}(Q))\}.$$

According to the discussion around (2.3) and Proposition 2.4(1), $\pi_{\mathcal{P}}$ implements a bijection

$$S_{\mathcal{P}} \cap \Phi^{-1}_{\mathcal{B}}(\mathcal{O}_K) \to \Phi^{-1}_{\mathcal{P}}(\mathcal{O}_K).$$

We will count the left-hand side if K is connected. If K is disconnected, there are a few subtleties (none of which are very serious) which are best treated later.

Consider the top integral Borel-Moore homology of the conormal variety $T_K^*\mathcal{P}$. Since we have assumed K is connected, the closures of the individual conormal bundles exhaust the irreducible components of $T_K^*\mathcal{P}$, and their classes form a basis of the homology,

$$\mathrm{H}^{\infty}_{\mathrm{top}}\left(T_K^*\mathcal{P}, \mathbb{Z}\right) = \bigoplus_{Q \in K\backslash\mathcal{P}} [\overline{T_Q^*\mathcal{P}}].$$

If $\mathcal{P} = \mathcal{B}$, Rossmann [R] (extending earlier work of Kazhdan-Lusztig [KL]) described a construction giving an action of W on this homology space. The action is graded in the following sense that if $Q \in \Phi_{\mathcal{B}}^{-1}(\mathcal{O}_K)$, then

$$w \cdot [\overline{T_Q^*\mathcal{B}}]$$

is a linear combination of conormal bundles to orbits in fibers $\Phi_{\mathcal{B}}^{-1}(\mathcal{O}'_K)$ with $\mathcal{O}'_K \subset \overline{\mathcal{O}_K}$. Hence if we set

$$\Phi_{\mathcal{B}}^{-1}(\mathcal{O}_K, \leq) = \bigcup_{\mathcal{O}'_K \subseteq \overline{\mathcal{O}_K}} \Phi_{\mathcal{B}}^{-1}(\mathcal{O}'_K)$$

and

$$\Phi_{\mathcal{B}}^{-1}(\mathcal{O}_K, <) = \bigcup_{\mathcal{O}'_K \subsetneq \overline{\mathcal{O}_K}} \Phi_{\mathcal{B}}^{-1}(\mathcal{O}'_K)$$

then

$$\mathbf{M}(\mathcal{O}_K) \ := \bigoplus_{Q \in \Phi_{\mathcal{B}}^{-1}(\mathcal{O}_K, \leq)} [\overline{T_Q^*\mathcal{B}}] \Bigg/ \bigoplus_{Q \in \Phi_{\mathcal{B}}^{-1}(\mathcal{O}_K, <)} [\overline{T_Q^*\mathcal{B}}]$$

is a W module with basis indexed by $\Phi_{\mathcal{B}}^{-1}(\mathcal{O}_K)$. Rossmann's construction shows that

$$\mathbf{M}(\mathcal{O}_K) \simeq \mathrm{Sp}(\xi)^{A_K},$$

where $\xi \in \mathcal{O}_K$ as above. This proves the proposition for $\mathcal{P} = \mathcal{B}$. For the general case, we must identify $S_{\mathcal{P}}$ in terms of the Weyl group action. It follows from Rossmann's constructions that

$$s_\alpha \cdot [\overline{T_Q^*\mathcal{B}}] = -[T_Q^*\mathcal{B}]$$

if and only if

$$\dim \pi_\alpha^{-1}\left(\pi_\alpha(Q)\right) = \dim(Q).$$

Thus (2.3) implies that $S_{\mathcal{P}} \cap \Phi_{\mathcal{B}}^{-1}(\mathcal{O}_K)$ indexes exactly the basis elements of $\mathbf{M}(\mathcal{O}_K)$ which transform by the sign representation of the Weyl group of type $\mathcal{P}$. The proposition thus follows in the case of K connected. (A complete proof in the disconnected case is discussed after Proposition 2.13.) ☐

The above proof is extrinsic, in the sense that it is deduced from a statement about the $\mathcal{P} = \mathcal{B}$ case. We may argue more intrinsically (without reference to $\mathcal{B}$) using results of Borho-MacPherson [BM] as follows.

Fix $\xi \in \mathcal{N}_{\mathcal{P}}^\theta$ and consider $\mu_{\mathcal{P}}^{-1}(\xi)$. In terms of the identification around (1.1),

$$\mu_{\mathcal{P}}^{-1}(\xi) = \{(\mathfrak{p}, \xi) \mid \xi \in (\mathfrak{g}/\mathfrak{p})^*\}.$$

(Borho-MacPherson write $\mathcal{P}_\xi^0$ for $\mu_{\mathcal{P}}^{-1}(\xi)$ and call it a Spaltenstein variety.) Clearly $A_G(\xi)$, and hence $A_K(\xi)$, act on the set of irreducible components $\mathrm{Irr}(\mu_{\mathcal{P}}^{-1}(\xi))$. Fix $C \in \mathrm{Irr}(\mu_{\mathcal{P}}^{-1}(\xi))$, and consider $Z(C) := K \cdot C \subset T^*\mathcal{P}$. Since $\xi \in \mathcal{N}_{\mathcal{P}}^\theta \subset \mathcal{N}(\mathfrak{g}/\mathfrak{k})^*$, it follows from (1.2) that $Z(C)$ is in fact contained in the conormal variety

$$Z(C) \subset T_K^*\mathcal{P},$$

which is of course pure-dimensional of dimension $\dim(\mathcal{P})$. Hence

$$\dim(Z(C)) \leq \dim(\mathcal{P}).$$

But clearly

$$\dim(Z(C)) = \dim(K\cdot\xi) + \dim(C),$$

and thus

$$\dim(C) \leq \dim(\mathcal{P}) - \dim(K\cdot\xi). \tag{2.9}$$

Write $\mathrm{Irr}_{\max}(\mu_{\mathcal{P}}^{-1}(\xi))$ for those irreducible components whose dimensions actually achieve the upper bound. (This set could be empty, for instance, as we shall see in Example 3.3 below when $\mathcal{P} = \mathcal{P}_\beta$ and ξ is a representative of a minimal nilpotent orbit.)

Proposition 2.10. *Fix $\xi \in \mathcal{N}_{\mathcal{P}}^{\theta}$, set $\mathcal{O}_K = K\cdot\xi$, assume $\Phi_{\mathcal{P}}^{-1}(\mathcal{O}_K)$ is nonempty, and fix $Q \in \Phi_{\mathcal{P}}^{-1}(\mathcal{O}_K)$. Then*

$$C(Q) := \overline{T_Q^*\mathcal{P}} \cap \mu_{\mathcal{P}}^{-1}(\xi)$$

is an $A_K(\xi)$ orbit on $\mathrm{Irr}_{\max}(\mu_{\mathcal{P}}^{-1}(\xi))$. The assignment $Q \mapsto C(Q)$ gives a bijection

$$\Phi_{\mathcal{P}}^{-1}(\mathcal{O}_K) \longrightarrow A_K(\xi)\backslash\mathrm{Irr}_{\max}(\mu_{\mathcal{P}}^{-1}(\xi)). \tag{2.11}$$

Proof. Fix $C \in \mathrm{Irr}_{\max}(\mu_{\mathcal{P}}^{-1}(\xi))$. Then $\dim(Z(C)) = \dim(\mathcal{P})$ by definition. Notice that $Z(C)$ is nearly irreducible (and it is if K is connected). In general, the component group of K (which is finite by hypothesis) acts transitively on the irreducible components of $Z(C)$. But from the definition of $T_K^*\mathcal{P}$, the closure of each conormal bundle $T_Q^*\mathcal{P}$ consists of a subset of irreducible components of $T_K^*\mathcal{P}$ on which the component group of K acts transitively. Since $\dim(Z(C)) = \dim(T_K^*\mathcal{P})$, it follows that there is some Q such that

$$Z(C) = \overline{T_Q^*\mathcal{P}};$$

moreover Q must be an element of $\Phi_{\mathcal{P}}^{-1}(\mathcal{O}_K)$. Clearly $Z(C) = Z(C')$ if and only if C and C' are in the same $A_K(\xi)$ orbit. The assignment $C \mapsto Q$ gives a bijection $A_K(\xi)\backslash\mathrm{Irr}_{\max}(\mu_{\mathcal{P}}^{-1}(\xi)) \to \Phi_{\mathcal{P}}^{-1}(\mathcal{O}_K)$ which, by construction, is the inverse of the map in (2.11). This completes the proof. ☐

Corollary 2.12. *Let $\xi_1, \ldots, \xi_k$ be representatives of the K orbits on $\mathcal{N}_{\mathcal{P}}^{\theta}$. Then the map*

$$Q \longrightarrow \left(\Phi_{\mathcal{P}}(Q), \overline{T_Q^*\mathcal{P}} \cap \mu_{\mathcal{P}}^{-1}(\xi_i)\right)$$

for i the unique index such that $K\cdot\xi_i$ dense in $\Phi_{\mathcal{P}}(Q)$ implements a bijection

$$K\backslash\mathcal{P} \longrightarrow \coprod_i A_K(\xi_i)\backslash\mathrm{Irr}_{\max}(\mu_{\mathcal{P}}^{-1}(\xi_i)).$$

Thus everything reduces to understanding the irreducible components of $\mu_{\mathcal{P}}^{-1}(\xi)$ of maximal possible dimension. For this we need some nontrivial results of Borho-MacPherson. [BM, Theorem 3.3] shows that the fundamental classes of the elements of $\mathrm{Irr}_{\max}(\mu_{\mathcal{P}}^{-1}(\xi))$ index a basis of $\mathrm{Hom}_{W(\mathcal{P})}(\mathrm{Sp}(\xi), \mathrm{sgn})$. Actually, to be precise, their condition for C to belong to $\mathrm{Irr}_{\max}(\mu_{\mathcal{P}}^{-1}(\xi))$ is that

$$\dim(C) = \dim(\mathcal{P}) - \frac{1}{2}\dim(G\cdot\xi).$$

To square with (2.9), we need to invoke the result of Kostant-Rallis [KR] that $K\cdot\xi$ is Lagrangian in $G\cdot\xi$. In any case, because $A_G(\xi)$ acts on $\mathrm{Sp}(\xi)$ and commutes with the W action, $A_G(\xi)$ also acts on $\mathrm{Hom}_{W(\mathcal{P})}(\mathrm{sgn}, \mathrm{Sp}(\xi))$, and [BM, Theorem 3.3] shows that this action is compatible with the action of $A_G(\xi)$ on $\mathrm{Irr}(\mu_{\mathcal{P}}^{-1}(\xi))$. In particular this implies the following result.

Proposition 2.13. *Fix $\xi \in \mathcal{N}_{\mathcal{P}}^{\theta}$. Then the number of $A_K(\xi)$ orbits on $\mathrm{Irr}_{\max}(\mu_{\mathcal{P}}^{-1}(\xi))$ equals the dimension of*

$$\mathrm{Hom}_{W(\mathcal{P})}\left(\mathrm{sgn}, \mathrm{Sp}(\xi)^{A_K}\right).$$

Combining Proposition 2.10 and 2.13, we obtain an alternate proof of Proposition 2.8 which makes no assumption on the connectedness of K.

Remark 2.14. The $\mathcal{P} = \mathcal{B}$ case of Corollary 2.12 is due to Springer (unpublished). In this case, $W(\mathcal{B})$ is trivial, and thus $\Phi_{\mathcal{B}}^{-1}(\mathcal{O}_K)$ has order equal to the W-representation $\mathrm{Sp}(\xi)^{A_K}$.

It is of interest to compute the bijection of Corollary 2.12 as explicitly as possible. For instance, if $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{C})$ and $\mathcal{P} = \mathcal{B}$ consists of pairs of flags, the left-hand side of the bijection in Corollary 2.12 consists of elements of the symmetric group S_n. On the right-hand side, all A-groups are trivial, and the irreducible components in question amount to pairs of irreducible components of the usual Springer fiber. Such pairs are parametrized by same-shape pairs of standard Young tableaux. Steinberg [St] showed that the bijection of the corollary amounts to the classical Robinson-Schensted correspondence.

A few other classical cases have been worked out explicitly ([vL], [Mc1], [T1], [T3]). But general statements are lacking. For instance, given Q and Q', there is no known effective algorithm to decide if $\Phi_{\mathcal{P}}(Q) = \Phi_{\mathcal{P}}(Q')$. The next section is devoted to special cases of the parametrization which lead to nice general statements. It might appear that these special cases are too restrictive to be of much use. But it turns out that they encode exactly the geometry needed for the Adams-Barbasch-Vogan definition of Arthur packets. This is explained in Section 4.

3. $\mathcal{P}$-REGULAR K ORBITS

The main results of this section are Proposition 3.7(b) and Remark 3.10 which together give an effective computation of a portion of the bijection of Proposition 2.10 under the assumption that $\mu_{\mathcal{P}}$ is birational.

Definition 3.1 (see [ABV, Definition 20.17])**.** A nilpotent orbit $\mathcal{O}_K$ of K on $\mathcal{N}_{\mathcal{P}}^{\theta}$ is called $\mathcal{P}$-regular (or simply regular, if $\mathcal{P}$ is clear from the context) if $G \cdot \mathcal{O}_K$ is dense in $\mu_{\mathcal{P}}(T^*\mathcal{P})$. Since $\mathcal{O}_K$ is Lagrangian in $G \cdot \mathcal{O}_K$ [KR], this condition is equivalent to

$$\dim(\mathcal{O}_K) = \frac{1}{2}\dim \mu(T^*\mathcal{P}) = \dim(\mathfrak{g}/\mathfrak{p}),$$

for any $\mathfrak{p} \in \mathcal{P}$. In other words, $\mathcal{P}$-regular nilpotent K-orbits meet the complex Richardson orbit induced from $\mathfrak{p}$. An orbit Q of K on $\mathcal{P}$ is called $\mathcal{P}$-regular (or simply regular) if $\Phi_{\mathcal{P}}(Q)$ is a $\mathcal{P}$-regular nilpotent orbit. Note that regular $\mathcal{P}$-orbits need not exist in general (for instance, if $G_{\mathbb{R}}$ is compact and $\mathcal{P}$ is not trivial).

Since regular nilpotent K orbits are automatically maximal in the closure order on $\mathcal{N}_{\mathcal{P}}^{\theta}$, Proposition 2.4(2) shows that regular K orbits on $\mathcal{P}$ are minimal in the weak closure order:

Proposition 3.2. *Suppose Q is a regular K orbit on $\mathcal{P}$. Then Q is minimal in the weak closure order on $K\backslash\mathcal{P}$.*

The next example shows that regular K orbits on $\mathcal{P}$ need not be minimal in the full closure order (i.e. they need not be closed).

Example 3.3. Retain the notation of Example 2.5. Let $\mathcal{P}_\alpha$ (resp. $\mathcal{P}_\beta$) consist of parabolic subalgebras of type α (resp. β) and write π_α and π_β in place of $\pi_{\mathcal{P}_\alpha}$ and $\pi_{\mathcal{P}_\beta}$, and similarly for μ_α and μ_β. Then the closure order on $K\backslash\mathcal{P}_\alpha$ is obtained by the appropriate restriction from (2.6). (Subscripts now indicate dimensions; dashed edges are those covering relations present in the full order but not the weak one.)

(3.4)
$$\begin{tikzcd}
 & & \pi_\alpha(Q)_3 & & \\
 & \pi_\alpha(R_+)_2 \arrow[ur] & & \pi_\alpha(R_-)_2 \arrow[ul] & \\
\pi_\alpha(T'_+)_0 \arrow[ur] & & \pi_\alpha(S')_1 \arrow[ul, dotted] \arrow[ur, dotted] & & \pi_\alpha(T'_-)_0 \arrow[ul]
\end{tikzcd}$$

The closure order on $K\backslash\mathcal{P}_\beta$ is again obtained by restriction from (2.6). (Once again subscripts indicate dimensions.)

(3.5)
$$\begin{tikzcd}
 & \pi_\beta(Q)_3 & \\
 & \pi_\beta(S)_2 \arrow[u] & \\
\pi_\beta(T_+)_1 \arrow[ur, dotted] & & \pi_\beta(T_-)_1 \arrow[ul, dotted]
\end{tikzcd}$$

In this case $\mathcal{N}^\theta_\alpha = \mathcal{N}^\theta_\beta$, and the closure order on $K\backslash\mathcal{N}^\theta_\mathcal{P}$ is just the bottom three rows of (2.7),

(3.6)
$$\begin{tikzcd}
2^2_+ & & 2^1_+2^1_- & & 2^2_- \\
 & 2^1_+1^1_+1^1_- \arrow[ul] \arrow[ur] & & 2^1_-1^1_+1^1_- \arrow[ul] \arrow[ur] & \\
 & & 1^2_+1^2_- \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

From Proposition 5.2 below (for instance), both $\Phi_\alpha = \Phi_{\mathcal{P}_\alpha}$ and $\Phi_\beta = \Phi_{\mathcal{P}_\beta}$ are injective. There are enough edges in the weak closure order on $K\backslash\mathcal{P}_\alpha$ so that Proposition 2.4(1) allows one to conclude that Φ_α reverses the full closure order. In fact, Φ_α is the obvious order reversing bijection of (3.4) onto (3.6). Hence $\pi_\alpha(T'_\pm)$ and $\pi_\alpha(S')$ are $\mathcal{P}_\alpha$-regular.

By contrast, Φ_β does not invert the dashed edges in (3.5): Φ_β maps $\pi_\beta(Q)$ to the zero orbit, and the three remaining orbits to the three orbits of maximal dimension in $\mathcal{N}^\theta_\mathcal{P}$. Hence $\pi_\beta(T'_\pm)$ and $\pi_\beta(S)$ are $\mathcal{P}_\beta$-regular. In particular, $\pi_\beta(S)$ is a $\mathcal{P}_\beta$-regular orbit which is not closed.

Finally note that the fiber of Φ_α over $2^1_\pm 1^1_+ 1^1_-$ consists of a single element, while the corresponding fiber for Φ_β is empty. This is consistent with Proposition 2.8 since $\mathrm{Sp}(\xi)$ (for ξ a representative of these orbits) is a one dimensional representation on which the simple reflection s_α (resp. s_β) acts nontrivially (resp. trivially). $\square$

An essential difference in the two cases considered in Example 3.3 is that μ_α is birational, but μ_β has degree two.

Proposition 3.7 ([ABV, Theorem 20.18])**.** *Suppose $\mu_\mathcal{P}$ is birational onto its image. Then:*

(a) *Any regular K orbit on $\mathcal{P}$ consists of θ-stable parabolic subalgebras (and hence is closed).*
(b) *$\Phi_\mathcal{P}$ is a bijection from the set of regular K orbits on $\mathcal{P}$ to the set of regular nilpotent K orbits on $\mathcal{N}^\theta_\mathcal{P}$.*

Proof. Fix a $\mathcal{P}$-regular nilpotent K orbit $\mathcal{O}_K$ in $\mathcal{N}^\theta_{\mathcal{P}}$, $\xi \in \mathcal{O}_K$, and $Q \in \Phi^{-1}_{\mathcal{P}}(\mathcal{O}_K)$. Since $\mu_{\mathcal{P}}$ is birational, the set $\mathrm{Irr}_{\max}(\mu^{-1}_{\mathcal{P}}(\xi))$ is a single point, and so Proposition 2.10 shows that Q is the unique orbit in $\Phi^{-1}_{\mathcal{P}}(\mathcal{O}_K)$. This gives (b).

Again since $\mu_{\mathcal{P}}$ is birational, there is a unique parabolic $\mathfrak{p} \in Q$ such that $\xi \in (\mathfrak{g}/\mathfrak{p})^*$. Since $\theta(\xi) = -\xi$, $\theta(\mathfrak{p})$ is also such a parabolic. So $\theta(\mathfrak{p}) = \mathfrak{p}$. Thus $Q = K \cdot \mathfrak{p}$ consists of θ-stable parabolic subalgebras. This gives the first part of (a). The same (well-known) proof of the fact that K orbits of θ-stable Borel subalgebras are closed (for example, [Mi, Lemma 5.8]), also applies to show that orbits of θ-stable parabolics are closed. (It is no longer true that a closed K orbit on $\mathcal{P}$ consists of θ-stable parabolic subalgebras. But if a θ-stable parabolic algebra in $\mathcal{P}$ exists, all closed orbits do indeed consist of θ-stable parabolic subalgebras.) ☐

Because of the good properties in Proposition 3.7, we will mostly be interested in $\mathcal{P}$-regular orbits when $\mu_{\mathcal{P}}$ is birational. For orientation (and later use in Section 4) it is worth recalling a sufficient condition for birationality from [He]; see also [CM, Theorem 7.1.6] and [ABV, Lemma 27.8].

Proposition 3.8. *Suppose $\mathcal{O}$ is an even complex nilpotent orbit. Let $\mathcal{P}$ denote the variety of parabolic subalgebras in $\mathfrak{g}$ corresponding to the subset of the simple roots labeled 0 in the weighted Dynkin diagram for $\mathcal{O}$ (e.g.* [CM, Section 3.5]*). Then $\mathcal{O}$ is dense in $\mu_{\mathcal{P}}(T^*\mathcal{P})$ and $\mu_{\mathcal{P}}$ is birational.*

☐

Return to Proposition 3.7(a). Example 5.12 below shows that if $\mu_{\mathcal{P}}$ is birational, then not every (necessarily closed) K orbit of θ-stable parabolic subalgebras on $\mathcal{P}$ need be regular. So the question becomes: can one give an effective procedure to select the regular K orbits on $\mathcal{P}$ from among all orbits of θ-stable parabolics (when $\mu_{\mathcal{P}}$ is birational)? This is only a small part of computing the parametrization of Corollary 2.12, so it is perhaps surprising that the answer we give after Proposition 3.9 depends on the power of the Kazhdan-Lusztig-Vogan algorithm for $G_{\mathbb{R}}$, the real form of G with complexified Cartan involution θ.

We need a few definitions. Recall that the associated variety of a two-sided ideal in $\mathrm{U}(\mathfrak{g})$ is the subvariety of $\mathfrak{g}^*$ cut out by the associated graded ideal $\mathrm{gr}I$ (with respect to the standard filtration on $\mathrm{U}(\mathfrak{g})$) in $\mathrm{grU}(\mathfrak{g}) = S(\mathfrak{g})$. (From [BB1], if I is primitive, then $\mathrm{AV}(I)$ is the closure of a single nilpotent coadjoint orbit.) Finally if $\mathfrak{p}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$, recall the irreducible $(\mathfrak{g}, K)$-module $A_{\mathfrak{p}}$ constructed in [VZ]. (It would be more customary to denote these modules $A_{\mathfrak{q}}$, but we have already used the letter Q for another purpose.)

Proposition 3.9. *Suppose $\mu_{\mathcal{P}}$ is birational. Fix a closed K orbit Q on $\mathcal{P}$ consisting of θ-stable parabolic subalgebras. Fix $\mathfrak{p} \in Q$. Then Q is $\mathcal{P}$-regular in the sense of Definition 3.1 if and only if*

$$\mathrm{AV}(\mathrm{Ann}(A_{\mathfrak{p}})) = \mu(T^*\mathcal{P}),$$

the closure of the complex Richardson orbit induced from $\mathfrak{p}$.

Remark 3.10. We remark that the condition of the proposition is effectively computable from a knowledge of the Kazhdan-Lusztig-Vogan polynomials for $G_{\mathbb{R}}$. More precisely, the results of Section 2 allow us to enumerate the closed orbits of K on $\mathcal{P}$ from the structure of K orbits on $\mathcal{B}$. In turn, the description of $K\backslash\mathcal{B}$ has been implemented in the command `kgb` in the software package `atlas` (available for download from `www.liegroups.org`). Moreover, from the description in [VZ] (implemented in the `atlas` command `blocku`), one may determine which closed orbits consist of θ-stable parabolic subalgebras. (Alternatively, one may implement the algorithms of [BH, Section 3.3], at least if K is connected.) For a representative $\mathfrak{p}$ of each such orbit, one then uses the command `wcells`, to enumerate the cell of Harish-Chandra modules containing the Vogan-Zuckerman module $A_{\mathfrak{p}}$. (The computation of cells relies on computing Kazhdan-Lusztig-Vogan polynomials.) Finally $\mathrm{AV}(\mathrm{Ann}(A_{\mathfrak{p}})) = \mu(T^*\mathcal{P})$ if and only if the cell containing $A_{\mathfrak{p}}$ affords the Weyl group representation $\mathrm{Sp}(\xi)^{A_G}$ (with notation as in Section 2), where ξ is an element of the Richardson orbit induced from $\mathfrak{p}$. Again, this is an effectively computable condition and is easy to implement from the output of `atlas`. Hence *if $\mu_{\mathcal{P}}$ is birational, there is an effective algorithm to enumerate the $\mathcal{P}$-regular orbits of K on $\mathcal{P}$.*

Remark 3.11. Suppose $\mathcal{O}$ is an even complex nilpotent orbit, so that Proposition 3.8 applies. Then Proposition 3.7(b) shows that the algorithm of Remark 3.10 also enumerates the K orbits in $\mathcal{O} \cap (\mathfrak{g}/\mathfrak{k})^*$. Using the Kostant-Sekiguchi correspondence, this amounts to the enumeration of the real forms of $\mathcal{O}$, i.e. $G_{\mathbb{R}}$ orbits on $\mathcal{O} \cap \mathfrak{g}_{\mathbb{R}}^*$. By contrast, if $\mathcal{O}$ is not even, the only known way to enumerate the real forms of $\mathcal{O}$ involves case-by-case analysis.

Proposition 3.9 is known to experts, but we sketch a proof (of more refined results) below; see also [ABV, Chapter 20]. We begin with some representation-theoretic preliminaries. Let $\mathcal{D}_{\mathcal{P}}$ denote the sheaf of algebraic differential operators on $\mathcal{P}$, and let $D_{\mathcal{P}}$ denote its global section. Since the enveloping algebra $\mathrm{U}(\mathfrak{g})$ acts on $\mathcal{P}$ by differential operators, we obtain a map $\mathrm{U}(\mathfrak{g}) \to D_{\mathcal{P}}$. Let $I_{\mathcal{P}}$ denote its kernel, and $R_{\mathcal{P}}$ its image. By choosing a base-point $\mathfrak{p}_\circ \in \mathcal{P}$, it is easy to see that $I_{\mathcal{P}}$ is the annihilator of the irreducible generalized Verma module induced from $\mathfrak{p}_\circ \in \mathcal{P}$ with trivial infinitesimal character. We will be interested in studying Harish-Chandra modules whose annihilators contain $I_{\mathcal{P}}$, i.e. $(R_{\mathcal{P}}, K)$-modules. For orientation, note that if $\mathcal{P} = \mathcal{B}$, $I_{\mathcal{B}}$ is a minimal primitive ideal, and thus any Harish-Chandra module with trivial infinitesimal character contains it.

Unlike the case of $\mathcal{P} = \mathcal{B}$, $\mathrm{U}(\mathfrak{g})$ need not surject onto $D_{\mathcal{P}}$ in general, and so $R_{\mathcal{P}} \simeq \mathrm{U}(\mathfrak{g})/I_{\mathcal{P}}$ is generally a proper subring of $D_{\mathcal{P}}$. Thus the localization functor

$$\begin{aligned} R_{\mathcal{P}}\text{-mod} &\longrightarrow \mathcal{D}_{\mathcal{P}}\text{-mod} \\ X &\longrightarrow \mathcal{X} := \mathcal{D}_{\mathcal{P}} \otimes_{R_{\mathcal{P}}} X. \end{aligned}$$

need not be an equivalence of categories. But nonetheless we have that the appropriate irreducible objects match. (Much more conceptual statements of which the following proposition is a consequence have recently been established by S. Kitchen.)

Proposition 3.12. *Suppose X is an irreducible $(D_{\mathcal{P}}, K)$-module. Then its restriction to $R_{\mathcal{P}}$ is irreducible.*

Sketch. Irreducible $(D_{\mathcal{P}}, K)$-modules are parametrized by irreducible K equivariant flat connections on $\mathcal{P}$. We show that the irreducible $(R_{\mathcal{P}}, K)$-modules are also parametrized by the same set. The parametrizations have the property that support of the localization of either type of module parametrized by such a connection $\mathcal{L}$ is simply the closure of the support of $\mathcal{L}$. This implies there are the same number of such irreducible modules and hence implies the proposition.

Let X be an irreducible $(R_{\mathcal{P}}, K)$-module. Hence we may consider X as an irreducible $(\mathfrak{g}, K)$-module, say X', whose annihilator contains $I_{\mathcal{P}}$. By localizing on $\mathcal{B}$, we may consider the corresponding irreducible K equivariant flat connection on $\mathcal{B}$, say $\mathcal{L}'$, parametrizing X'. The condition that $\mathrm{Ann}(X') \supset I_{\mathcal{P}}$ can be translated into a geometric condition on $\mathcal{L}'$ using [LV, Lemma 3.5], the conclusion of which is that $\mathcal{L}'$ fibers over an irreducible flat K-equivariant connection on $\mathcal{P}$ (with fiber equal to the trivial connection on $\mathcal{B}_{\mathfrak{l}}$). This implies that irreducible $(R_{\mathcal{P}}, K)$-modules are also parametrized by K equivariant flat connections on $\mathcal{P}$, as claimed, and the proposition follows. $\square$

Remark 3.13. Proposition 3.12 need not hold when considering twisted sheaves of differential operators corresponding to singular infinitesimal characters.

Next suppose X is an irreducible $R_{\mathcal{P}}$ module. Let $(\mathcal{X}^i)$ denote a good filtration on its localization $\mathcal{X}$ compatible with the degree filtration on $\mathcal{D}_{\mathcal{P}}$. Let $\mathrm{CV}(X)$ denote the support of $\mathrm{gr}(\mathcal{X})$. This is well defined independent of the choice of filtration. Moreover, there is a subset $\mathrm{cv}(X) \subset K \backslash \mathcal{P}$ such that

$$\mathrm{CV}(X) = \bigcup_{Q \in \mathrm{cv}(X)} \overline{T_Q^* \mathcal{P}}.$$

The set $\mathrm{cv}(X)$ is difficult to understand, but there are two easy facts about it. First, if X is irreducible, there is a dense K orbit, say $\mathrm{supp}_\circ(X)$ in the support of $\mathcal{X}$; then $\mathrm{supp}_\circ(X) \in \mathrm{cv}(X)$. Moreover if $Q \in \mathrm{cv}(X)$, then $Q \in \overline{\mathrm{supp}_\circ(X)}$. So, for example, if $\mathrm{supp}_\circ(X)$ is closed, then $\mathrm{cv}(X) = \{\mathrm{supp}_\circ(X)\}$.

Finally we define

$$\mathrm{AV}(X) = \mu(\mathrm{CV}(X)).$$

(Alternatively one may define $\mathrm{AV}(X)$ as in [V3] without localizing. The fact that the two definitions agree follows from [BB3, Theorem 1.9(c)]) Clearly $\mathrm{AV}(X)$ is the union of closures of K orbits on $\mathcal{N}^\theta_{\mathcal{P}}$. We let $\mathrm{av}(X)$ denote the set of these orbits.

Here is how these invariants are tied together.

Theorem 3.14. *Retain the setting above. Then*

(1) $\mathrm{AV}(I_{\mathcal{P}}) = \mu(T^*\mathcal{P})$.

(2) *If X is an irreducible $(R_{\mathcal{P}}, K)$-module, then*

$$G \cdot \mathrm{AV}(X) = \mathrm{AV}(\mathrm{Ann}(X)) \subset \mathrm{AV}(I_{\mathcal{P}}).$$

Proof. Part (1) is Theorem 4.6 in [BB1]. The equality in part (2) is proved in [V3, Section 6]; the inclusion follows because X is an $R_{\mathcal{P}} = \mathrm{U}(\mathfrak{g})/I_{\mathcal{P}}$ module. ☐

Proposition 3.15. *Suppose X is an irreducible $(\mathcal{R}_{\mathcal{P}}, K)$-module such that there exists a $\mathcal{P}$-regular K orbit $Q \in \mathrm{cv}(X)$. (For instance, suppose $\mathrm{supp}_\circ(X)$ is $\mathcal{P}$-regular.) Then $\Phi_{\mathcal{P}}(Q)$ is a K orbit of maximal dimension in $\mathrm{AV}(X)$; that is, $\Phi_{\mathcal{P}}(Q) \in \mathrm{av}(X)$.*

Proof. Since $\mathrm{AV}(X) = \mu(\mathrm{CV}(X))$ and since $Q \in \mathrm{cv}(X)$,

$$\Phi_{\mathcal{P}}(Q) \subset \mathrm{AV}(X) \tag{3.16}$$

for any $(R_{\mathcal{P}}, K)$-module. If Q is $\mathcal{P}$-regular, then the G saturation of the left-hand side of (3.16) is dense in $\mu(T^*\mathcal{P})$. But by Theorem 3.14 the right-hand side of (3.16) is also contained in $\mu(T^*\mathcal{P})$. So the current proposition follows. ☐

Corollary 3.17. *Suppose X is an irreducible $(R_{\mathcal{P}}, K)$-module. Then the following are equivalent.*

(a) *there exist a $\mathcal{P}$-regular orbit $Q \in \mathrm{cv}(X)$;*

(b) *there exists a $\mathcal{P}$-regular orbit $\mathcal{O}_K \in \mathrm{av}(X)$;*

(c) $\mathrm{Ann}(X) = I_{\mathcal{P}}$;

(d) $\mathrm{AV}(\mathrm{Ann}(X)) = \mathrm{AV}(I_{\mathcal{P}})$, *i.e.* $\mathrm{AV}(\mathrm{Ann}(X)) = \mu(T^*\mathcal{P})$.

Proof. The equivalence of (a) and (b) follows from the definitions above. Since the annihilator of any $R_{\mathcal{P}}$ module contains $I_{\mathcal{P}}$, the equivalence of (c) and (d) follows from [BKr, 3.6]. Theorem 3.14 and the definitions gives the equivalence of (b) and (d). ☐

Proof of Proposition 3.9. If $\mathfrak{p} \in \mathcal{P}$ is a θ-stable parabolic, then the Vogan-Zuckerman module $A_{\mathfrak{p}}$ is the unique irreducible $(R_{\mathcal{P}}, K)$-module whose localization is supported on the closed orbit $K \cdot \mathfrak{p}$ and thus, as remarked above, $\mathrm{cv}(A_{\mathfrak{p}}) = \{K \cdot \mathfrak{p}\}$. So Proposition 3.9 is a special case of Corollary 3.17. ☐

4. APPLICATIONS TO SPECIAL UNIPOTENT REPRESENTATIONS

The purpose of this section is to explain how the algorithm of Remark 3.10 produces special unipotent representations. Much of this section is implicit in [ABV, Chapter 27].

Fix a nilpotent adjoint orbit $\mathcal{O}^\vee$ for $\mathfrak{g}^\vee$, the Langlands dual of $\mathfrak{g}$. Fix a Jacobson-Morozov triple $\{e^\vee, h^\vee, f^\vee\}$ for $\mathcal{O}^\vee$, and set

$$\chi(\mathcal{O}^\vee) = (1/2)h^\vee.$$

Then $\chi(\mathcal{O}^\vee)$ is an element of some Cartan subalgebra $\mathfrak{h}^\vee$ of $\mathfrak{g}^\vee$. There is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h}^\vee$ canonically identifies with $\mathfrak{h}^*$. Hence we may view

$$\chi(\mathcal{O}^\vee) \in \mathfrak{h}^*.$$

There were many choices made in the definition of $\chi(\mathcal{O}^\vee)$. But nonetheless the infinitesimal character corresponding to $\chi(\mathcal{O}^\vee)$ is well-defined; i.e. $\chi(\mathcal{O}^\vee)$ is well-defined up to $G^\vee$ conjugacy and thus (via Harish-Chandra's theorem) specifies a well-defined maximal ideal $Z(\mathcal{O}^\vee)$ in the center of $\mathrm{U}(\mathfrak{g})$. We call $\chi(\mathcal{O}^\vee)$ the unipotent infinitesimal character attached to $\mathcal{O}^\vee$.

By a result of Dixmier [Di], there exists a unique maximal primitive ideal in $\mathrm{U}(\mathfrak{g})$ containing $Z(\mathcal{O}^\vee)$. Denote it by $I(\mathcal{O}^\vee)$, and let $d(\mathcal{O}^\vee)$ denote the dense nilpotent coadjoint orbit in $\mathrm{AV}(I(\mathcal{O}^\vee))$. The orbit $d(\mathcal{O}^\vee)$ is called the Spaltenstein dual of $\mathcal{O}^\vee$ (after Spaltenstein who first defined it in a different way); see [BV, Appendix A].

Fix $G_\mathbb{R}$ as above, and define

$$\mathrm{Unip}(\mathcal{O}^\vee) = \{X \text{ an irreducible } (\mathfrak{g}, K) \text{ module} \mid \mathrm{Ann}(X) = I(\mathcal{O}^\vee)\}.$$

This is the set of special unipotent representations for $G_\mathbb{R}$ attached to $\mathcal{O}^\vee$. Since the annihilator of such a representation X is the maximal primitive ideal containing $Z(\mathcal{O}^\vee)$, X is as small as the (generally singular) infinitesimal character $\chi(\mathcal{O}^\vee)$ allows. These algebraic conditions are conjectured to have implications about unitarity.

Conjecture 4.1 (Arthur, Barbasch-Vogan [BV])**.** *The set* $\mathrm{Unip}(\mathcal{O}^\vee)$ *consists of unitary representations.*

We are going to produce certain special unipotent representations from the regular orbits of Definition 3.1. In order to do so, we need to shift our perspective and work on side of the Langlands dual $\mathfrak{g}^\vee$. So let $G'_\mathbb{R}$ be a real form of a connected reductive algebraic group with Lie algebra $\mathfrak{g}^\vee$ and let K' denote the complexification of a maximal compact subgroup in $G'_\mathbb{R}$. Fix an *even* nilpotent coadjoint orbit $\mathcal{O}^\vee$. (This is equivalent to requiring that $\chi(\mathcal{O}^\vee)$ is integral.) Define $\mathcal{P}^\vee$ as in Proposition 3.8. Thus the main results of Section 3 are available in this setting.

Let X' denote an irreducible $(R_{\mathcal{P}^\vee}, K')$-module, and let X denote the Vogan dual of X' in the sense of [V2]. Thus X is an irreducible Harish-Chandra module for a group $G_\mathbb{R}$ arising as the real points of a connected reductive algebraic group with Lie algebra $\mathfrak{g}$. Moreover, X has trivial infinitesimal character.

Recall that we are interested in representations with infinitesimal character $\chi(\mathcal{O}^\vee)$. In order to pass to this infinitesimal character, we need to introduce certain translation functors. There are technical complications that arise in this setting since $G_\mathbb{R}$ need not be connected (although it is in Harish-Chandra's class by our hypothesis). See [KnV, Section VII.14]. For the sake of exposition, we will proceed as if $G_\mathbb{R}$ were connected, and leave it to the reader to supply the details arising from disconnectedness issues.

Fix a representative $\rho \in \mathfrak{h}^*$ representing the trivial infinitesimal character. Choose a representative $\chi \in \mathfrak{h}^*$ representing the (integral) infinitesimal character $\chi(\mathcal{O}^\vee)$ so that χ and ρ lie in the same closed Weyl chamber. Let $\nu = \rho - \chi$. After possibly passing to an appropriate cover, let F^ν denote the finite-dimensional representation of $G_\mathbb{R}$ with extremal weight ν. Using it, define the translation functor $\psi = \psi_\rho^\chi$ (as in [KnV, Section VII.13]) from the category of Harish-Chandra modules with trivial infinitesimal character to the category of Harish-Chandra modules with infinitesimal character $\chi(\mathcal{O}^\vee)$.

Theorem 4.2 (cf. [ABV, Chapter 27])**.** *Retain the notation introduced after Conjecture 4.1. In particular, fix an even nilpotent orbit* $\mathcal{O}^\vee$*, and let* $\mathcal{P}^\vee$ *denote the variety of parabolic subalgebras corresponding to the nodes labeled 0 in the weighted Dynkin diagram for* $\mathcal{O}^\vee$*. Let* X' *be an irreducible* $(R_{\mathcal{P}^\vee}, K')$*-module. Let* $Z = \psi(X)$ *denote the translation functor to infinitesimal character* $\chi(\mathcal{O}^\vee)$ *applied to the Vogan dual* X *of* X'*. Then the following are equivalent:*

(a) *Z is a (nonzero) special unipotent representation attached to $\mathcal{O}^\vee$.*
(b) *there exists a $\mathcal{P}^\vee$-regular orbit $Q^\vee \in \mathrm{cv}(X')$.*

Proof. From the properties of the duality explained in [V2, Section 14] (and the translation principle), Z is nonzero with infinitesimal character $\chi(\mathcal{O}^\vee)$ if and only if X' is annihilated by $I_{\mathcal{P}^\vee}$, i.e. if and only if X' descends to a $(\mathcal{D}_{\mathcal{P}^\vee}, K)$-module. Moreover Z is annihilated by a maximal primitive ideal if and only if the $D_{\mathcal{P}^\vee}$-module X' has minimal possible annihilator, namely $I_{\mathcal{P}^\vee}$. The conclusion is that Z is special unipotent attached to $\mathcal{O}^\vee$ if and only if X' is a $(\mathcal{D}_{\mathcal{P}^\vee}, K)$-module annihilated by $I_{\mathcal{P}^\vee}$. So the the theorem follows from the equivalence of (a) and (c) in Corollary 3.17. ☐

Since the duality of [V2] is effectively computable, and since the same is true of the translation functors ψ, the theorem shows Remark 3.10 translates into an effective construction of special unipotent representations. More precisely, one uses Remark 3.10 to enumerate the relevant $\mathcal{P}^\vee$-regular orbits, and for each one constructs the representation $X' = A_{\mathfrak{p}}$ of Proposition 3.9. As remarked in the proof of Proposition 3.9, X' satisfies condition (b) of Theorem 4.2. Applying the construction of the theorem gives special unipotent representations.

In fact, this construction may be understood further in light of the following refinement. In the setting of Theorem 4.2, fix a $\mathcal{P}^\vee$-regular orbit $Q^\vee$, and define $\mathbb{A}(Q^\vee)$ be the set of special unipotent representations attached to $\mathcal{O}^\vee$ produced by applying Theorem 4.2 to all modules X' with $Q^\vee \in \mathrm{cv}(X')$. Then the theorem implies

$$\mathrm{Unip}(\mathcal{O}^\vee) = \bigcup \mathbb{A}(Q^\vee),$$

where the (not necessarily disjoint) union is over all $\mathcal{P}^\vee$-regular orbits.

The sets $\mathbb{A}(Q^\vee)$ are the Arthur packets defined in [ABV, Chapter 27]. While there are effective algorithms to enumerate $\mathrm{Unip}(\mathcal{O}^\vee)$, there are no such algorithms for individual packets $\mathbb{A}(Q^\vee)$ (except in favorable cases). In any event, the discussion of the previous paragraph shows that *Remark 3.10 leads to an effective algorithm to enumerate an element of each Arthur packet of integral special unipotent representations.* These representatives are necessarily distinct.

5. EXAMPLES

Example 5.1 (Maximal parabolic subalgebras for classical groups). Suppose G is classical and $\mathcal{P}$ consists of maximal parabolic subalgebra. Then it is well-known that

$$\mathrm{ind}_{W(\mathcal{P})}^{W}(\mathrm{sgn})$$

decomposes multiplicity freely as a W-module. Thus if $\mathrm{Sp}(\xi)^{A_K}$ is irreducible as a W-module, then Proposition 2.8 implies $\Phi_{\mathcal{P}}^{-1}(\mathcal{O}_K)$ is a single orbit. In particular if the orbits of $A_K(\xi)$ and $A_G(\xi)$ on irreducible components of the Springer fiber $\mu_{\mathcal{B}}^{-1}(\xi)$ coincide (for instance, if $A_K(\xi)$ surjects onto $A_G(\xi)$ for each ξ), then $\mathrm{Sp}(\xi)^{A_K} = \mathrm{Sp}(\xi)^{A_G}$ is irreducible and $\Phi_{\mathcal{P}}$ is injective.

Proposition 5.2. *Suppose the real form $G_{\mathbb{R}}$ of G corresponding to θ is a classical semisimple Lie group with no complex factors whose Lie algebra has no simple factor isomorphic to $\mathfrak{so}^*(2n)$ or $\mathfrak{sp}(p,q)$. If $\mathcal{P}$ consists of maximal parabolic subalgebras, then $\Phi_{\mathcal{P}}$ is injective.*

Proof. Unfortunately this follows from a case-by-case analysis of the classical groups. First note that the orbits of $A_K(\xi)$ and $A_G(\xi)$ on $\mu_{\mathcal{B}}^{-1}(\xi)$ are insensitive to the isogeny class of $G_{\mathbb{R}}$. So, by the remarks preceding the proposition, it is enough to examine when the two kinds of orbits coincide for a simply connected group $G_{\mathbb{R}}$ with simple Lie algebra. In type A, all A-groups are trivial (up to isogeny) so there is nothing to check. It follows from direct computation that $A_K(\xi)$ surjects on $A_G(\xi)$ for $G_{\mathbb{R}} = \mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{SO}(p,q)$, but that the image of $A_K(\xi)$ in $A_G(\xi)$ is always trivial for $\mathrm{Sp}(p,q)$ and $\mathrm{SO}^*(2n)$. This completes the case-by-case analysis and hence the proof.

Remark 5.3. For the groups in Proposition 5.2, the map $\Phi_{\mathcal{B}}$ is computed explicitly in [T1] and [T3]. Using Proposition 2.4(1) this gives one (rather roundabout) way to compute $\Phi_{\mathcal{P}}$ in these cases. For exceptional groups, the injectivity of the proposition fails. See Example 5.12 below.

Example 5.4. Suppose now $G_{\mathbb{R}} = \mathrm{Sp}(2n, \mathbb{R})$ and $\mathcal{P}$ consists of maximal parabolic of type corresponding to the subset of simple roots obtained by deleting the long one. (So if $n = 2$, $\mathcal{P} = \mathcal{P}_\alpha$ in Example 3.3.) Then the analysis of the preceding example extends to show that $\Phi_{\mathcal{P}}$ is an order-reversing bijection. The closure order on $K\backslash\mathcal{N}^\theta_{\mathcal{P}}$ (and hence $K\backslash\mathcal{P}$) is as follows.

$$\begin{tikzcd}
2^n_+ & & 2^{n-1}_+ 2^1_- & \cdots & 2^{n-1}_- 2^1_+ & & 2^n_- \\
& \cdots \arrow[ul] \arrow[ur] & & \cdots \arrow[ul] \arrow[ur] & & \cdots \arrow[ul] \arrow[ur] & \\
& & 2^1_+ 1^{n-1}_+ 1^{n-1}_- \arrow[ul, dash] \arrow[ur, dash] & & 2^1_- 1^{n-1}_+ 1^{n-1}_- \arrow[ul, dash] \arrow[ur, dash] & & \\
& & & 1^n_+ 1^n_- \arrow[ul] \arrow[ur] & & &
\end{tikzcd} \tag{5.5}$$

Here, as before, we are using the parametrization of $K\backslash\mathcal{N}^\theta_{\mathcal{P}}$ given in [CM, Theorem 9.3.5]. There are thus $n+1$ orbits which are $\mathcal{P}$-regular, all of which are closed according to Proposition 3.7(a) (which applies since $\mathcal{P}$ is attached via Proposotion 3.8 to the even complex orbit with partition 2^n). Theorem 4.2 produces $n+1$ special unipotent representations for $\mathrm{SO}(n, n+1)$.

Example 5.6. Suppose $G_{\mathbb{R}} = \mathrm{U}(n, n)$ and $\mathcal{P}$ corresponds to the subset of simple roots obtained by deleting the middle simple root in the Dynkin diagram of type A_{2n-1}. Then $\Phi_{\mathcal{P}}$ is an order reversing bijection, and the partially ordered sets in question again look like that (5.5) using the parametrization of $K\backslash\mathcal{N}^\theta_{\mathcal{P}}$ given in [CM, Theorem 9.3.3]. Again there are $n+1$ orbits which are $\mathcal{P}$-regular. Theorem 4.2 produces $n+1$ special unipotent representation for $\mathrm{GL}(2n, \mathbb{R})$, each of which turns out to be a constituent of maximal Gelfand-Kirillov dimension in the degenerate principal series for $\mathrm{GL}(2n, \mathbb{R})$ induced from a one-dimensional representation of a Levi factor isomorphic to a product of n copies of $\mathrm{GL}(2, \mathbb{R})$.

In terms of representation theory of $G_{\mathbb{R}} = \mathrm{U}(n, n)$, it is well-known that the enveloping algebra in this case does surject on the ring of global differential operators on $\mathcal{P}$ (e.g. the discussion of [T2, Remark 3.3]) and localization is an equivalence of categories. Because all Cartan subgroups in $\mathrm{U}(n, n)$ are connected, the only irreducible flat K-equivariant connections on $\mathcal{P}$ are the trivial ones supported on single K orbits. The map $Q \mapsto \Phi_{\mathcal{P}}(Q)$ coincides with the map which sends the unique irreducible $(R_{\mathcal{P}}, K)$-module supported on the closure of Q to the dense orbit in its (irreducible) associated variety, and is a bijection between such irreducible modules and the K orbits on $\mathcal{N}^\theta_{\mathcal{P}}$. It would be interesting to see if this observation could be used to give a geometric explanation of the computation of composition series of certain degenerate principal series for $\mathrm{U}(n, n)$ first given in [Sa] and later reproved in [Le]. (See, for instance, Sahi's module diagrams reproduced in [Le, Figure 7], for example.)

Example 5.7. Suppose $G_{\mathbb{R}} = \mathrm{Sp}(1, 1)$, a real form of $G = \mathrm{Sp}(4, \mathbb{C})$. If $\mathcal{O}$ is the subregular nilpotent orbit for $\mathfrak{g}$ and $\xi \in \mathcal{O} \cap (\mathfrak{g}/\mathfrak{k})^*$, then $A_K(\xi)$ is trivial, but $A_G(\xi) \simeq \mathbb{Z}/2$. So the proof of Proposition 5.2 does not apply. Let α denote the short simple root and β the long one. The closure order on

$K\backslash\mathcal{B}$ is given by

(5.8)

$$\begin{tikzcd}
& Q & \\
& R \arrow[u, "\beta"] & \\
S_+ \arrow[ur, "\alpha"] & & S_- \arrow[ul, "\alpha"']
\end{tikzcd}$$

The picture for $K\backslash\mathcal{P}_\alpha$ is

(5.9)

$$\begin{tikzcd}
\pi_\alpha(Q)_3 \\
\pi_\alpha(R)_2 \arrow[u]
\end{tikzcd}$$

and for $K\backslash\mathcal{P}_\beta$

(5.10)

$$\begin{tikzcd}
& \pi_\beta(Q)_3 & \\
\pi_\beta(S_+)_2 \arrow[ur] & & \pi_\beta(S_-)_2 \arrow[ul]
\end{tikzcd}$$

Here $\mathcal{N}^\theta_\alpha = \mathcal{N}^\theta_\beta = \mathcal{N}^\theta_\mathcal{B}$, and the closure order of K orbits is simply

(5.11)

$$\begin{tikzcd}
2^1_+2^1_- \\
1^2_+1^2_- \arrow[u]
\end{tikzcd}$$

in the notation of [CM, Theorem 9.3.5]. Then Φ_α is an order reversing bijection, but Φ_β is two-to-one over $1^2_+1^2_-$. The reason is that

$$\mathrm{Sp}(\xi) = \mathrm{std} \oplus \chi,$$

where std is the two-dimensional standard representation of W and χ is a character on which the simple reflection s_α acts trivially and on which s_β acts nontrivially. The orbit $\pi_\alpha(R)$ is $\mathcal{P}_\alpha$-regular, and the orbits $\pi_\beta(S_\pm)$ are $\mathcal{P}_\beta$-regular.

Example 5.12. As an example of what can happen in the exceptional cases, let G be the (simply connected) connected complex group of type F_4 and θ correspond to the split real form $G_\mathbb{R}$ of G. (So K is a quotient of is $\mathrm{Sp}(3,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ by $\mathbb{Z}/2$.) Then the corresponding real form $G_\mathbb{R}$ is split. Let $\mathcal{P}$ denote the variety of maximal parabolic obtained by deleting the middle long root from the Dynkin diagram, and let $\mathcal{O}$ denote the corresponding Richardson orbit. Then $\mathcal{O}$ is 40 dimensional and is labeled $\mathrm{F}_4(\mathrm{A}_3)$ in the Bala-Carter classification. Moreover $\mathcal{O}$ is the unique orbit which is fixed under Spaltenstein duality. (Here we are of course identifying $\mathfrak{g}$ and $\mathfrak{g}^\vee$.) For $\xi \in \mathcal{O}$, $A_G(\xi) = S_4$, the symmetric group on four letters. The weighted Dynkin diagram of $\mathcal{O}$ has the middle long root labeled 2 and all others nodes labeled 0. So $\mathcal{P}$ corresponds to $\mathcal{O}$ as in Proposition 3.8.

From results of Djoković (recalled in [CM, Section 9.6]) there are 19 orbits of K on $\mathcal{N}^\theta_\mathcal{P}$. They are labeled 0–18; the orbit corresponding to label i will be denoted $\mathcal{O}^i_K$, and ξ^i will denote an element of $\mathcal{O}^i_K$. Orbits $\mathcal{O}^{16}_K, \mathcal{O}^{17}_K$, and $\mathcal{O}^{18}_K$ are the three K orbits on $\mathcal{O} \cap (\mathfrak{g}/\mathfrak{k})^*$. From the discussion leading to [Ki, Table 2], it follows that $A_K(\xi^i)$ surjects onto $A_G(\xi^i)$ for $i = 0, \ldots, 15$. In each of these cases, $A_G(\xi)$ is either trivial or $\mathbb{Z}/2$. We also have $A_K(\xi^{16}) = A_G(\xi^{16}) = S_4$. But $A_K(\xi^{17}) = D_4$, the dihedral group with eight elements, and $A_K(\xi^{17}) \to A_G(\xi^{17})$ is the natural inclusion into S_4. Finally, $A_K(\xi^{18}) = \mathbb{Z}/2 \times \mathbb{Z}/2$ which injects into $A_G(\xi^{18})$.

For $i = 17$ and 18, it is not immediately obvious how to read off $\mathrm{Sp}(\xi^i)^{A_K(\xi^i)}$ from, say, the tables of [Ca]. But for $i = 0, \ldots, 16$, the component group calculations of the previous paragraph imply that $\mathrm{Sp}(\xi^i)^{A_K(\xi^i)} = \mathrm{Sp}(\xi^i)^{A_G(\xi^i)}$, and such representations are indeed tabulated in [Ca]. Applying Proposition 2.8, it is then not difficult to show that

$$\#\Phi^{-1}(\mathcal{O}_K^i) = 1 \text{ if } i \in \{0,1,2,3\} \cup \{9,10,\ldots,16\}$$

and

$$\#\Phi^{-1}(\mathcal{O}_K^i) = 2 \text{ if } i \in \{4,5,6,7,8\}.$$

In more detail, the G-saturation of $\mathcal{O}_K^4$ and $\mathcal{O}_K^5$ is the complex orbit $\mathrm{A}_1 \times \widetilde{\mathrm{A}_1}$ in the Bala-Carter labeling, while $\mathcal{O}_K^6$, $\mathcal{O}_K^7$, and $\mathcal{O}_K^8$ have G saturation labeled by A_2. The corresponding irreducible Weyl group representations in these two cases both appear with multiplicity two in $\mathrm{ind}_{W(\mathcal{P})}^{W}(\mathrm{sgn})$. All other relevant multiplicities are one.

We thus conclude that there are 22 orbits of K on $\mathcal{P}$ which map via $\Phi_{\mathcal{P}}$ to some $\mathcal{O}_K^i$ for $i = 0,\ldots,15$. Meanwhile, using the software program `atlas`, one can compute the closure order of K on $\mathcal{B}$, and thus (as explained in Section 2), the closure order on $K\backslash\mathcal{P}$. Figure 4.1 gives the full closure order for $K\backslash\mathcal{P}$. Vertices are labeled according to their dimension. (The edges in Figure 4.1 do *not* distinguish between the weak and full closure order. Doing so would make the picture significantly more complicated and difficult to draw.) There are thus 24 orbits of K on $\mathcal{P}$. Since 22 have been shown to map to $\mathcal{O}_K^i$ for $i = 0,\ldots,15$, one concludes that the the fiber of $\Phi_{\mathcal{P}}$ over $\mathcal{O}^i$ for $i = 16$ and 17 must consist of just one element in each case.

In particular there are three $\mathcal{P}$-regular K orbits on $\mathcal{P}$ which are bijectively matched via Proposition 3.7(b) to $\mathcal{O}_K^{16}$, $\mathcal{O}_K^{17}$, and $\mathcal{O}_K^{18}$. But from the `atlas` computation of the closure order on $K\backslash\mathcal{P}$, there are *four* closed orbits of K on $\mathcal{P}$. (These are in fact exactly the four orbits which are minimal in the weak closure order.) See Figure 5.12. The `atlas` labels of the closed orbits are 3, 22, 31, and 47. Their respective dimensions are 0, 1, 2, and 3. Applying the algorithm of Remark 3.10, one deduces that the three $\mathcal{P}$-regular orbits are 3, 31, and 47. Theorem 4.2 thus produces three distrinct special unipotent representations, one in each of the three Arthur packets for $\mathcal{O} = d(\mathcal{O})$.

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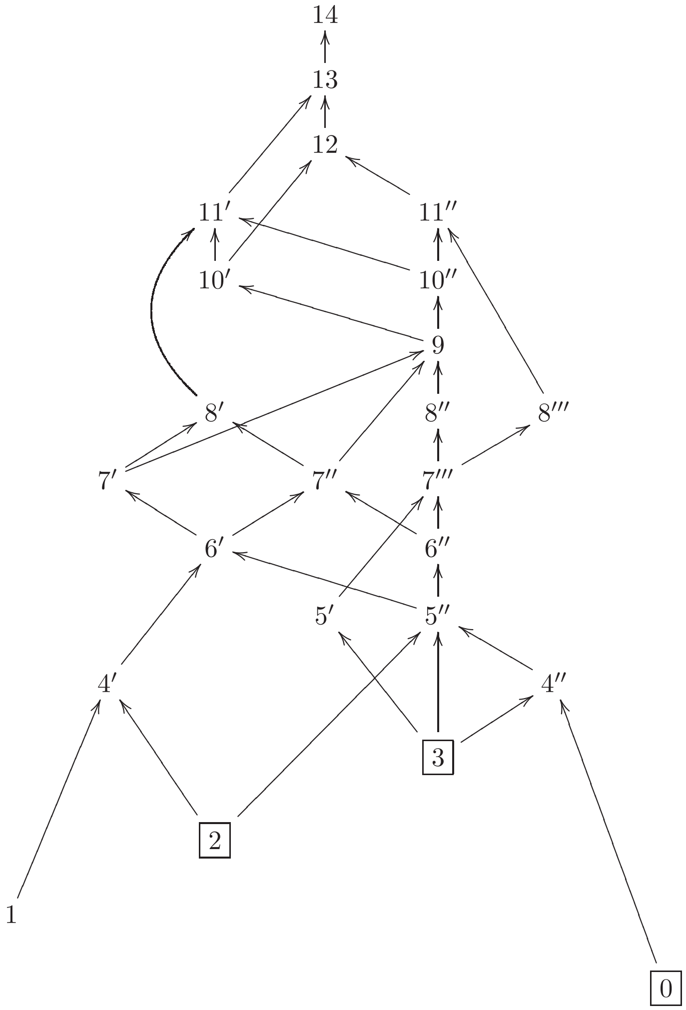

FIGURE 5.1. The full closure ordering of K-orbits on $\mathcal{P}$ for $G_{\mathbb{R}} = \mathrm{F}_4$ and $\mathcal{O} = \mathrm{F}_4(\mathrm{A}_3)$. Vertices are labeled according to their dimensions and boxed vertices are $\mathcal{P}$-regular. Note, in particular, that not every closed orbit is $\mathcal{P}$-regular.

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