UPSC: Problem Set 1

Opens: 3 p.m. Monday February 13th, 2023

Due: 12 p.m. Friday February 24th, 2023

- You must work independently.

- Write your solutions clearly and show all of your work.

- Include your name, student ID number, and email address.

- Email a pdf file of your solution to `ugrad_services@math.utah.edu` by the deadline.

- A winner will be decided on the basis of the best solution submitted. If no best solution can be determined (i.e. there exist relatively identical solutions), the winner will be the student who submitted their solution first.

- Each submission will be given 3 points for a fully correct solution and 1-2 points for a partially correct solution. The winner of each problem set will get a bonus of ϵ points.

- Please don't just search online for a solution - that isn't the point of this contest.

- Enjoy the problems!

Problem 1 [1 point]:

(a) Alice and Bob play a game with a pile of n stones. Each turn, a player may remove $1, 2, 3, 4$, or 5 stones from the pile. Whoever takes the last stone wins, and Alice plays first. For what values of n does Alice have a winning strategy?

(b) Alice and Bob decide to modify their game. Now each turn, each player may remove either $1/3$ or $1/2$ of the stones each turn, rounded up. For instance, if the pile has 7 stones, a player may either remove $\lceil 7/3 \rceil = 3$ stones or $\lceil 7/2 \rceil = 4$ stones on their turn. Describe the values of n such that Alice has a winning strategy.

Problem 2 [1 point]: Find the least possible area of a quadrilateral with a vertex on both branches of the hyperbola $xy = 1$ and both branches of the hyperbola $xy = -1$.

Problem 3 [1 point]: Let A be an $m \times n$ matrix with positive integer coefficients such that $m + n$ entries are distinct prime numbers. Show that the rank of A is at least 2.