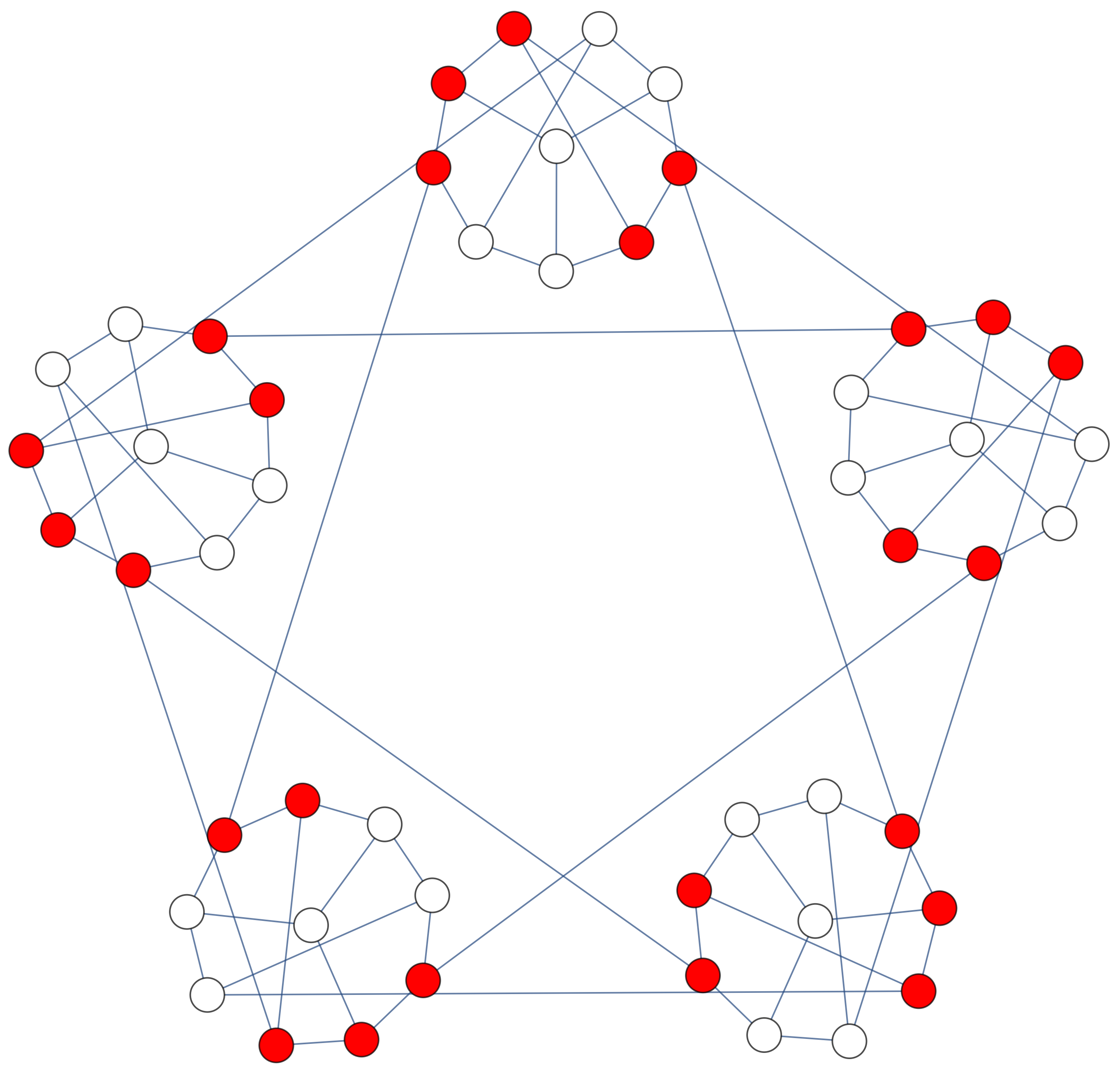
Graphical Designs



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Joint work with:

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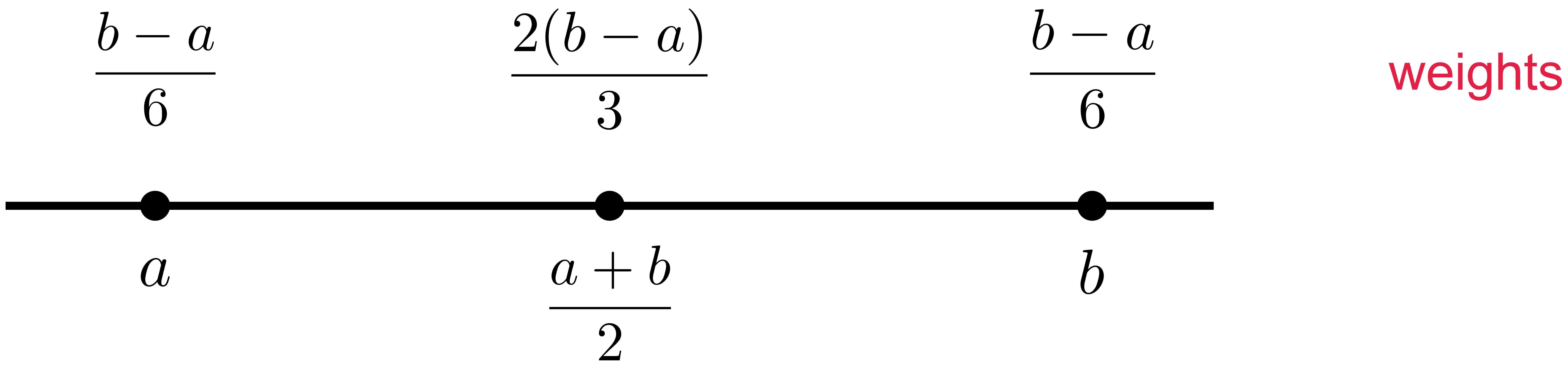
Caltech

Stefan Steinerberger
University of Washington

What are graphical designs?

SIMPSON'S RULE

$$\int_a^b f(x)dx \sim \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$



SPHERICAL DESIGNS

A spherical quadrature rule is a set of points $\{x_1, \ldots, x_n\} \subset \mathbb{S}^d$ and weights $a_i \in \mathbb{R}$ such that for sufficiently smooth functions f

$$\frac{1}{|\mathbb{S}^d|} \int_{\mathbb{S}^d} f(x)\ dx \approx \sum_{i=1}^{n} a_i f(x_i)$$

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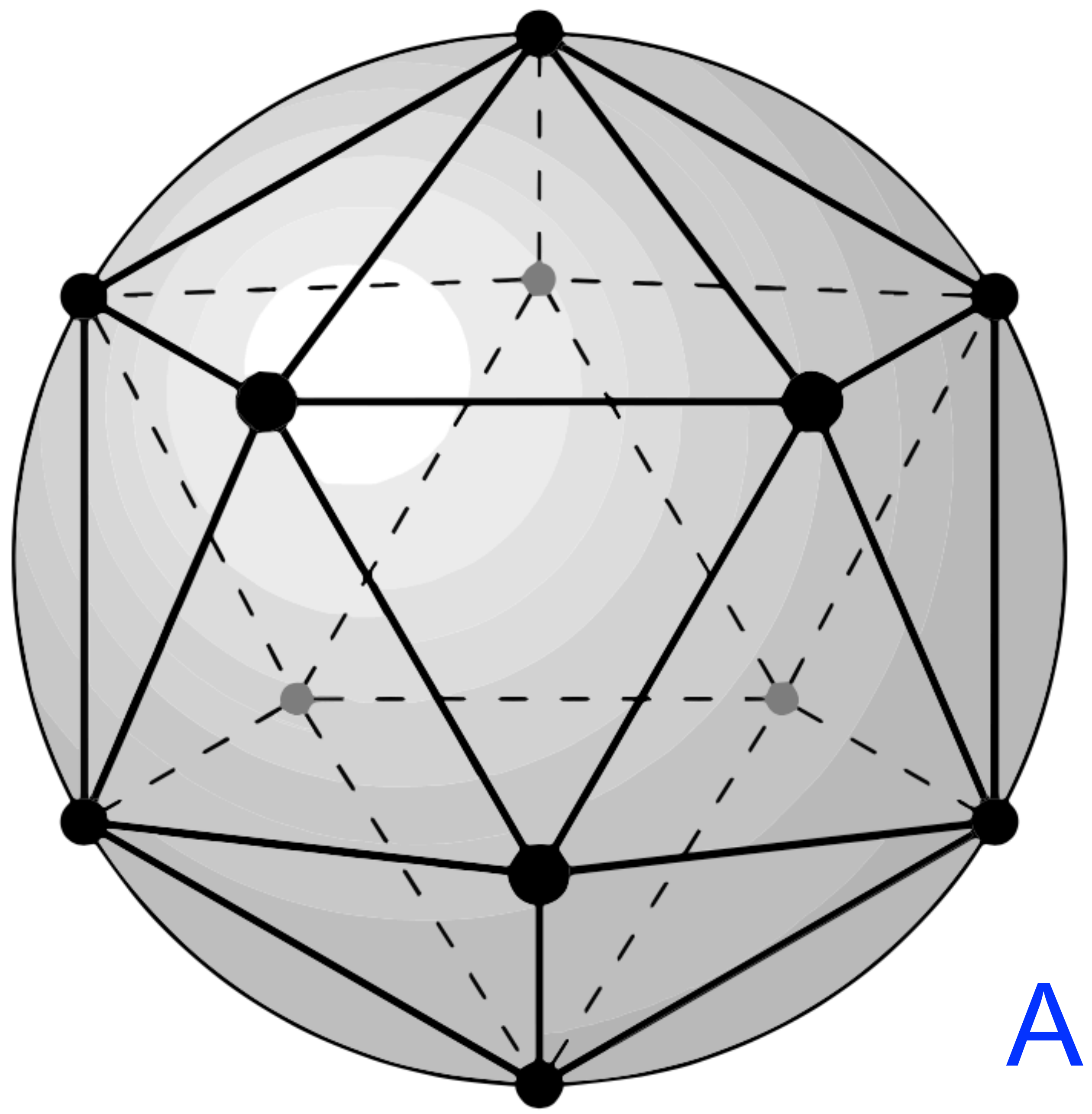

A spherical 5-design

A spherical t-design integrates all polynomials of degree at most t

PASSING TO GRAPHS

Definition. $G = (V, E)$ finite, simple, connected graph.
$W \subset V$ with weights $a_w \in \mathbb{R}$ averages a function $f : V \to \mathbb{R}$ if

$$\sum_{w \in W} a_w f(w) = \frac{1}{|V|} \sum_{v \in V} f(v).$$

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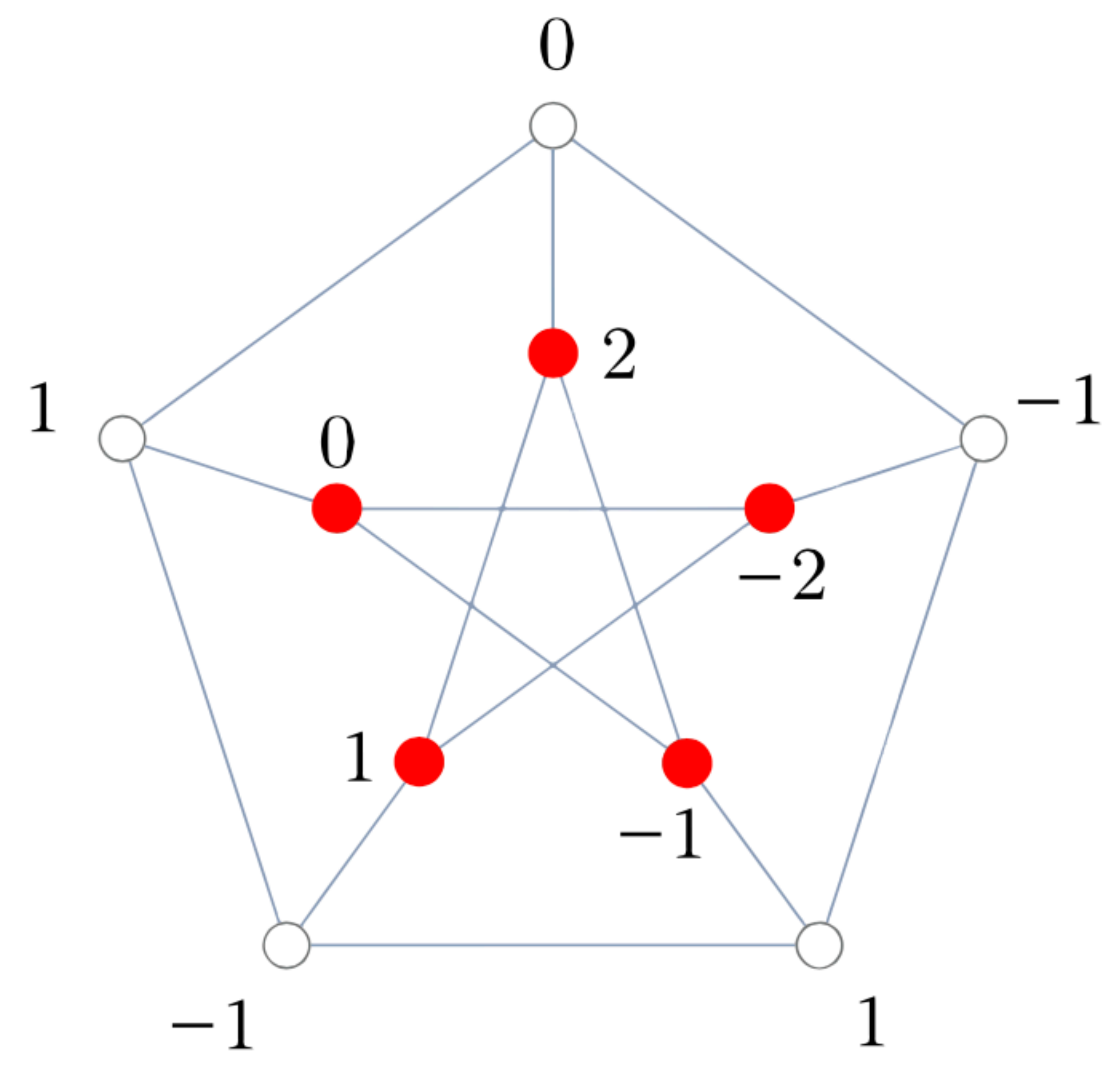
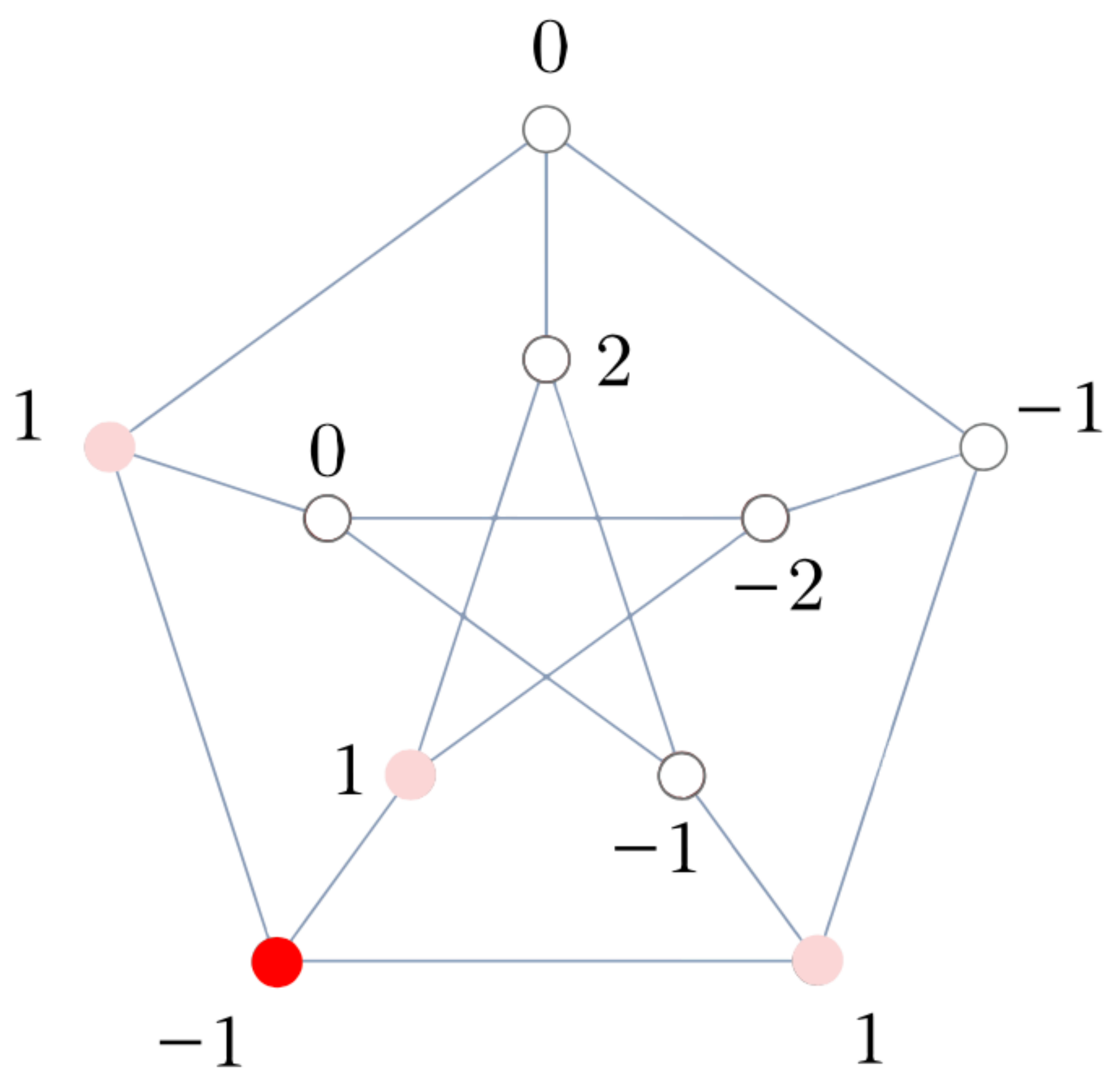
$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ -1 \\ 0 \\ 2 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$



$a = (0, 0, 0, 0, 0, 1, 1, 1, 1, 1)$

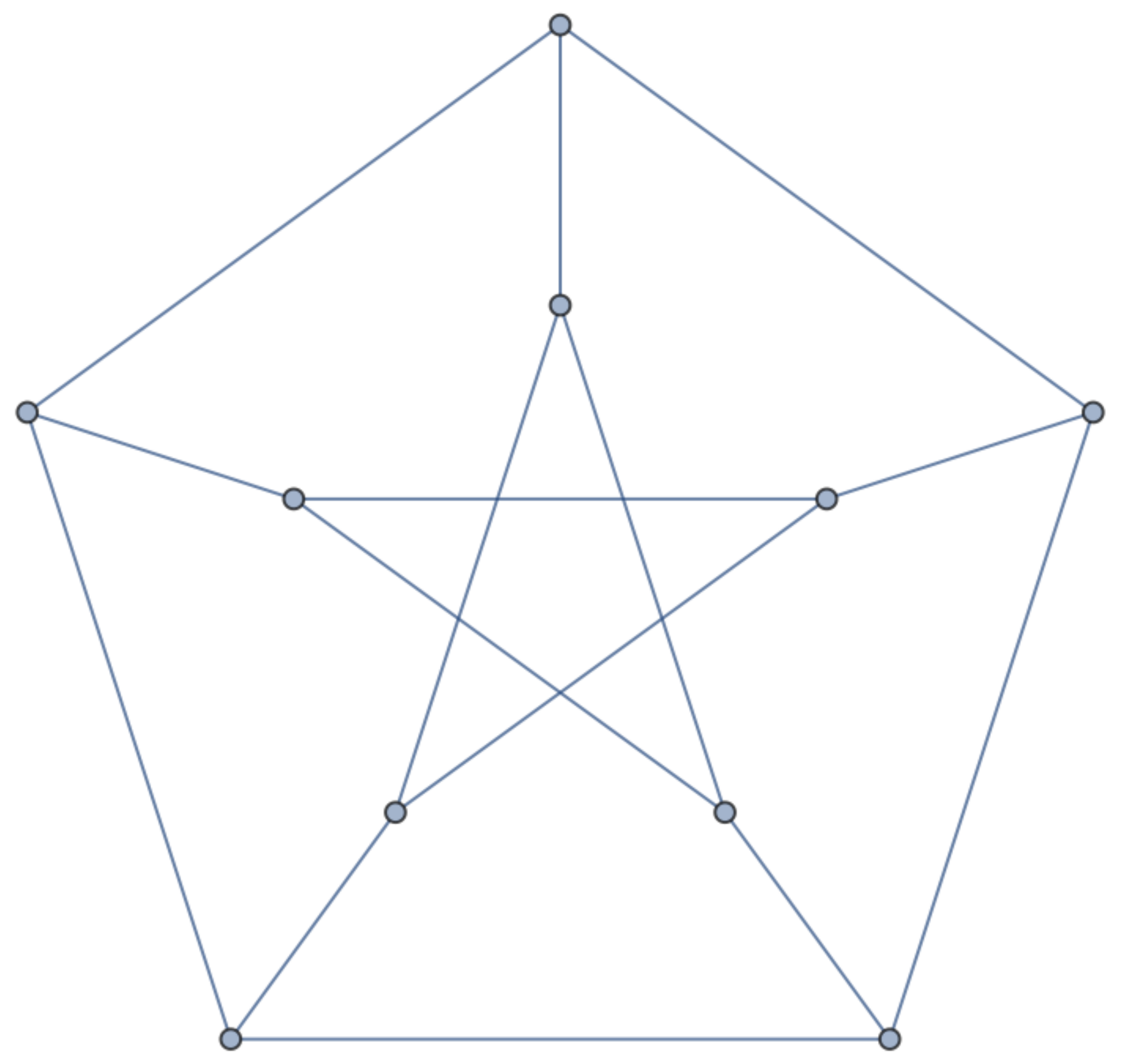
$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ -1 \\ 0 \\ 2 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$



$$a = \left(\frac{1}{3}, 0, 0, \frac{1}{3}, 1, 0, 0, 0, 0, \frac{1}{3}\right)$$

WHICH FUNCTIONS TO AVERAGE?

Assume G is regular from now on



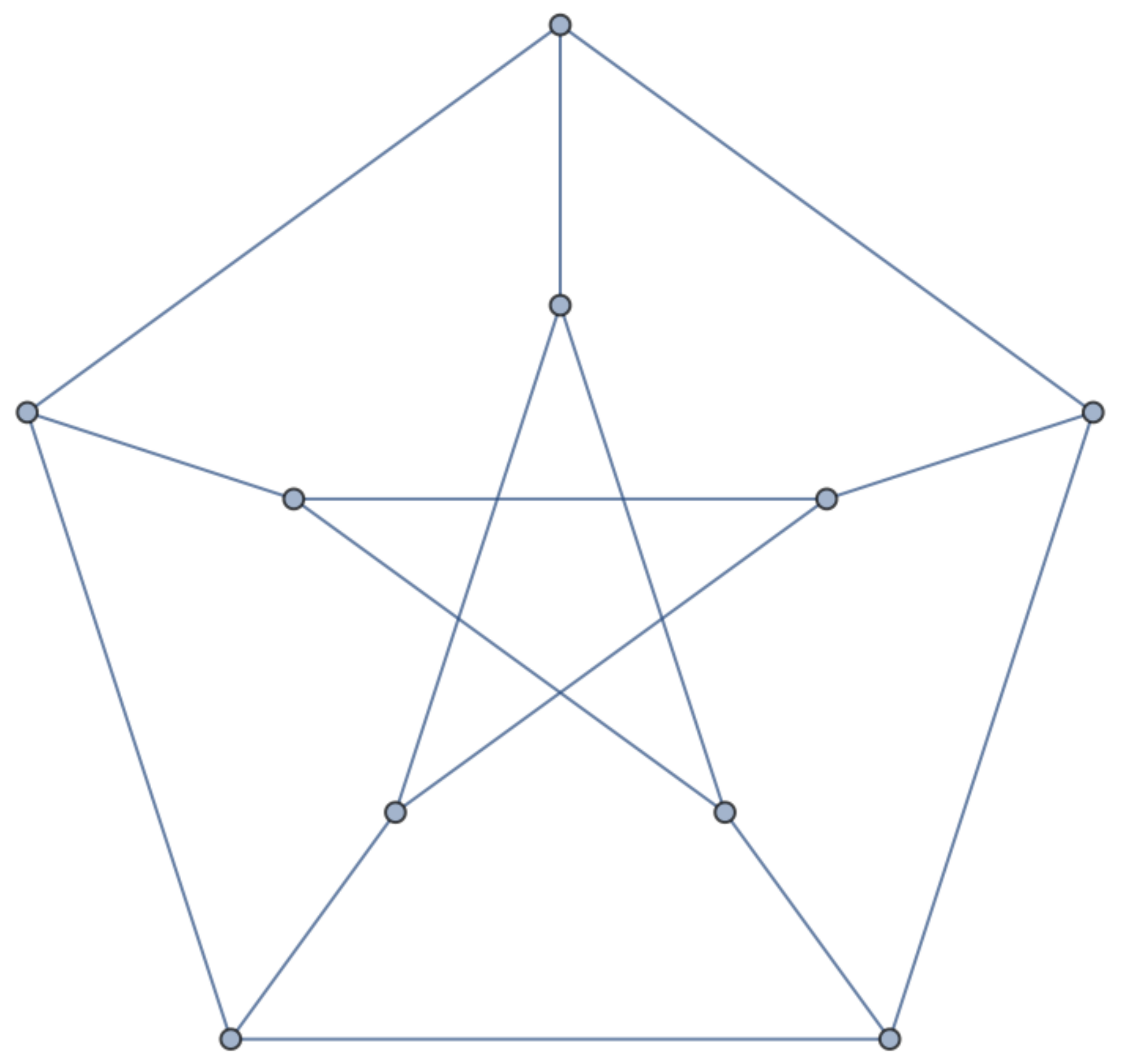
3-regular

WHICH FUNCTIONS TO AVERAGE?

Assume G is regular from now on

$A \in \{0,1\}^{|V|\times|V|}$ adjacency matrix of G, $A_{ij} = 1 \Leftrightarrow ij \in E$

$D \in \mathbb{R}^{|V|\times|V|}$ diagonal matrix $D_{vv} = \deg(v) \;\forall\, v \in V$



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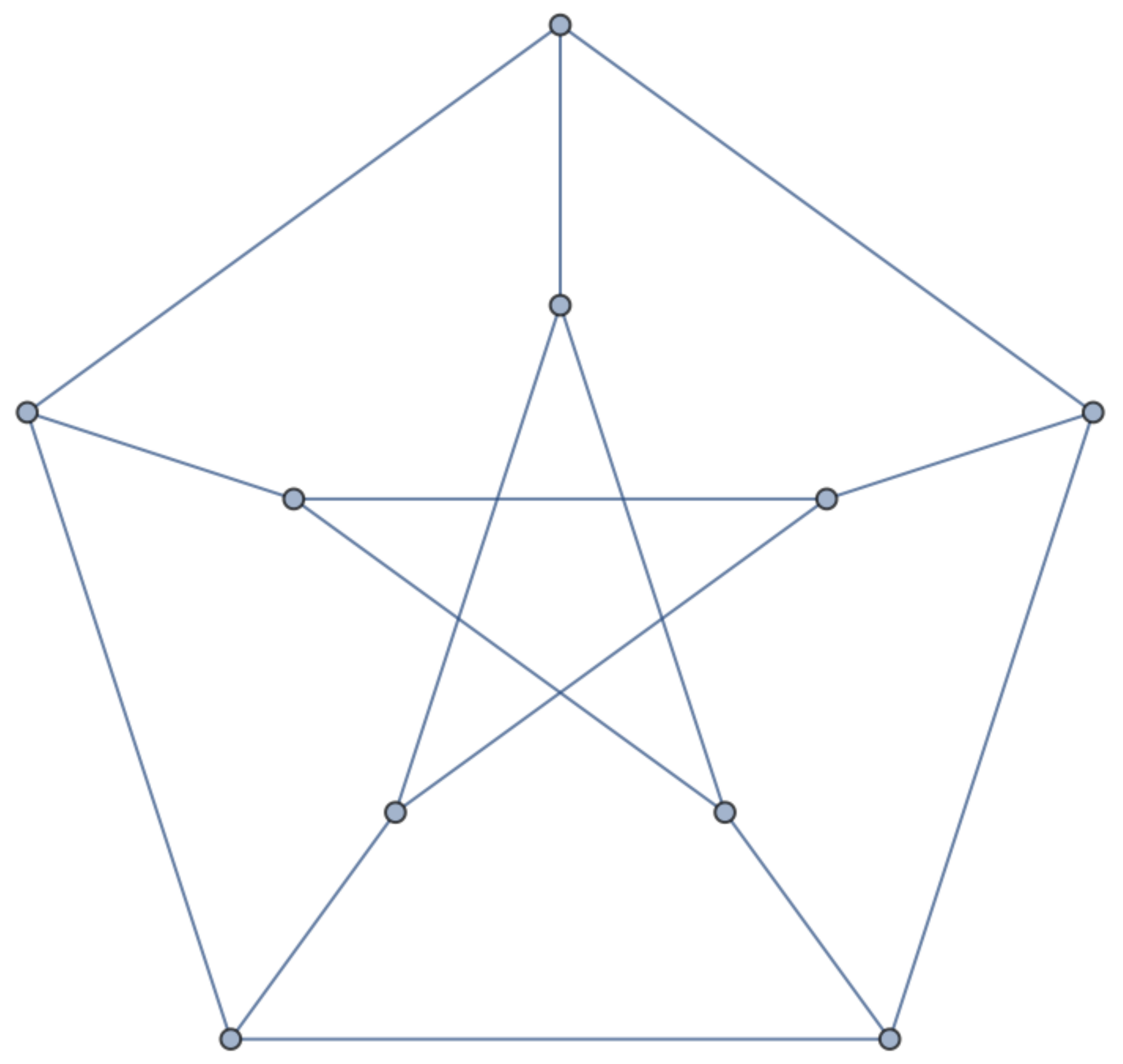
$$D = \frac{1}{3} I \in \mathbb{R}^{10\times 10}$$



3-regular

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$AD^{-1} = \frac{1}{3} A$$

Eigenvectors of AD^{-1}

AD^{-1} symmetric, nonnegative, doubly stochastic

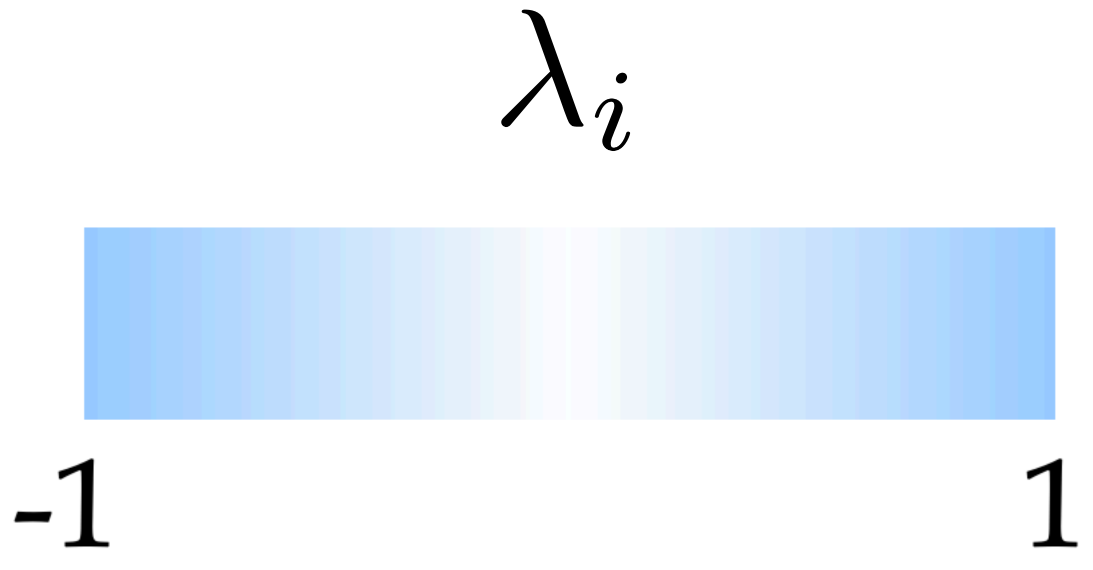
Eigenvectors of AD^{-1}

AD^{-1} symmetric, nonnegative, doubly stochastic

$\Rightarrow$ all eigenvalues are real

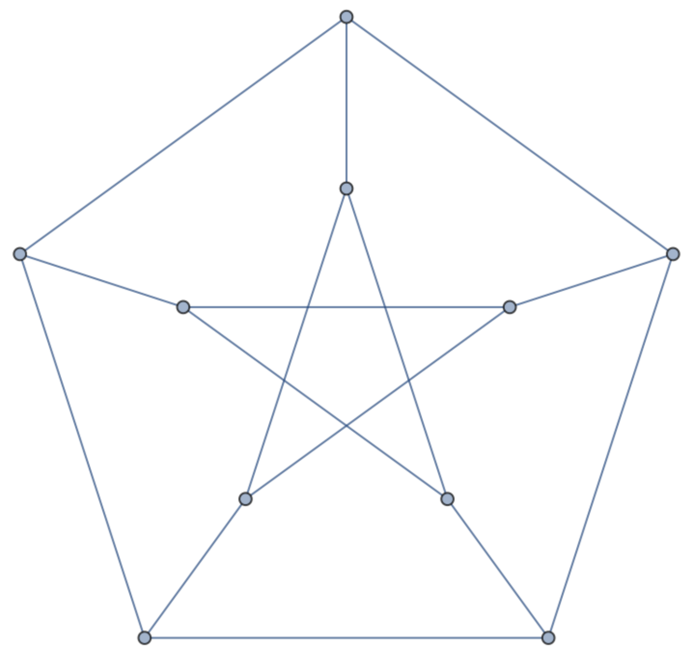
AD^{-1} has an orthonormal basis of eigenvectors

all eigenvalues are in $[-1, 1]$



$\lambda_{max} = 1$ with eigenvector $\mathbf{1} = (1, 1, 1, \dots, 1)$

PETERSEN GRAPH

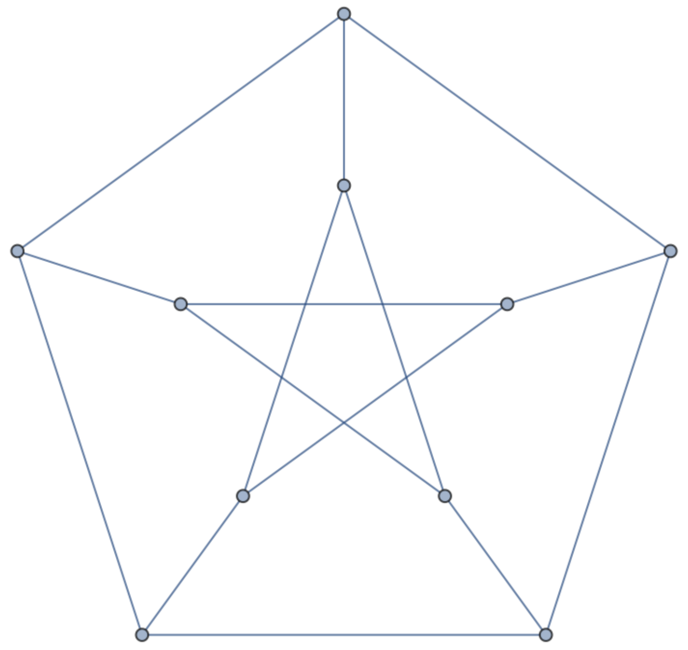


eigenvalues

$$\left\{1, -\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$$

orthonormal eigenbasis

$$\begin{pmatrix}
\frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\
\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{\sqrt{2}}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & 0 & \frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & \frac{1}{3\sqrt{2}} \\
\frac{1}{3\sqrt{10}} & -\frac{1}{3}\left(2\sqrt{\frac{2}{5}}\right) & \frac{1}{3\sqrt{10}} & \frac{1}{3\sqrt{10}} & \frac{1}{3\sqrt{10}} & -\frac{1}{3}\left(2\sqrt{\frac{2}{5}}\right) & \sqrt{\frac{2}{5}} & -\frac{1}{3}\left(2\sqrt{\frac{2}{5}}\right) & \frac{1}{3\sqrt{10}} & \frac{1}{3\sqrt{10}} \\
-\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\
-\frac{1}{2\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} \\
0 & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 \\
-\frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & 0 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{3} \\
\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}}
\end{pmatrix}$$



PETERSEN GRAPH

eigenvalues $\left\{1, -\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$

$$\begin{array}{c} 1 \\ \\ \\ -\frac{2}{3} \\ \\ \\ \\ \\ \frac{1}{3} \\ \\ \end{array} \begin{pmatrix}
\frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\
\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{\sqrt{2}}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & 0 & \frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & \frac{1}{3\sqrt{2}} \\
\frac{1}{3\sqrt{10}} & -\frac{1}{3}\left(2\sqrt{\frac{2}{5}}\right) & \frac{1}{3\sqrt{10}} & \frac{1}{3\sqrt{10}} & \frac{1}{3\sqrt{10}} & -\frac{1}{3}\left(2\sqrt{\frac{2}{5}}\right) & \sqrt{\frac{2}{5}} & -\frac{1}{3}\left(2\sqrt{\frac{2}{5}}\right) & \frac{1}{3\sqrt{10}} & \frac{1}{3\sqrt{10}} \\
-\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\
-\frac{1}{2\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} \\
0 & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 \\
-\frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & 0 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{3} \\
\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}}
\end{pmatrix}$$

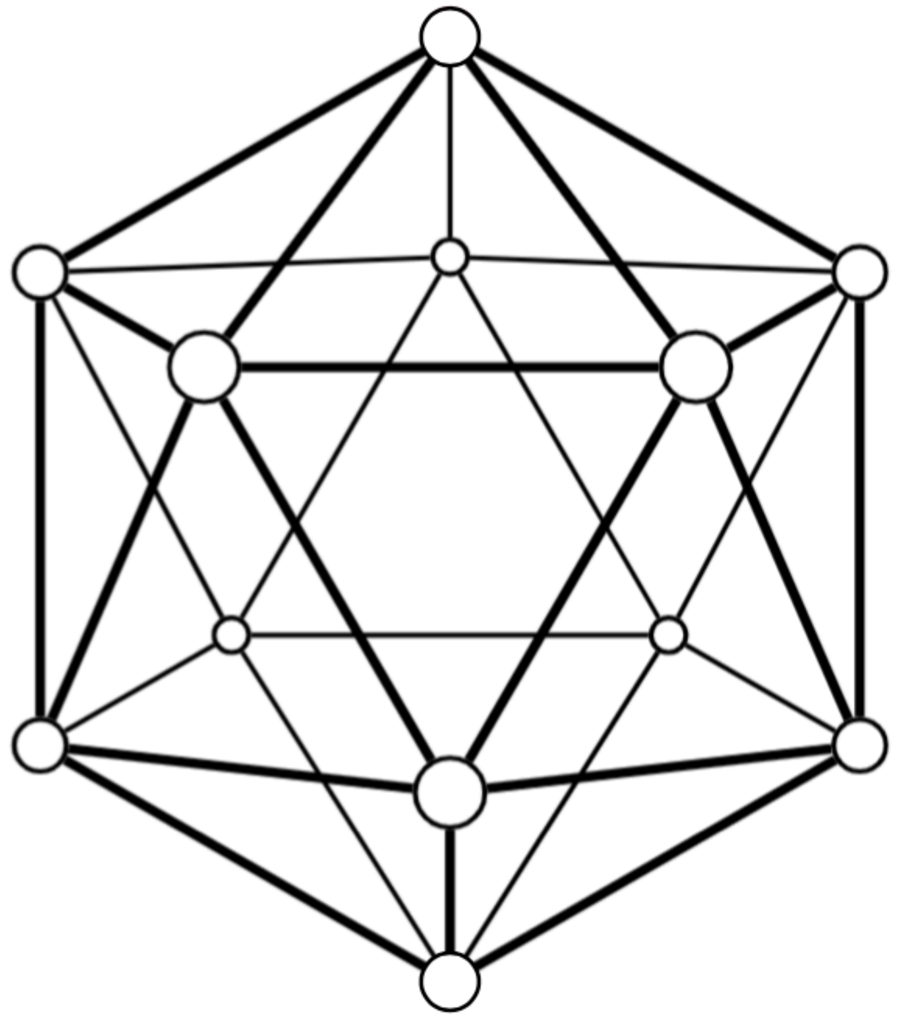
WHICH FUNCTIONS TO AVERAGE?

$$\boxed{\text{Eigenvectors of } AD^{-1}}$$

$$\{f : V \to \mathbb{R}\} = \mathbb{R}^V = \text{span}(\text{eigenvectors of } AD^{-1})$$

Ordering of eigenvalues orders the eigenvectors, so we might try to average eigenvectors in increasing order

WHICH FUNCTIONS TO AVERAGE?

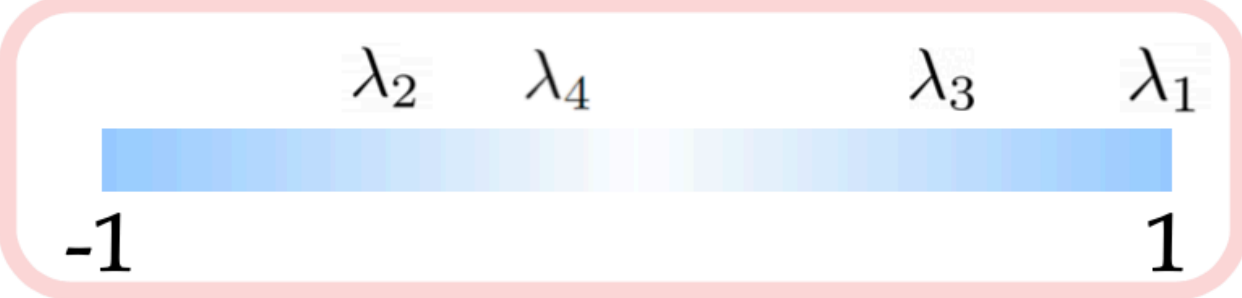


Eigenvectors of AD^{-1}

$\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$

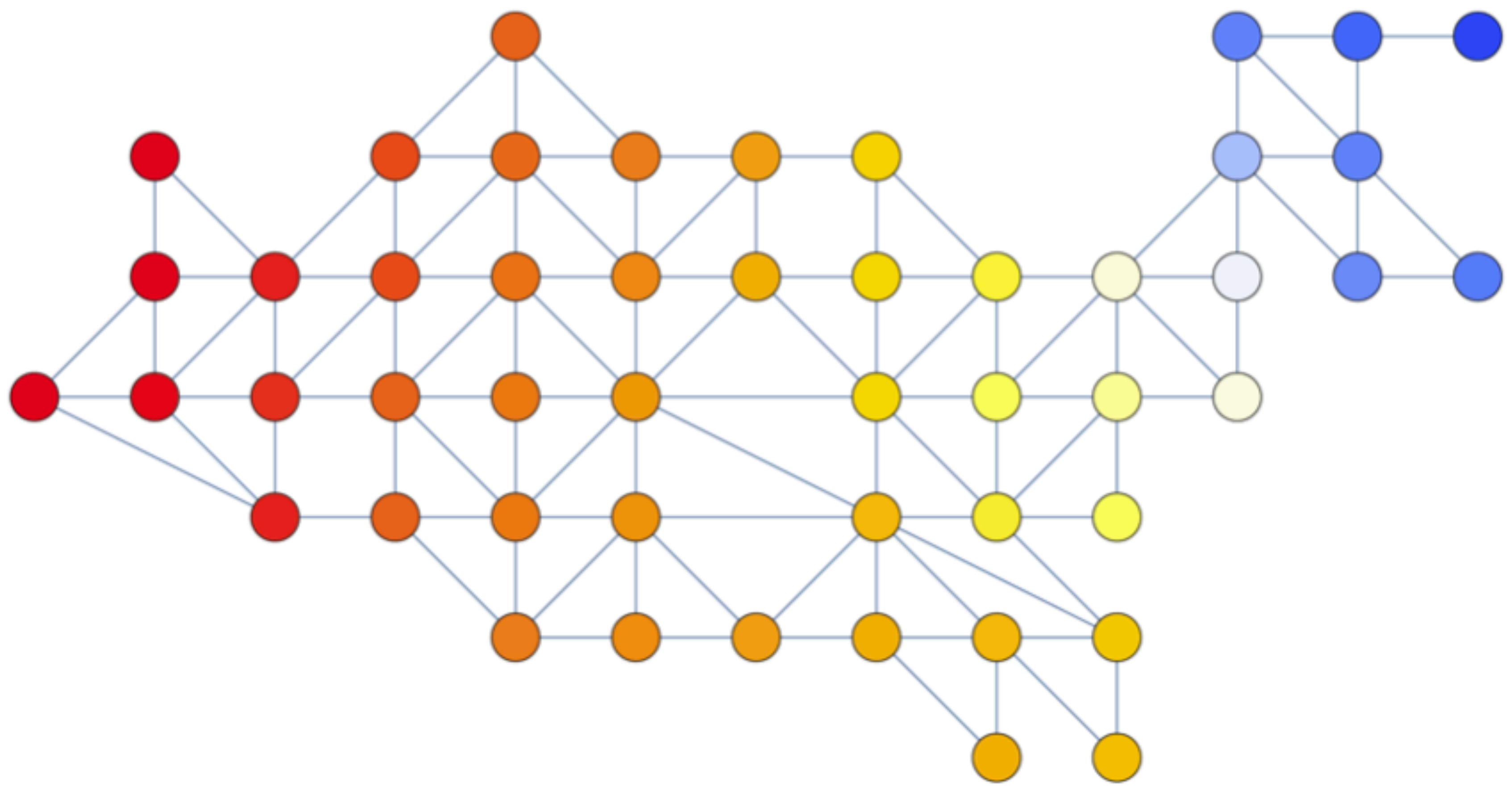
U

$\lambda_1 = 1$	1	1	1	1	1	1	1	1	1	1	1	1
	φ	$-\varphi$	$-\varphi$	φ	-1	-1	1	1	0	0	0	0
$\lambda_2 = -.4472$	-1	1	φ	$-\varphi$	0	φ	$-\varphi$	0	0	-1	1	0
	φ	$-\varphi$	-1	1	0	$-\varphi$	φ	0	-1	0	0	1
	ψ	$-\psi$	$-\psi$	ψ	-1	-1	1	1	0	0	0	0
$\lambda_3 = .4472$	-1	1	ψ	$-\psi$	0	ψ	$-\psi$	0	0	-1	1	0
	ψ	$-\psi$	-1	1	0	$-\psi$	ψ	0	-1	0	0	1
	-1	-1	1	1	0	0	0	0	0	0	0	0
	-1	-1	0	0	0	1	1	0	0	0	0	0
$\lambda_4 = -.2$	-1	-1	0	0	1	0	0	1	0	0	0	0
	-1	-1	0	0	0	0	0	0	0	1	1	0
	-1	-1	0	0	0	0	0	0	1	0	0	1

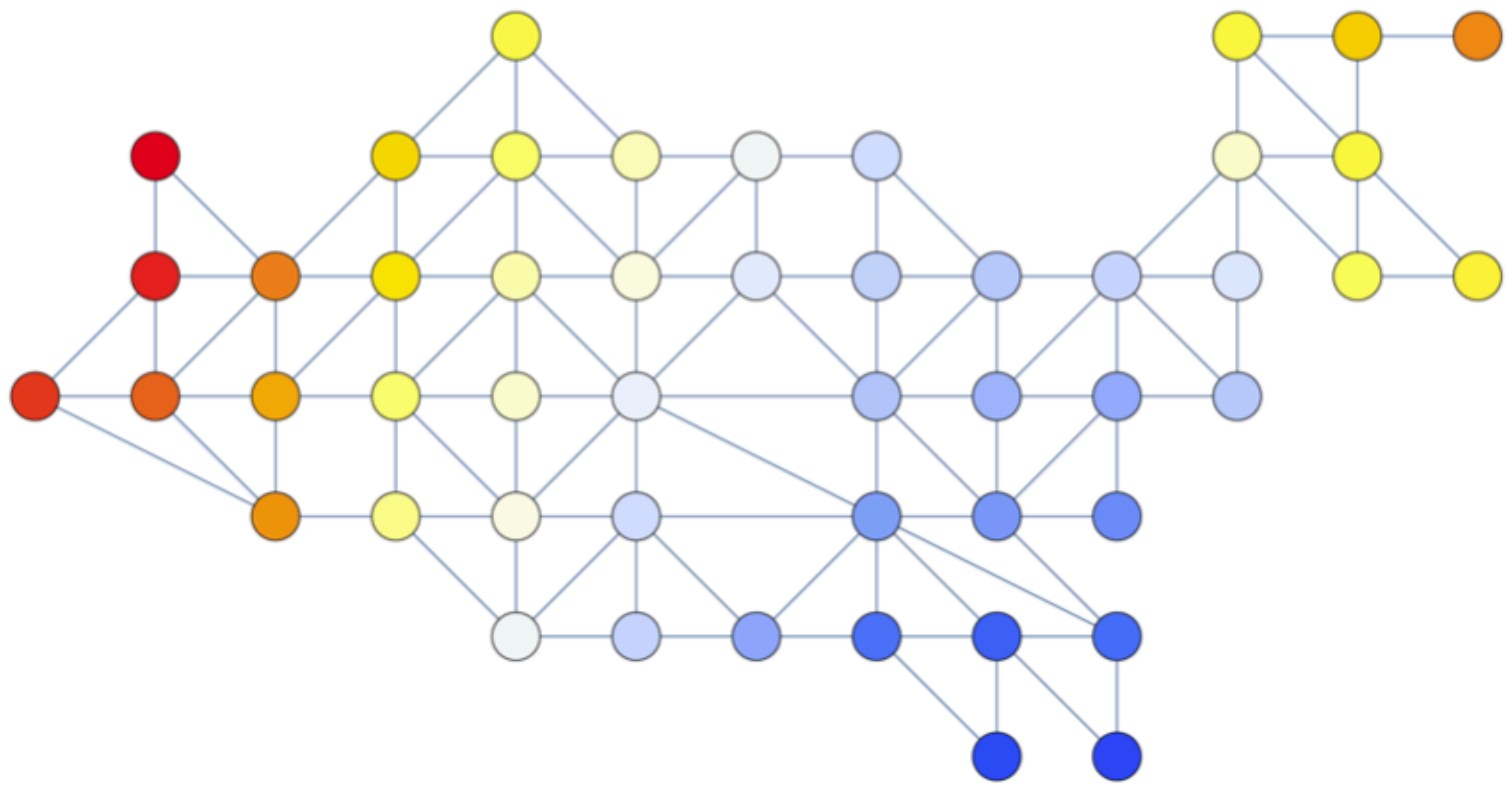


$$f_1 = \mathbf{1} \text{ and } \mathbf{1}^\top f_j = 0 \;\; \forall\, j \geq 2$$

FREQUENCY ORDERING OF EIGENVECTORS

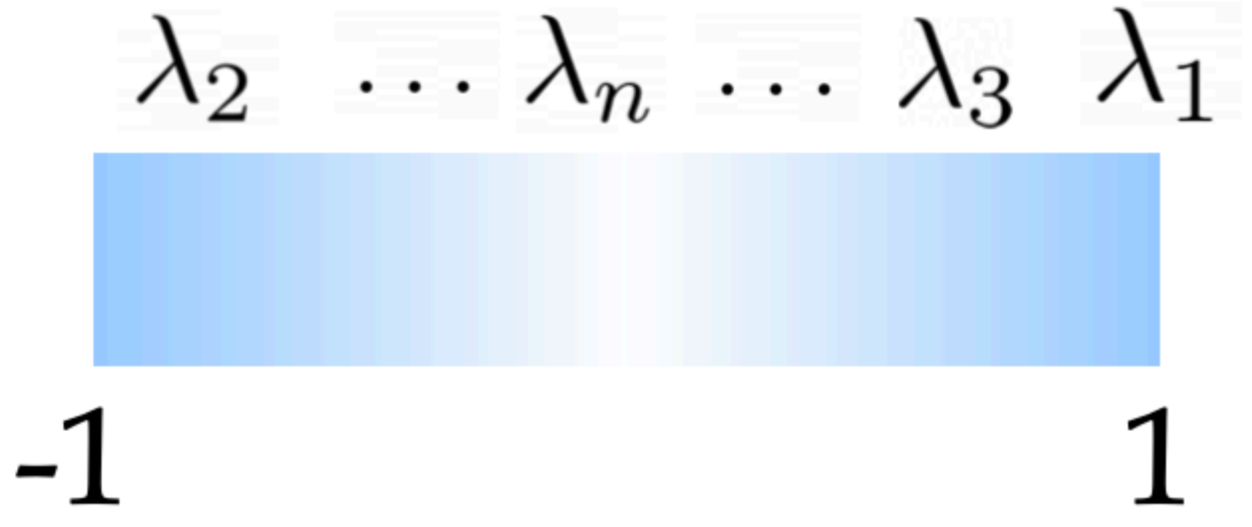


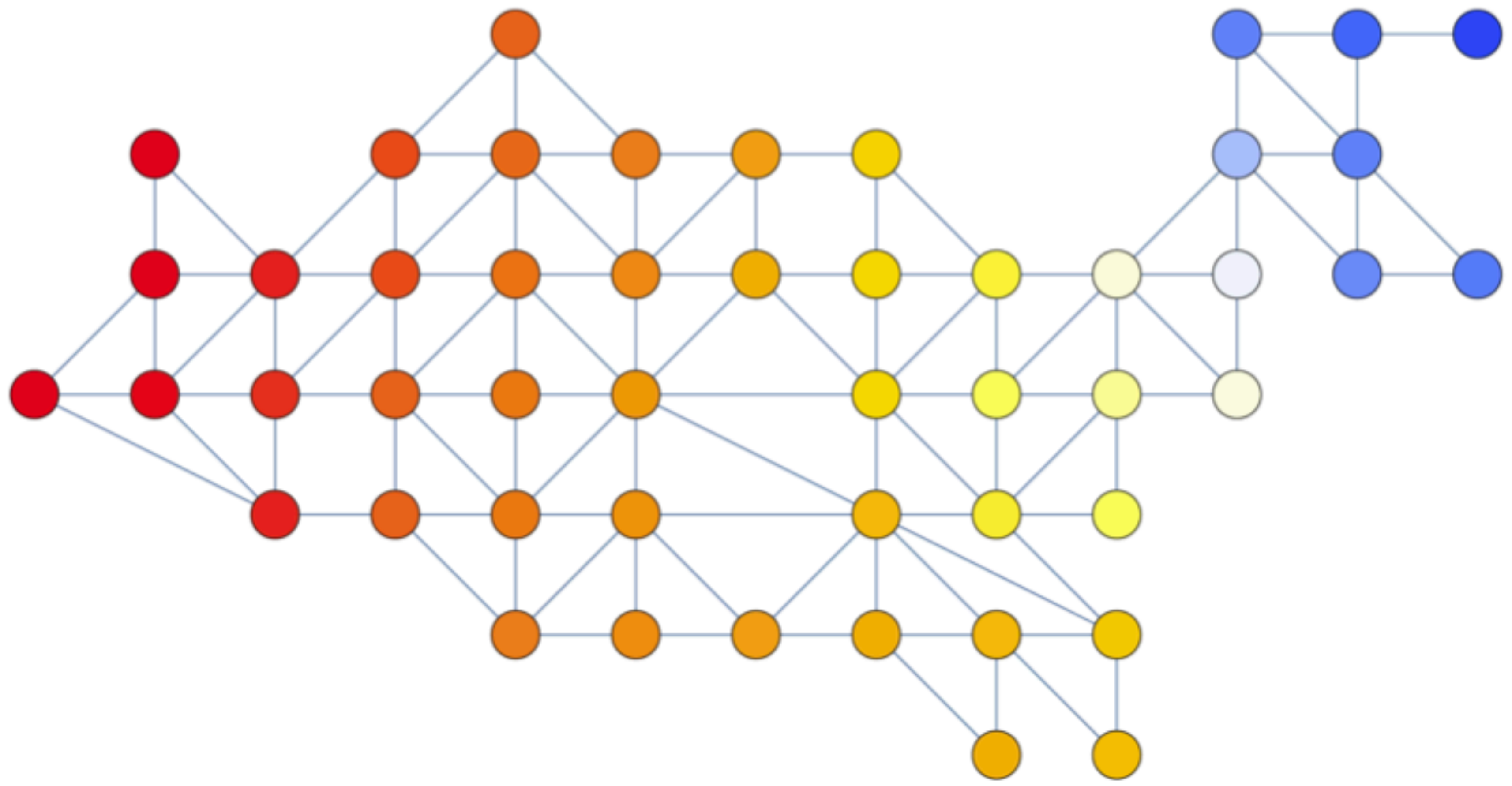
The first eigenvector by frequency



The second eigenvector by frequency

$$1 = |\lambda_1| \geq |\lambda_2| \geq \cdots |\lambda_n| \geq 0$$





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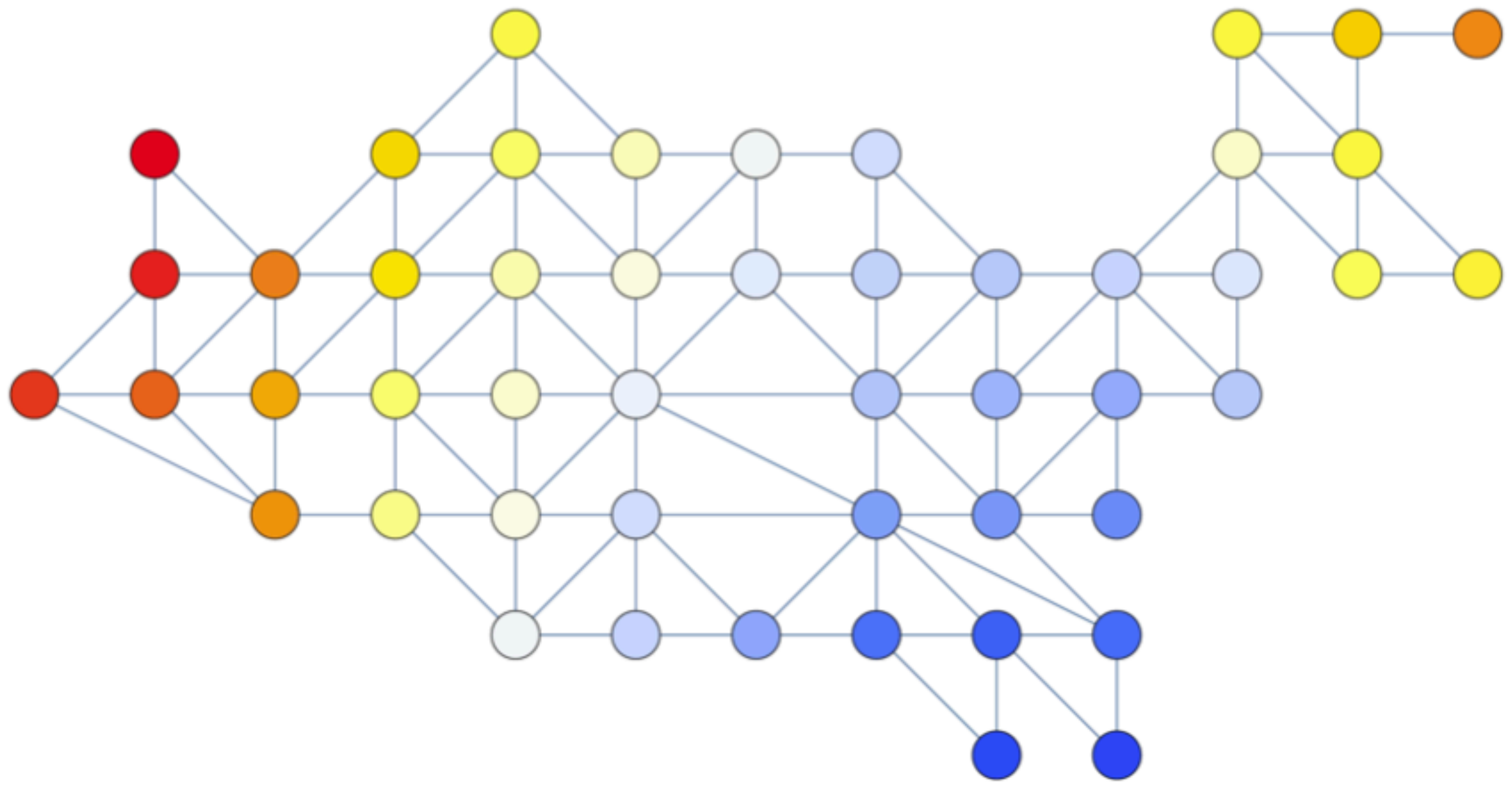
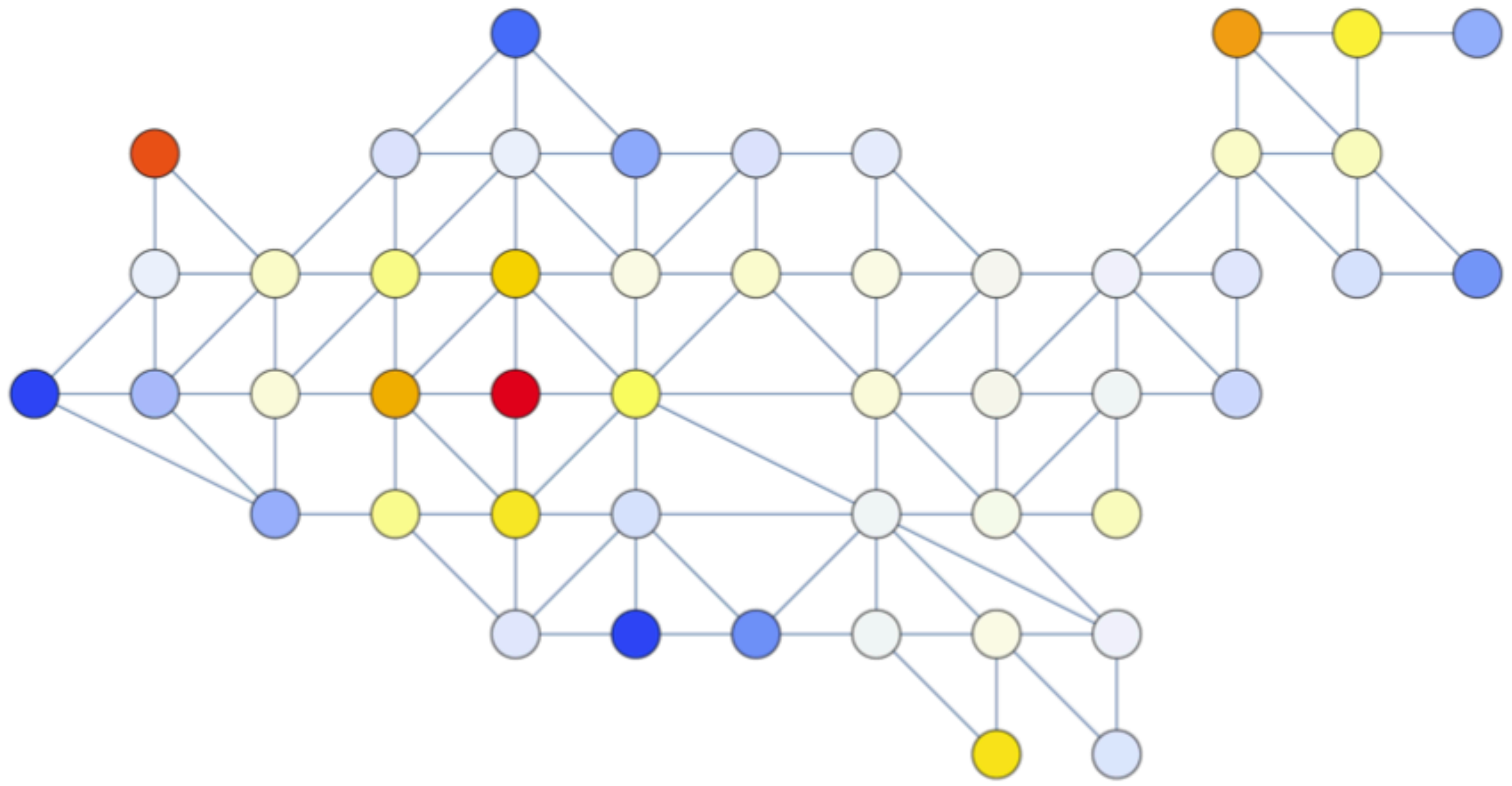
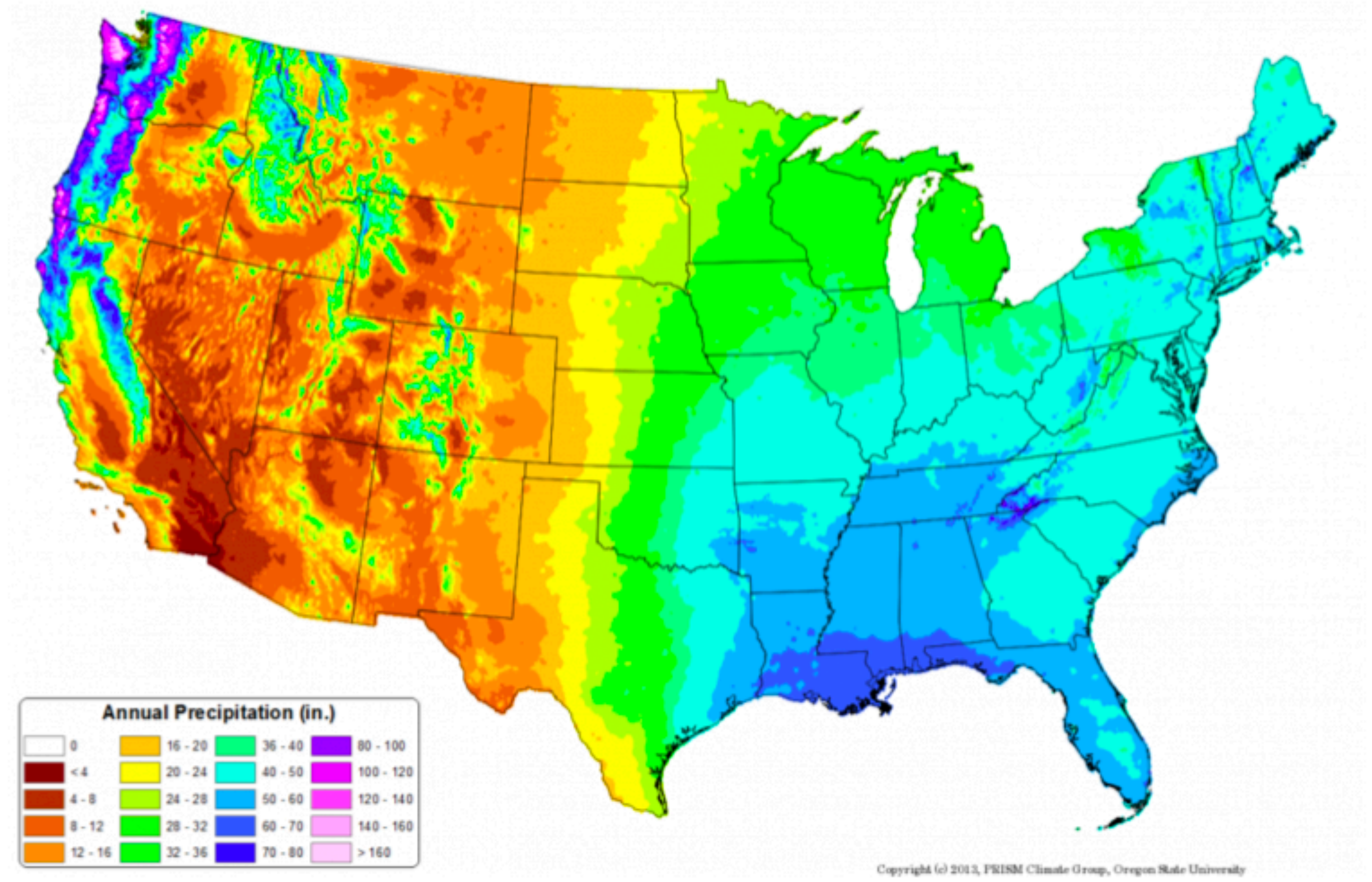


The second eigenvector by frequency



The eleventh eigenvector by frequency



Average annual precipitation, 1981-2010

GRAPHICAL DESIGNS

$$G = (V, E), \text{ order eigenspaces } \Lambda_1 < \Lambda_2 < \cdots < \Lambda_m$$

A k-graphical design in G is a $W \subset V$ and weights $a_w \in \mathbb{R}$ that averages $\Lambda_1, \ldots \Lambda_k$ with these weights.

Stefan Steinerberger (2020)

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Stefan Steinerberger (2020)

$a_w \in \mathbb{R}$	weighted design
$a_w \geq 0$	positively weighted design
$a_w \in \{0, 1\}$	combinatorial design

EXTREMAL DESIGNS

$$G = (V, E), \text{ order eigenspaces } \Lambda_1 < \Lambda_2 < \cdots < \Lambda_m$$

An extremal design is a $(m-1)$ design.

Konstantin Golubev (2020)

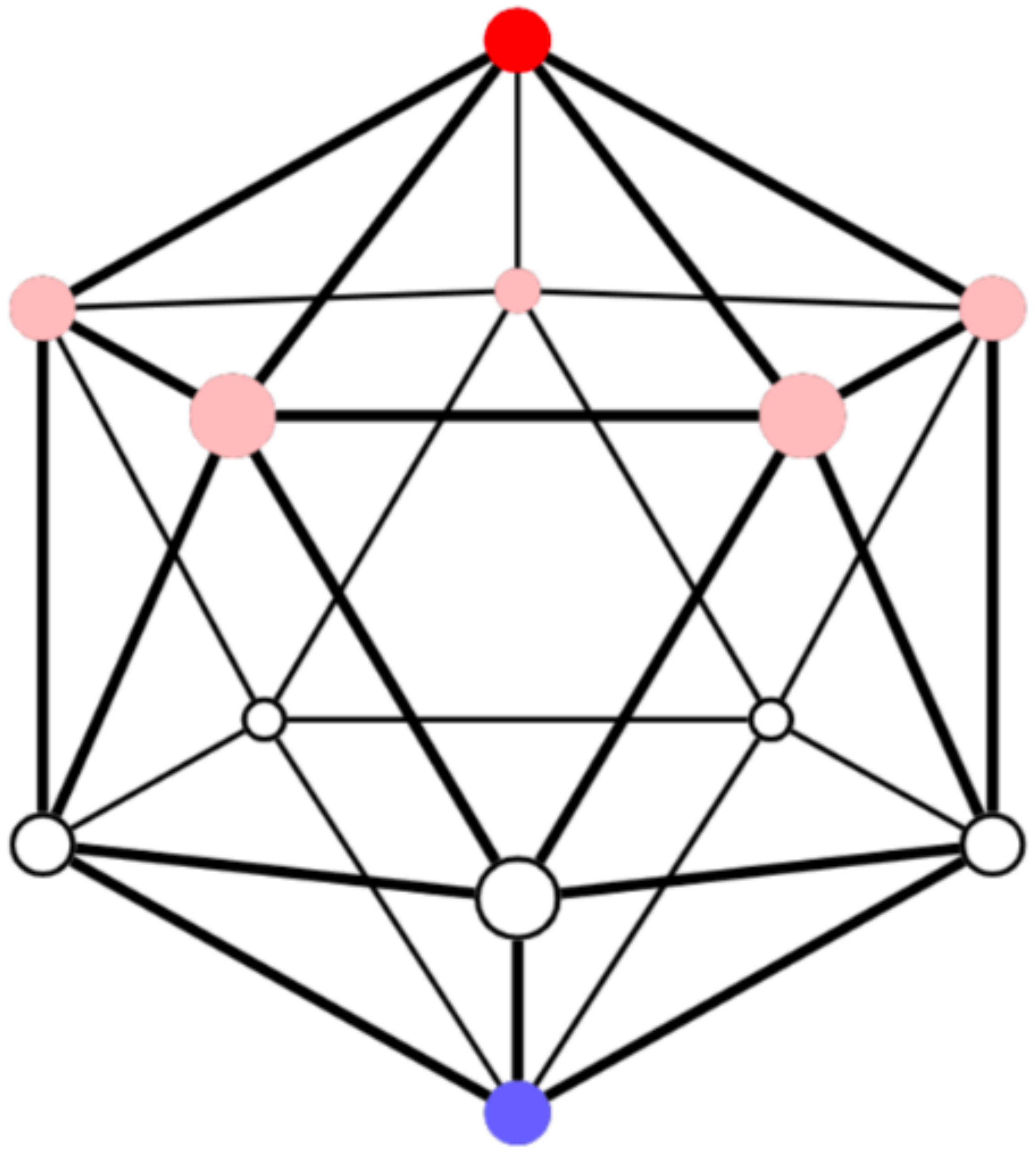
(No proper subset can average all eigenspaces)

QUESTIONS

- Do k-designs always exist in a graph G?
- How does one compute k-designs in G?
- How does one compute smallest k-designs in G?
- Is there a way to organize all k-designs?

Are the answers different for the
different types of designs?

GRAPHICAL DESIGNS



An arbitrarily weighted 3-design

$$a_w \in \mathbb{R}$$

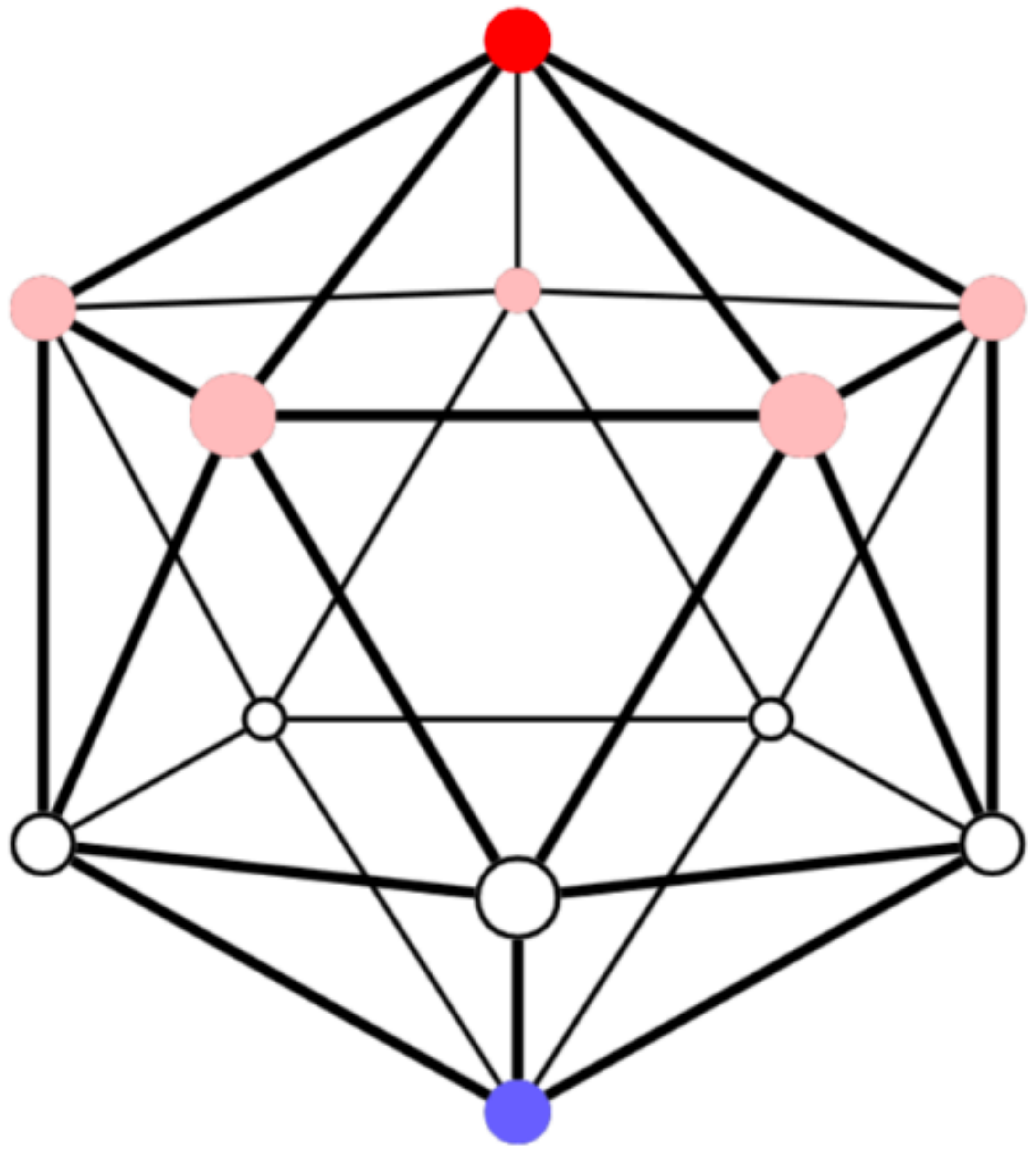
The icosahedral graph with

$$\Lambda_1 < \Lambda_4 < \Lambda_3 < \Lambda_2.$$

GRAPHICAL DESIGNS

The icosahedral graph with

$$\Lambda_1 < \Lambda_4 < \Lambda_3 < \Lambda_2.$$



An arbitrarily weighted 3-design

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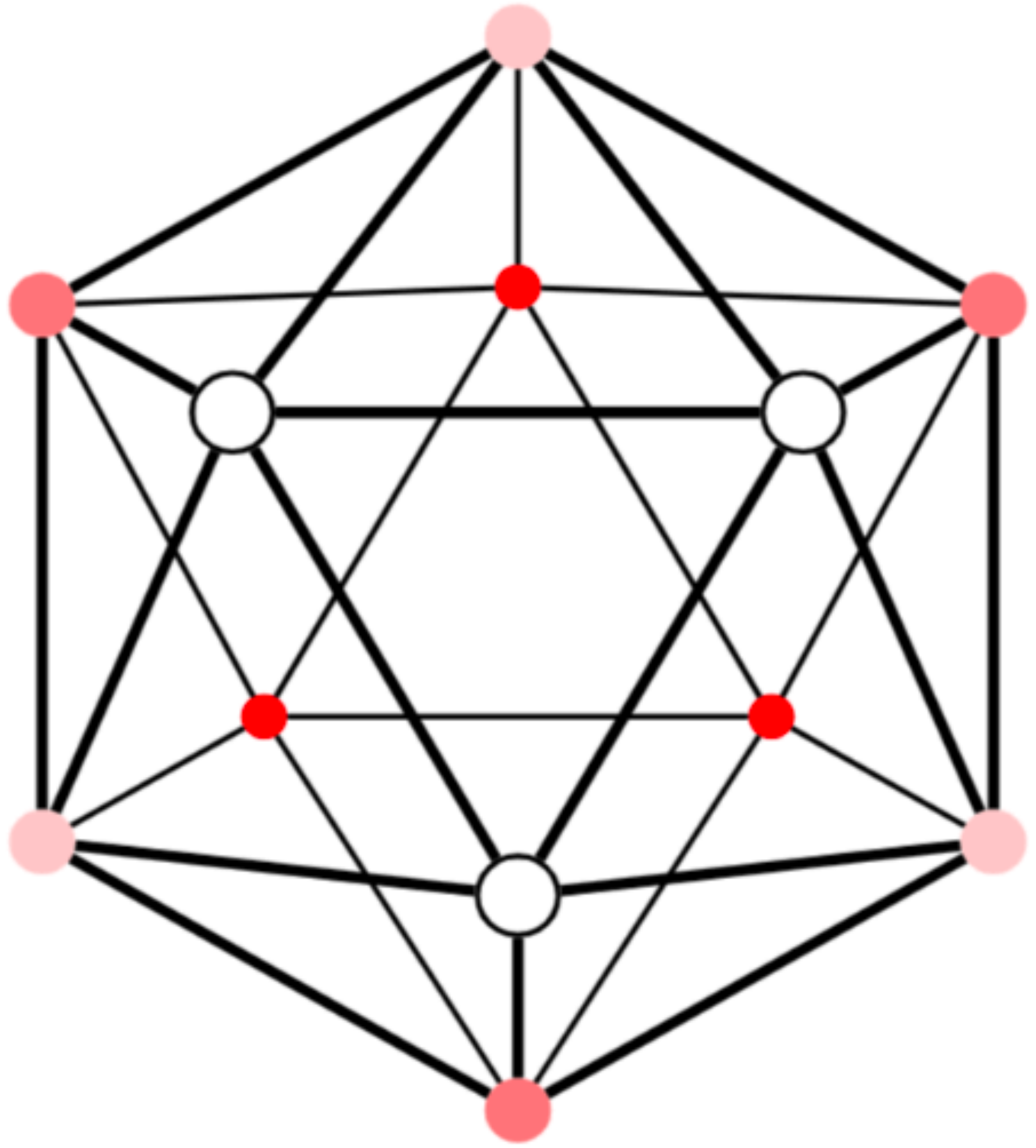


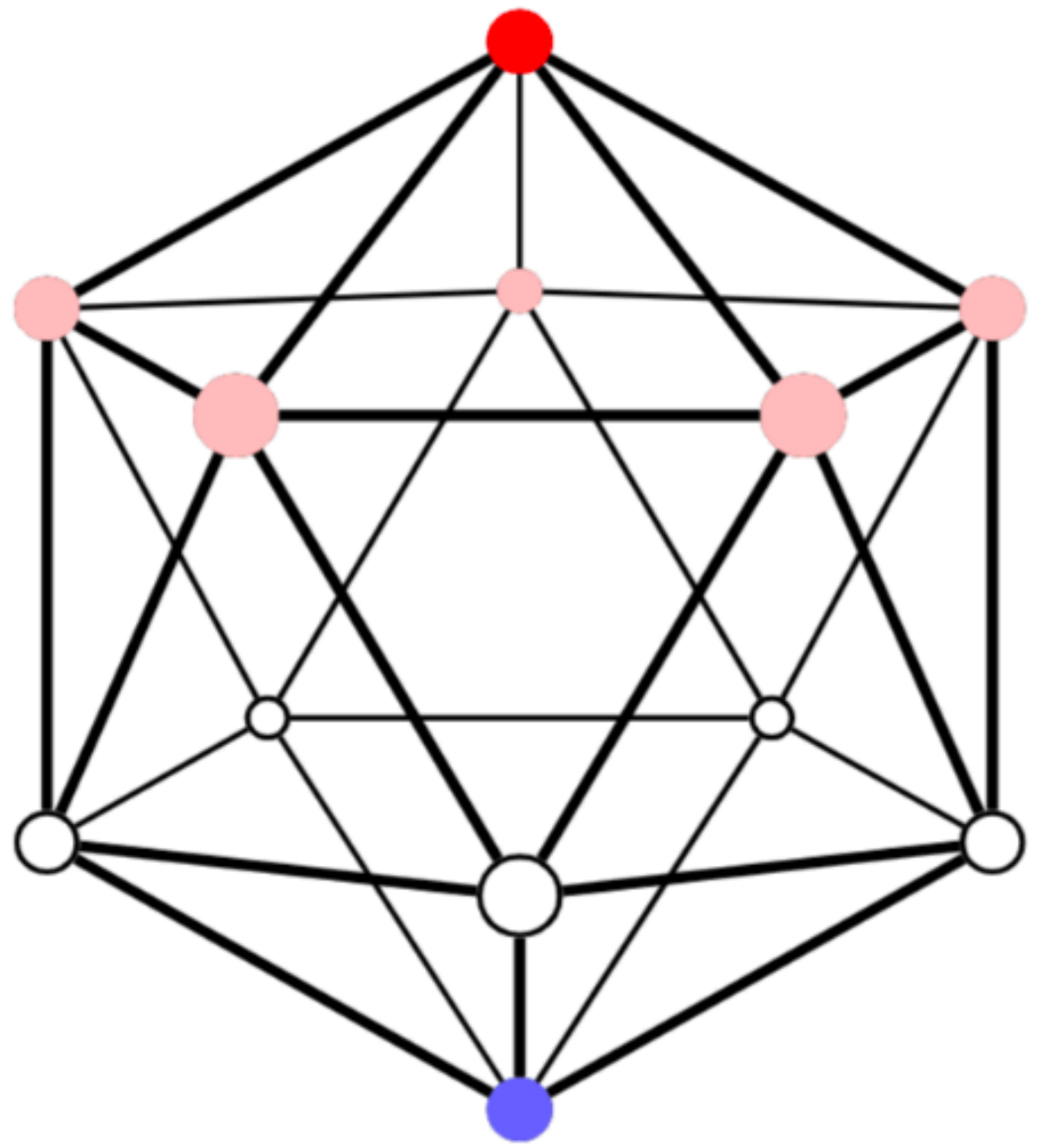
A positively weighted 3-design

$$a_w \geq 0$$

GRAPHICAL DESIGNS

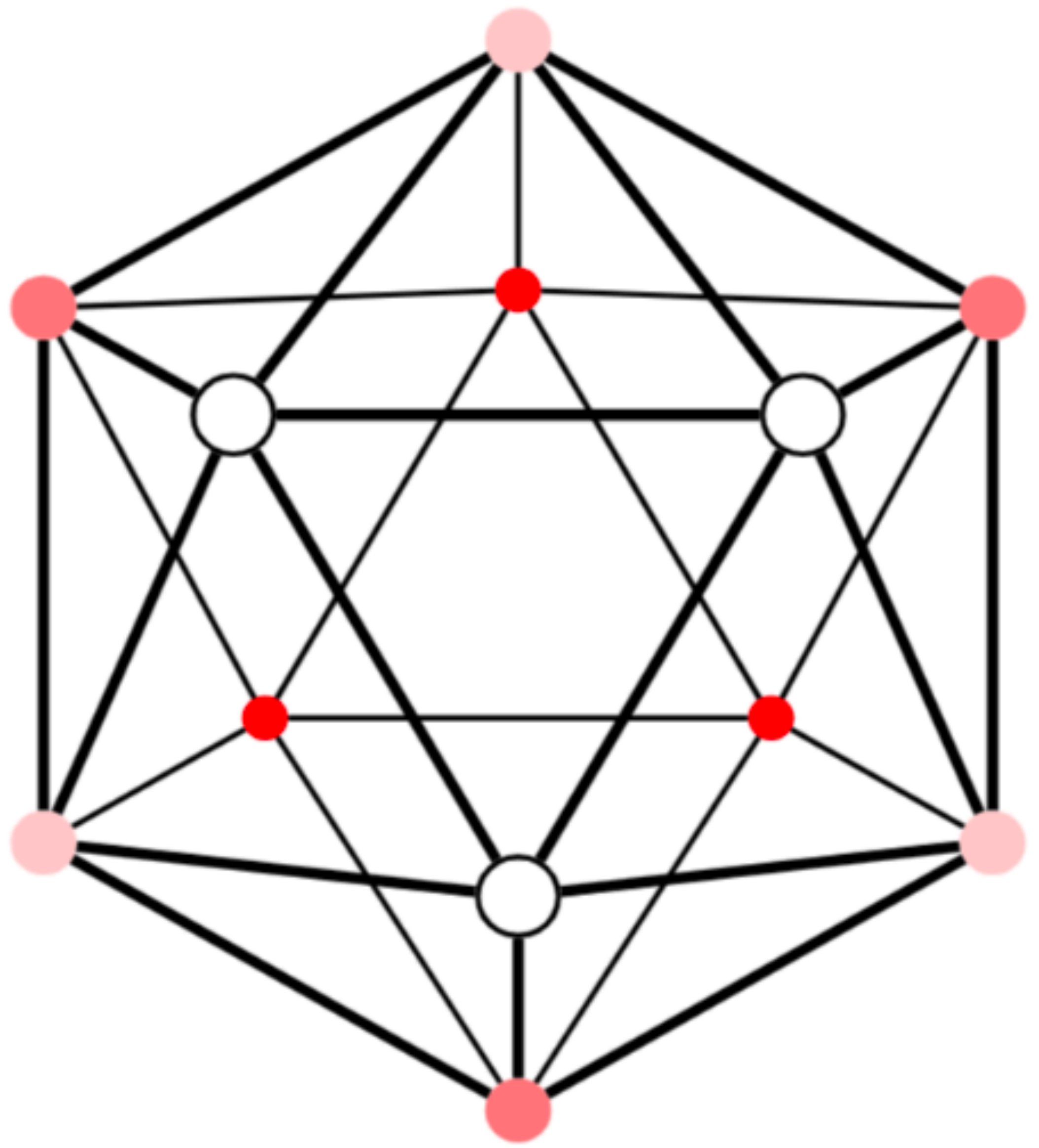
The icosahedral graph with

$$\Lambda_1 < \Lambda_4 < \Lambda_3 < \Lambda_2.$$



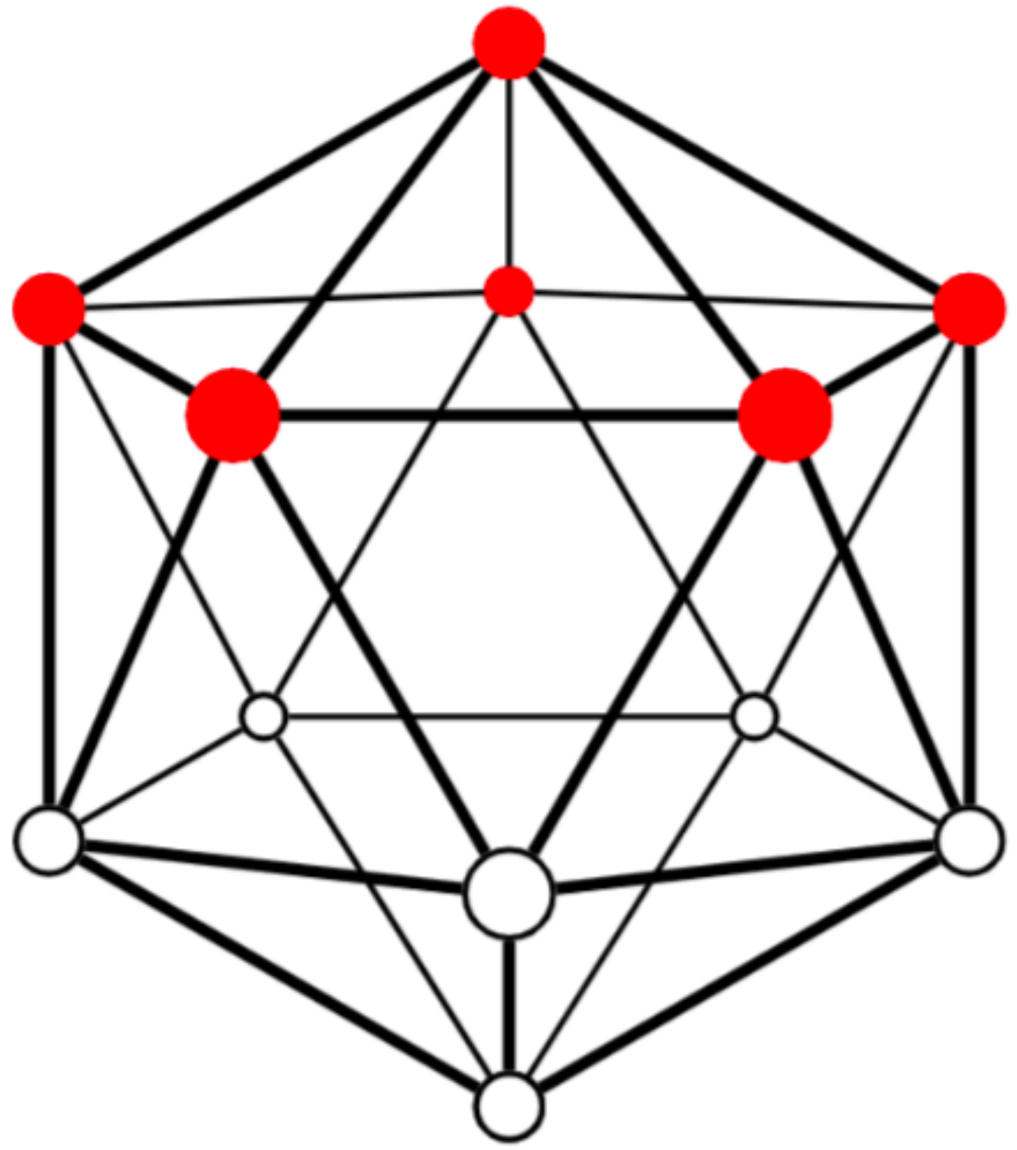
An arbitrarily weighted 3-design

$a_w \in \mathbb{R}$



A positively weighted 3-design

$a_w \geq 0$



A combinatorial 2-design

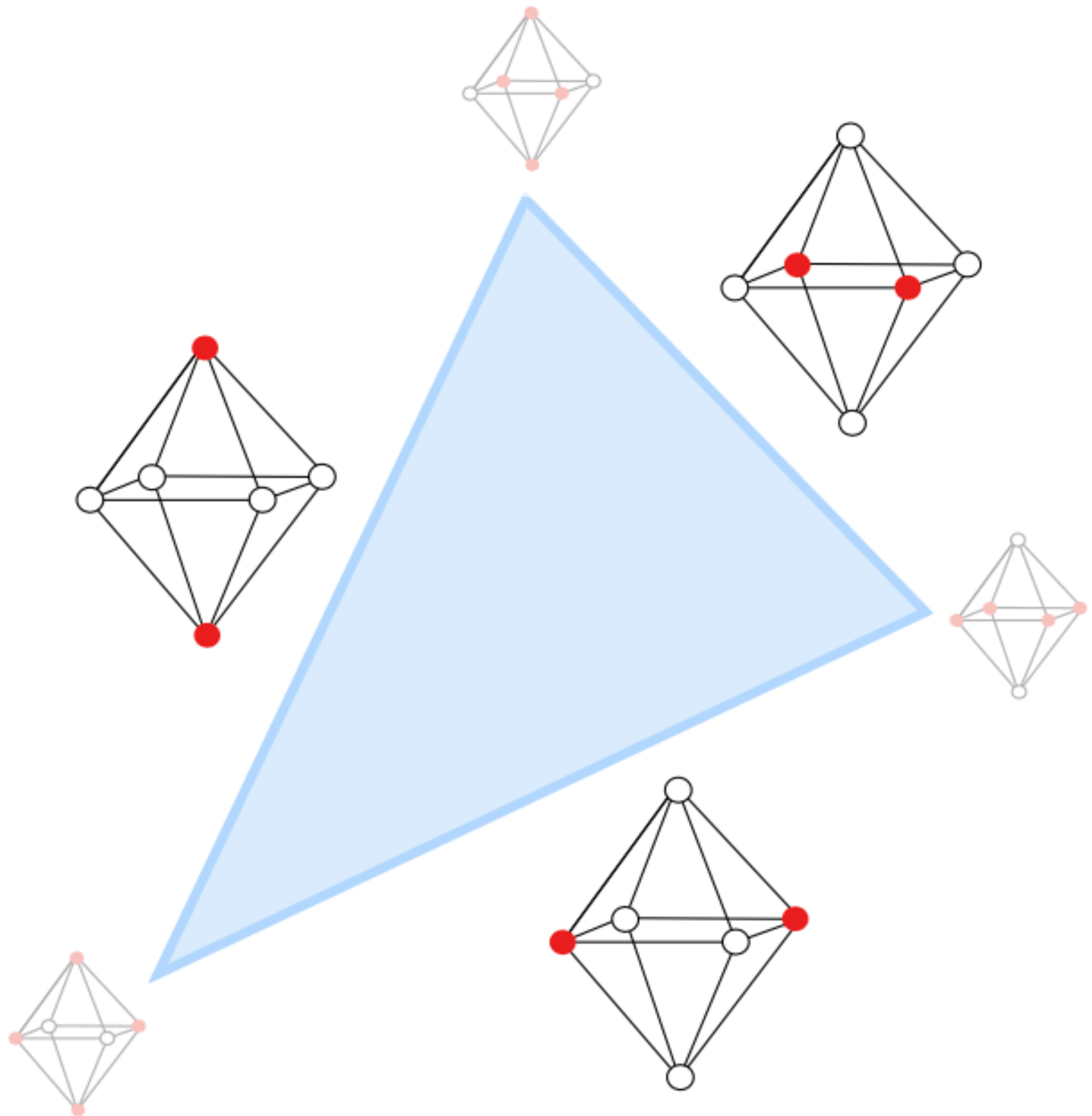
$a_w \in \{0, 1\}$

EXISTENCE OF POSITIVELY WEIGHTED DESIGNS

STRUCTURE THEOREM
(Babecki-T. 2022)

$\left\{ \begin{array}{c} \text{Minimal positively} \\ \text{weighted k-designs} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Facets of} \\ P_{\overline{k}} = \operatorname{conv}(\mathcal{U}_{\overline{k}}). \end{array} \right\}$

$W \subset V$ k-design $\longleftrightarrow$ $V \setminus W$ facet



EIGENPOLYTOPES

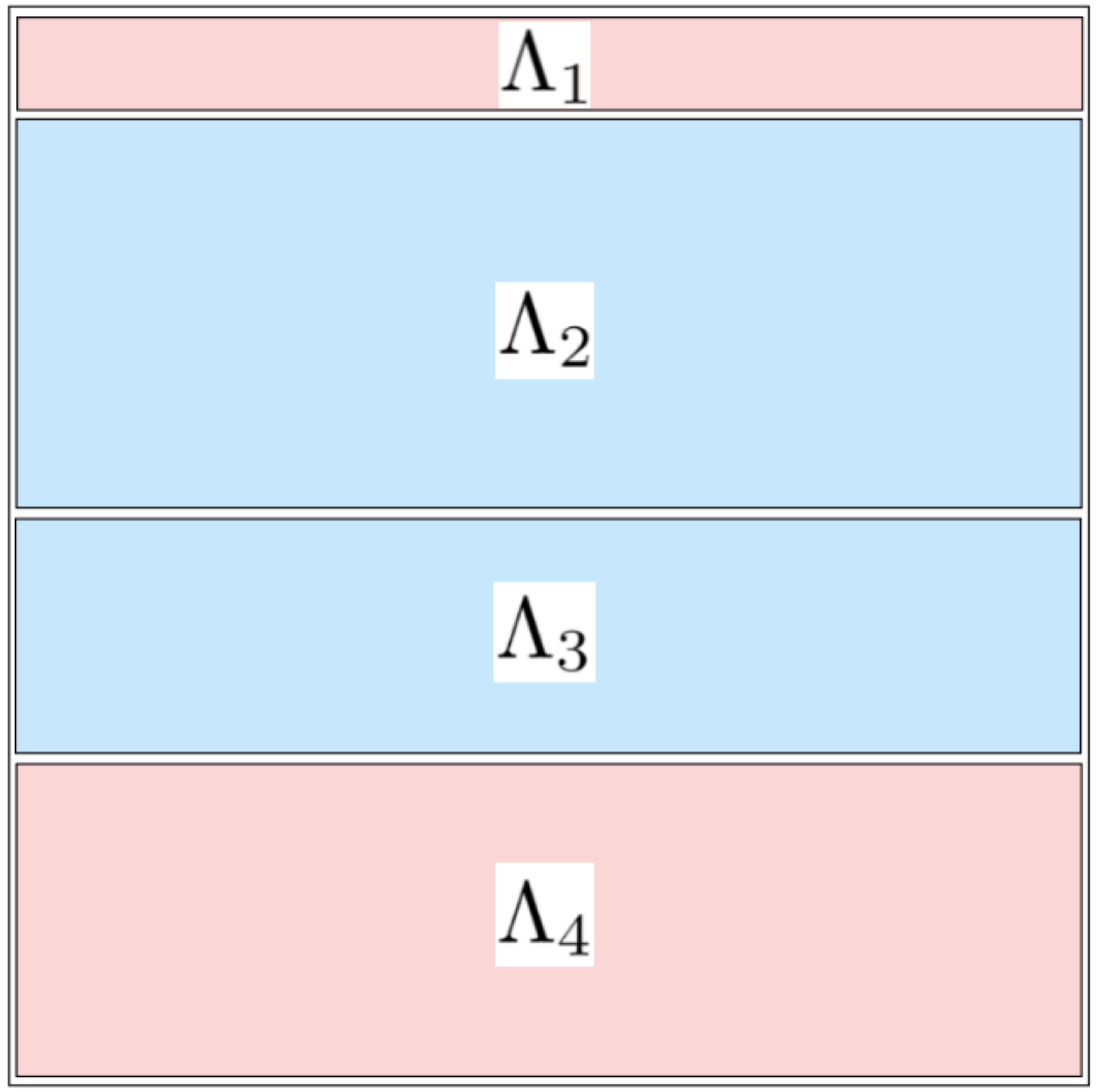
$$\mathbf{k} = \{\lambda_2, \ldots, \lambda_k\}$$

$$\bar{\mathbf{k}} = \{\lambda_1\} \cup \{\lambda_{k+1}, \ldots, \lambda_m\}$$

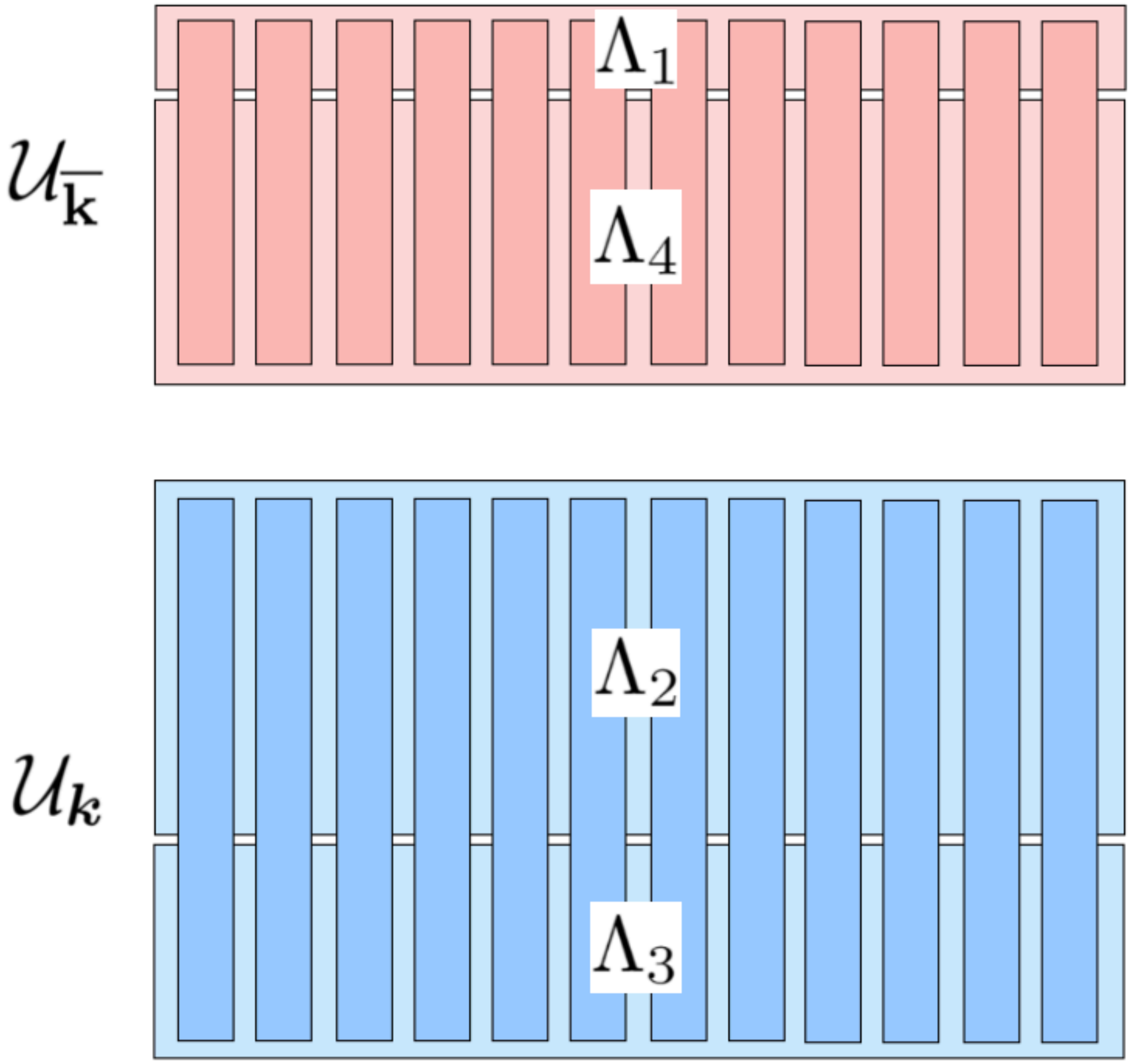
(Godsil 1978)

$$P_{\bar{\mathbf{k}}} = \mathrm{conv}(\mathcal{U}_{\bar{\mathbf{k}}})$$

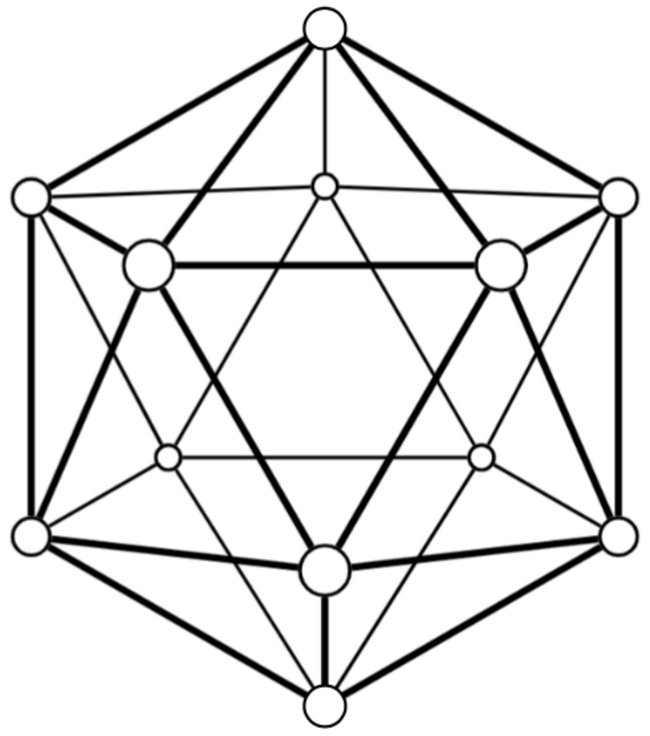
$U =$



m = 4, k = 3



3-DESIGNS IN THE ICOSAHEDRAL GRAPH

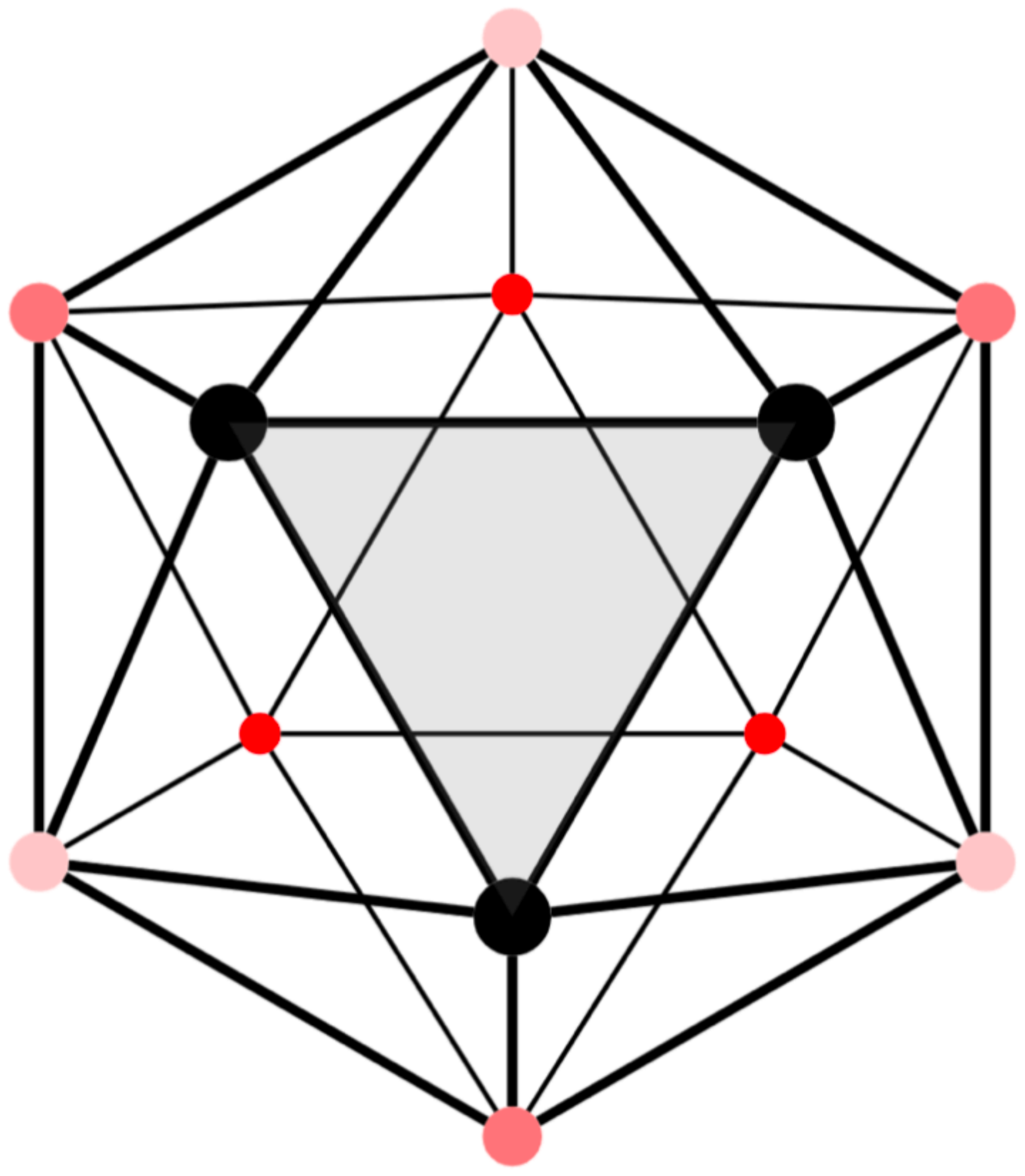


$\bar{k}$ k $\bar{k}$

$$\Lambda_1 < \Lambda_4 < \Lambda_3 < \Lambda_2.$$

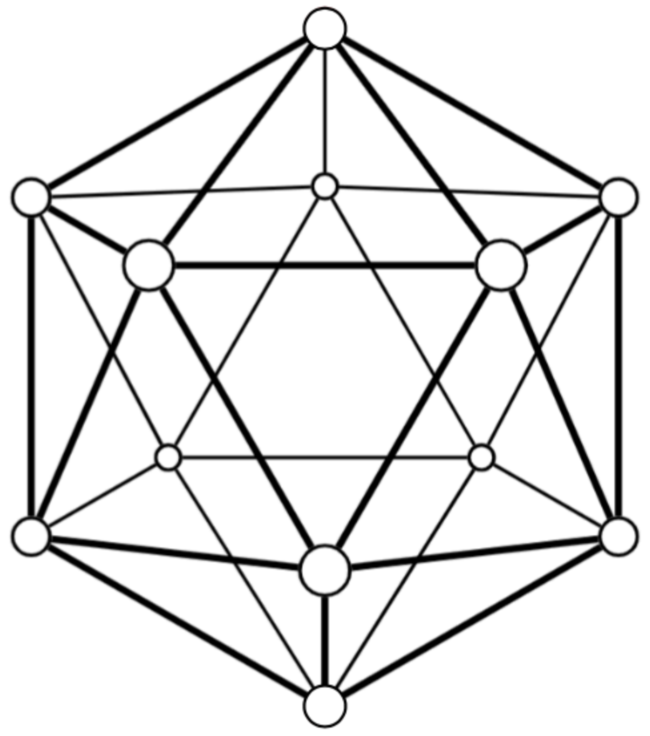
$\lambda_1 = 1$	1	1	1	1	1	1	1	1	1	1	1	1
$\lambda_2 = -.4472$	φ	$-\varphi$	$-\varphi$	φ	-1	-1	1	1	0	0	0	0
	-1	1	φ	$-\varphi$	0	φ	$-\varphi$	0	0	-1	1	0
	φ	$-\varphi$	-1	1	0	$-\varphi$	φ	0	-1	0	0	1
$\lambda_3 = .4472$	ψ	$-\psi$	$-\psi$	ψ	-1	-1	1	1	0	0	0	0
	-1	1	ψ	$-\psi$	0	ψ	$-\psi$	0	0	-1	1	0
	ψ	$-\psi$	-1	1	0	$-\psi$	ψ	0	-1	0	0	1
$\lambda_4 = -.2$	-1	-1	1	1	0	0	0	0	0	0	0	0
	-1	-1	0	0	0	1	1	0	0	0	0	0
	-1	-1	0	0	1	0	0	1	0	0	0	0
	-1	-1	0	0	0	0	0	0	0	1	1	0
	-1	-1	0	0	0	0	0	0	1	0	0	1

3-DESIGNS IN THE ICOSAHEDRAL GRAPH



$P_{\bar{3}}$ is an icosahedron!

$\Rightarrow$ minimal 3-designs have size 9



$$\underbrace{\Lambda_1}_{\bar{k}} < \underbrace{\Lambda_4 < \Lambda_3}_{k} < \underbrace{\Lambda_2}_{\bar{k}}.$$

$\lambda_1 = 1$	1	1	1	1	1	1	1	1	1	1	1	1
$\lambda_2 = -.4472$	φ	$-\varphi$	$-\varphi$	φ	-1	-1	1	1	0	0	0	0
	-1	1	φ	$-\varphi$	0	φ	$-\varphi$	0	0	-1	1	0
	φ	$-\varphi$	-1	1	0	$-\varphi$	φ	0	-1	0	0	1
$\lambda_3 = .4472$	ψ	$-\psi$	$-\psi$	ψ	-1	-1	1	1	0	0	0	0
	-1	1	ψ	$-\psi$	0	ψ	$-\psi$	0	0	-1	1	0
	ψ	$-\psi$	-1	1	0	$-\psi$	ψ	0	-1	0	0	1
$\lambda_4 = -.2$	-1	-1	1	1	0	0	0	0	0	0	0	0
	-1	-1	0	0	0	1	1	0	0	0	0	0
	-1	-1	0	0	1	0	0	1	0	0	0	0
	-1	-1	0	0	0	0	0	0	0	1	1	0
	-1	-1	0	0	0	0	0	0	1	0	0	1

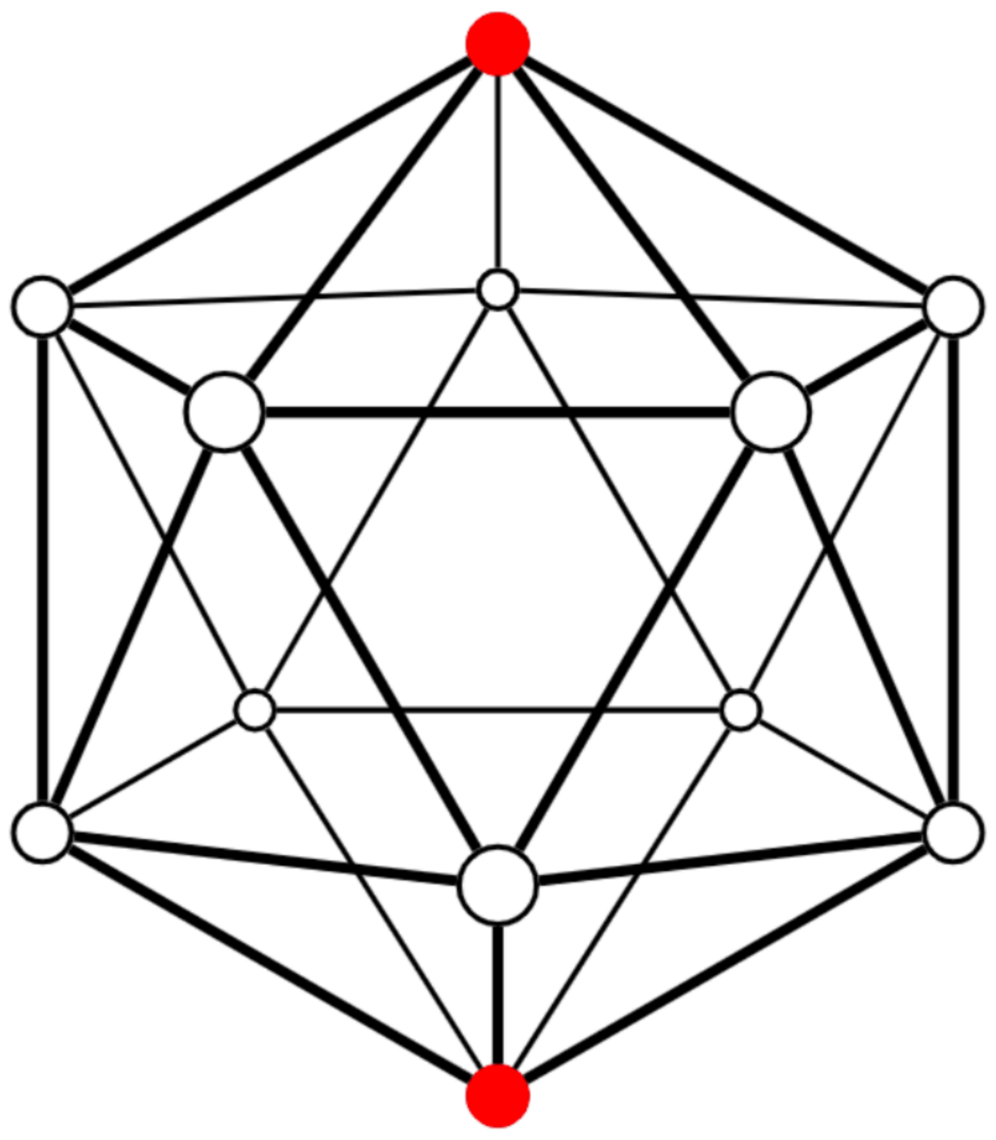
ICOSAHEDRAL GRAPH

$\bar{k}$ $\quad k \quad$ $\bar{k}$

$\Lambda_1 < \Lambda_2 \leq \Lambda_3 < \Lambda_4$

frequency order

$\lambda_1 = 1$	1	1	1	1	1	1	1	1	1	1	1	1
$\lambda_2 = -.4472$	φ	$-\varphi$	$-\varphi$	φ	-1	-1	1	1	0	0	0	0
	-1	1	φ	$-\varphi$	0	φ	$-\varphi$	0	0	-1	1	0
	φ	$-\varphi$	-1	1	0	$-\varphi$	φ	0	-1	0	0	1
$\lambda_3 = .4472$	ψ	$-\psi$	$-\psi$	ψ	-1	-1	1	1	0	0	0	0
	-1	1	ψ	$-\psi$	0	ψ	$-\psi$	0	0	-1	1	0
	ψ	$-\psi$	-1	1	0	$-\psi$	ψ	0	-1	0	0	1
$\lambda_4 = -.2$	-1	-1	1	1	0	0	0	0	0	0	0	0
	-1	-1	0	0	0	1	1	0	0	0	0	0
	-1	-1	0	0	1	0	0	1	0	0	0	0
	-1	-1	0	0	0	0	0	0	0	1	1	0
	-1	-1	0	0	0	0	0	0	1	0	0	1



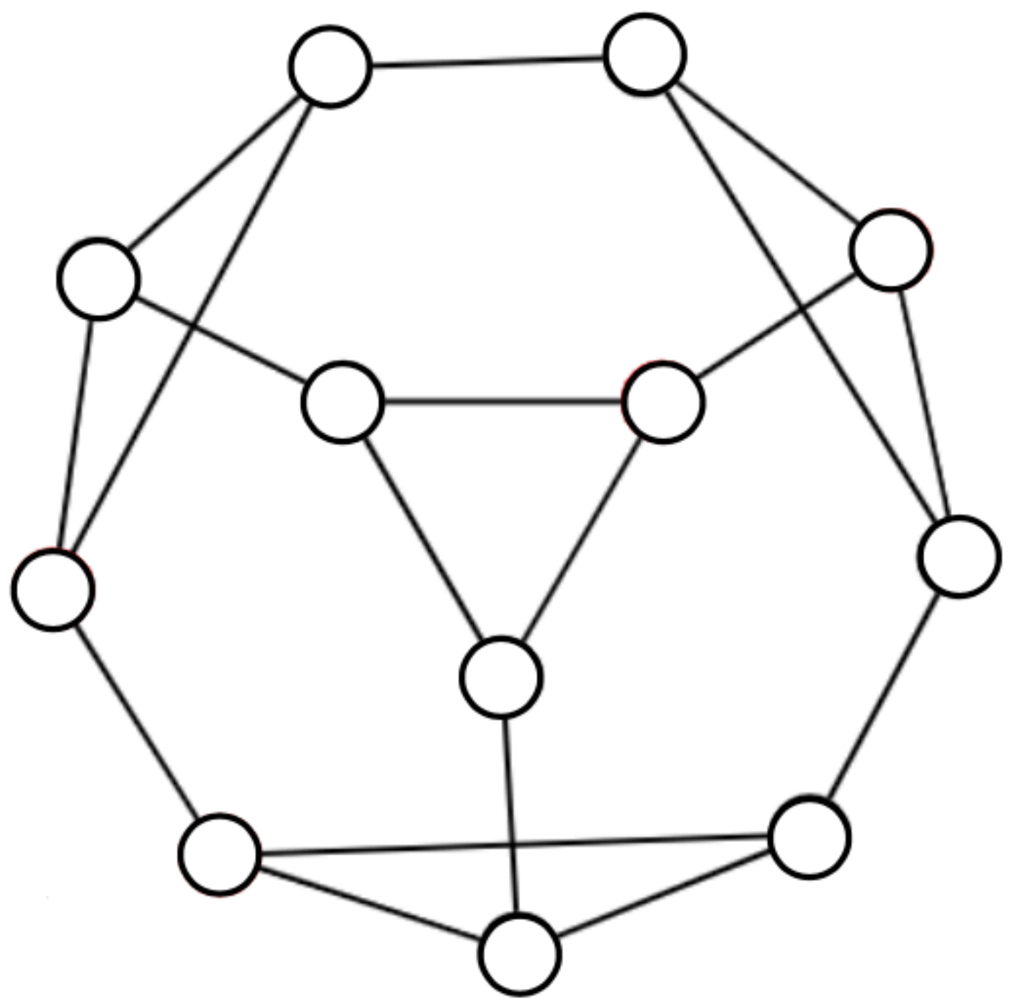
$P_{\bar{3}}$ is a 5-simplex with two elements of $\mathcal{U}_{\bar{3}}$ at each vertex

$\Rightarrow$ each facet has 5+5 elements of $\mathcal{U}_{\bar{3}}$

$\Rightarrow$ minimal 3-designs have size 2

TRUNCATED TETRAHEDRAL GRAPH

minimal 4-graphical designs



$$U = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline -1 & -1.5 & -0.5 & 1.5 & 2 & 1.5 & -0.5 & -1.5 & -1 & 1 & 0 & 0 \\ 2 & 1.5 & 1.5 & -0.5 & -1 & -1.5 & -1.5 & -0.5 & -1 & 0 & 1 & 0 \\ -1 & -0.5 & -1.5 & -1.5 & -1 & -0.5 & 1.5 & 1.5 & 2 & 0 & 0 & 1 \\ \hline 0 & -1 & 1 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & -1 & 0 & 1 \\ \hline -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ \hline 1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & -1 & 0 & 1 \end{bmatrix}$$

eigenvalue

1

2/3

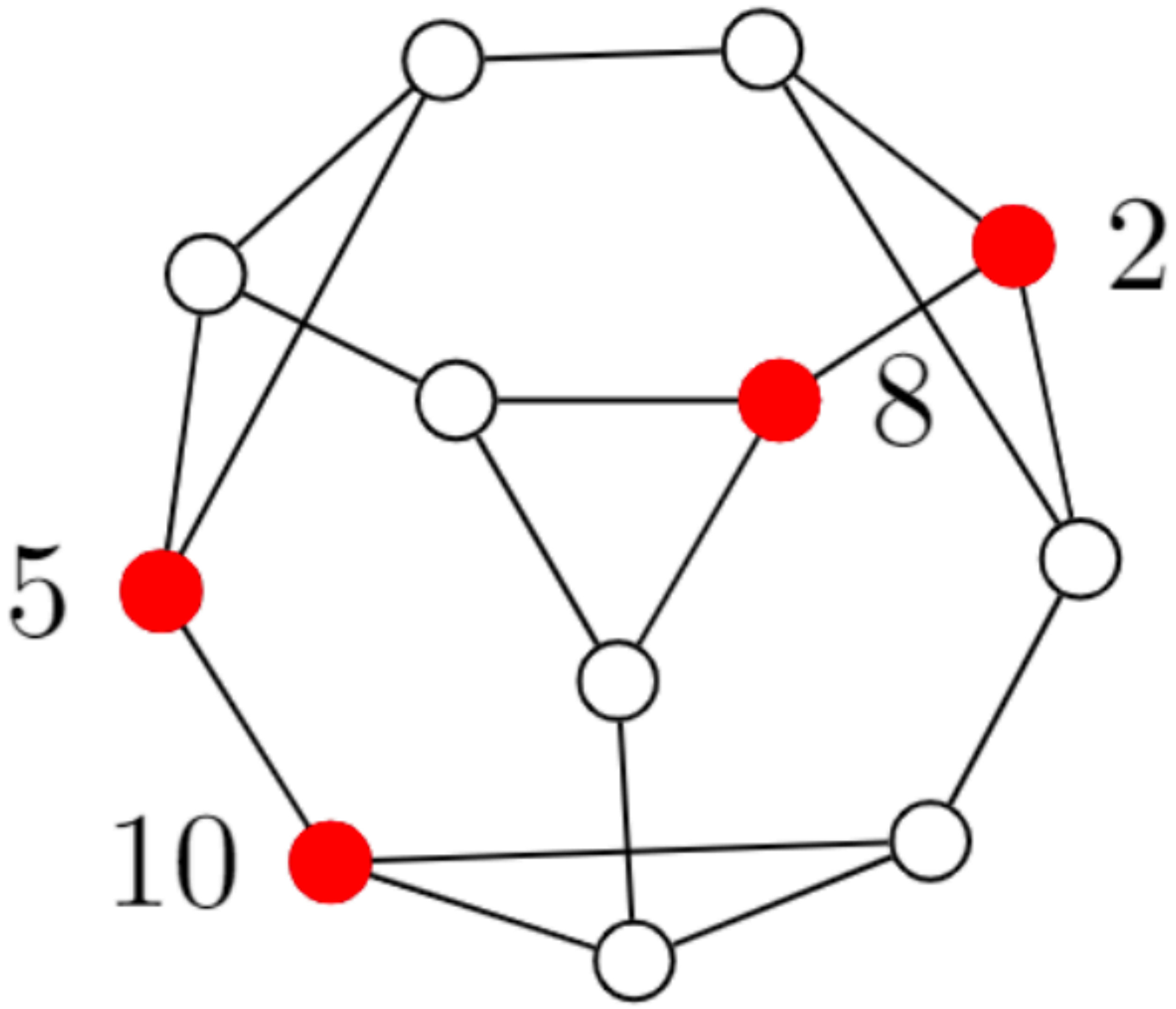
-2/3

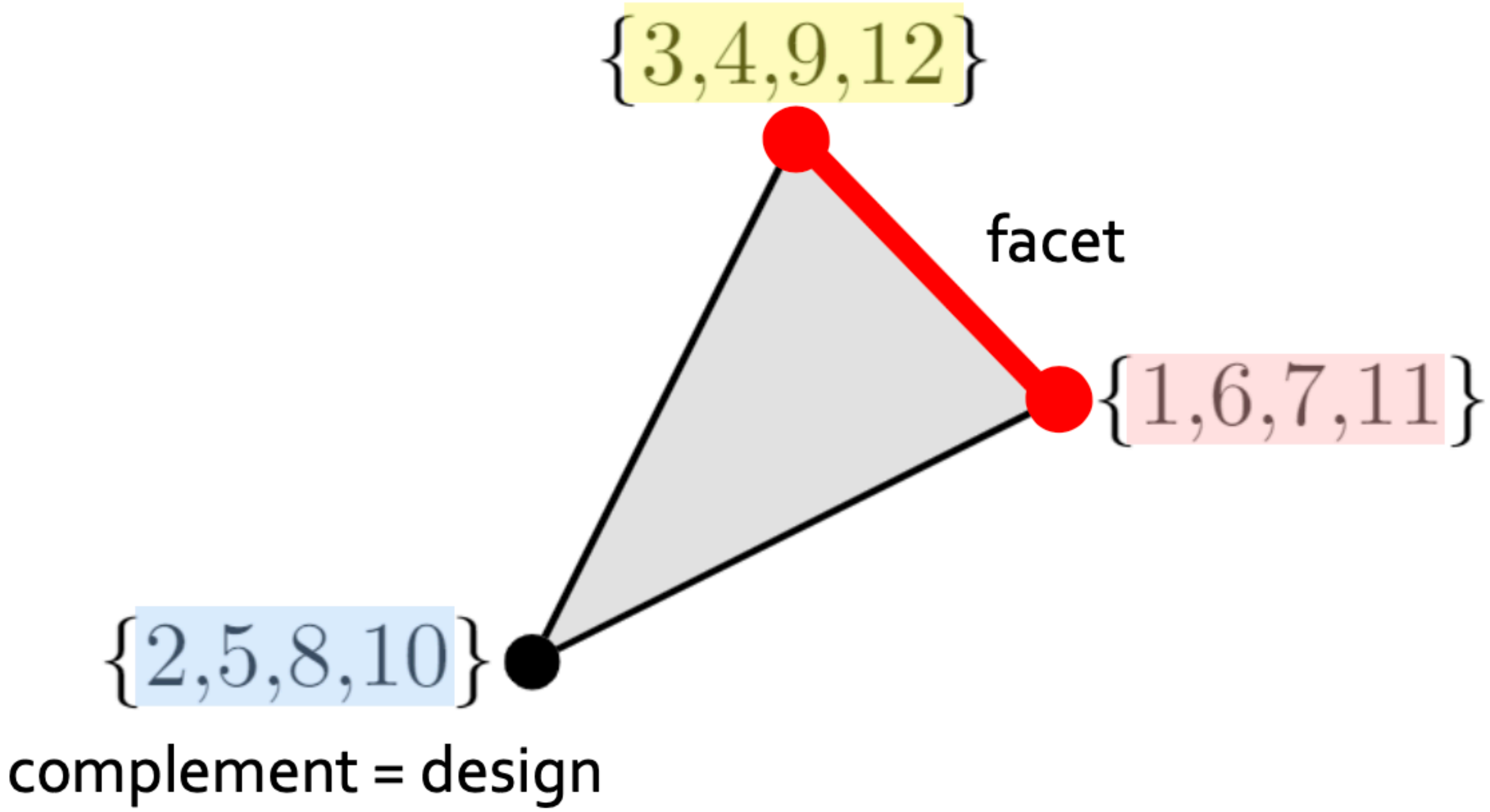
-1/3

0

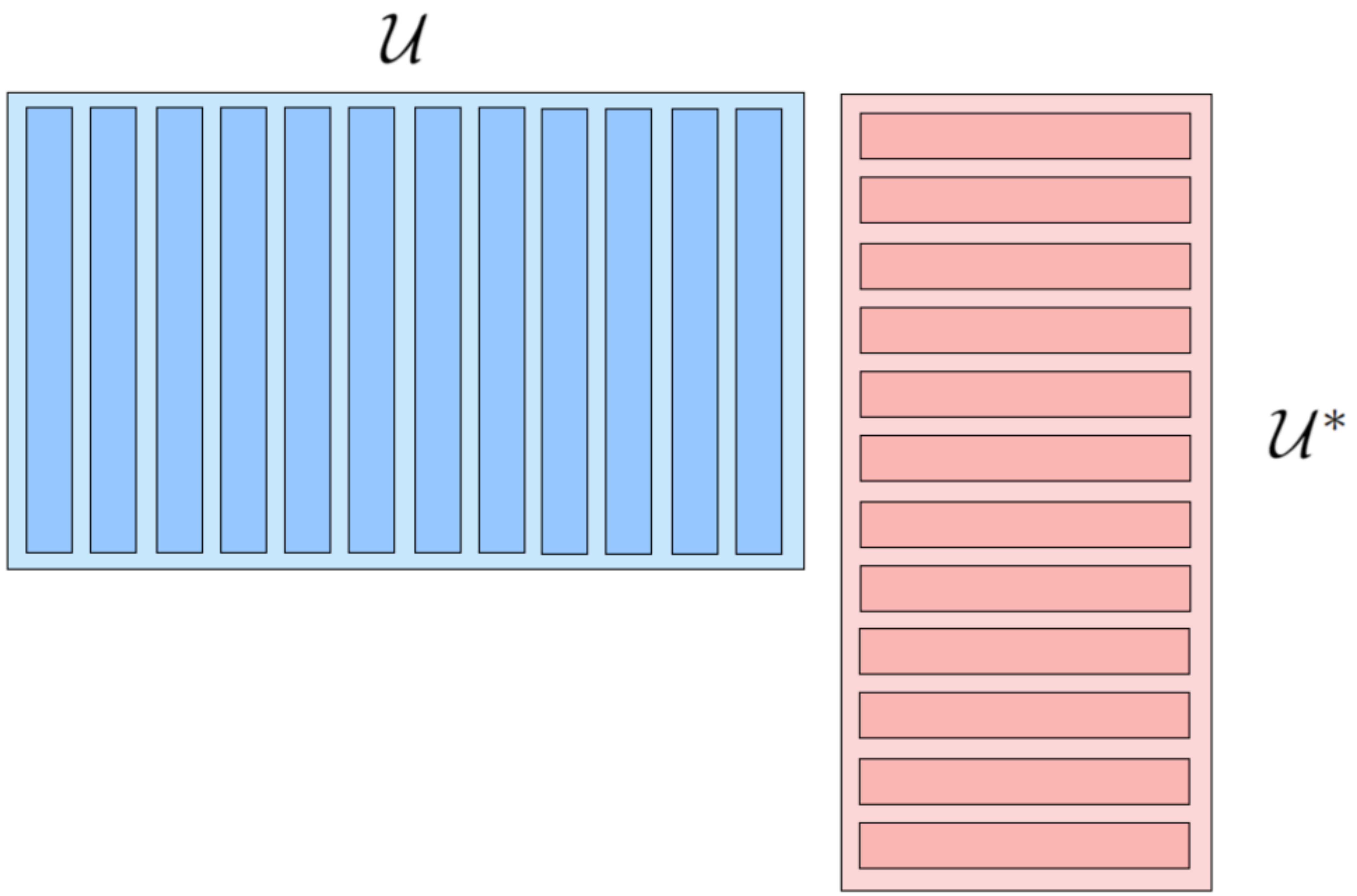
TRUNCATED TETRAHEDRAL GRAPH

$$U_{\bar{\mathbf{4}}} = \left[\begin{array}{cccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right].$$





WHY IT WORKS — ORIENTED MATROID DUALITY



$$= 0$$

$(\mathcal{U}, \mathcal{U}^*)$ are dual configurations

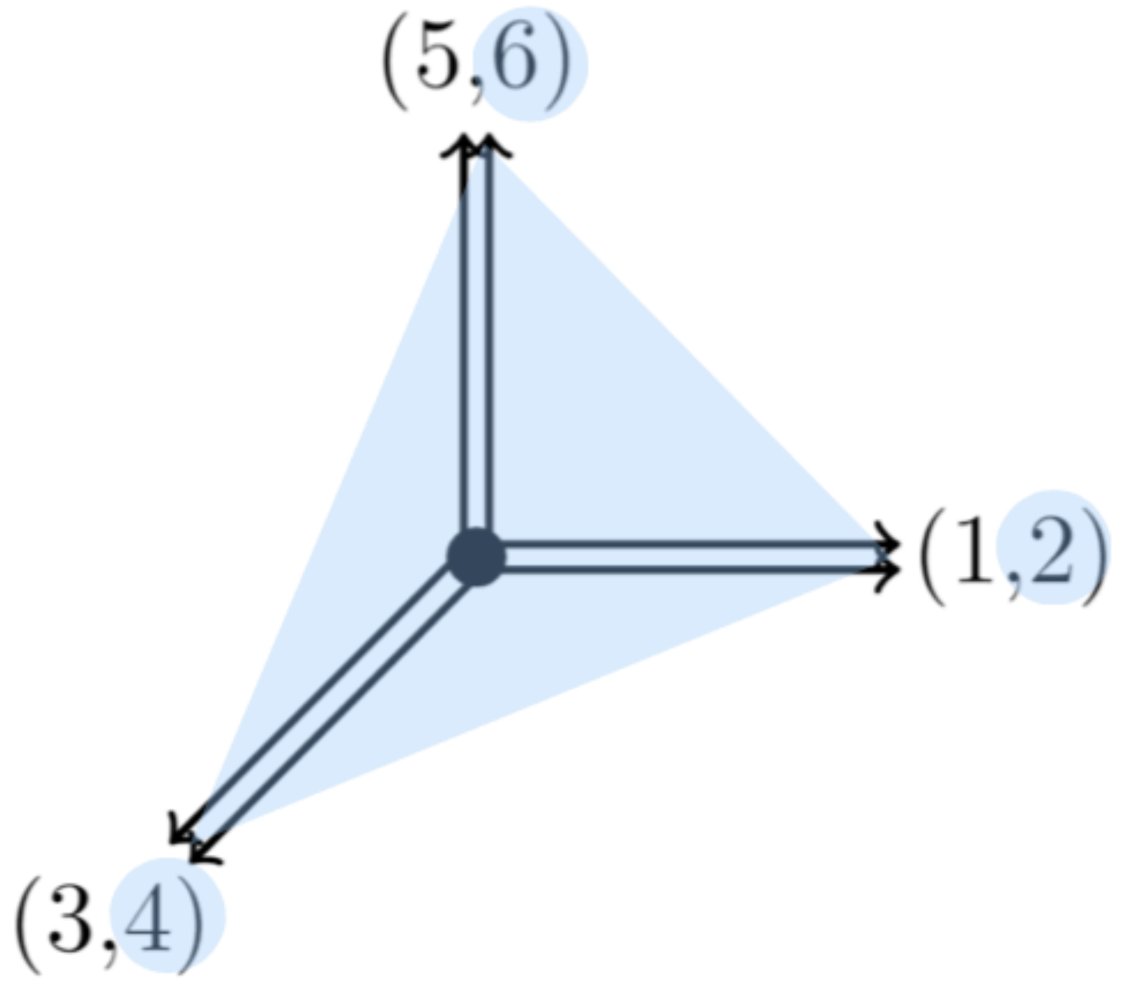
Gale duality for polytopes:

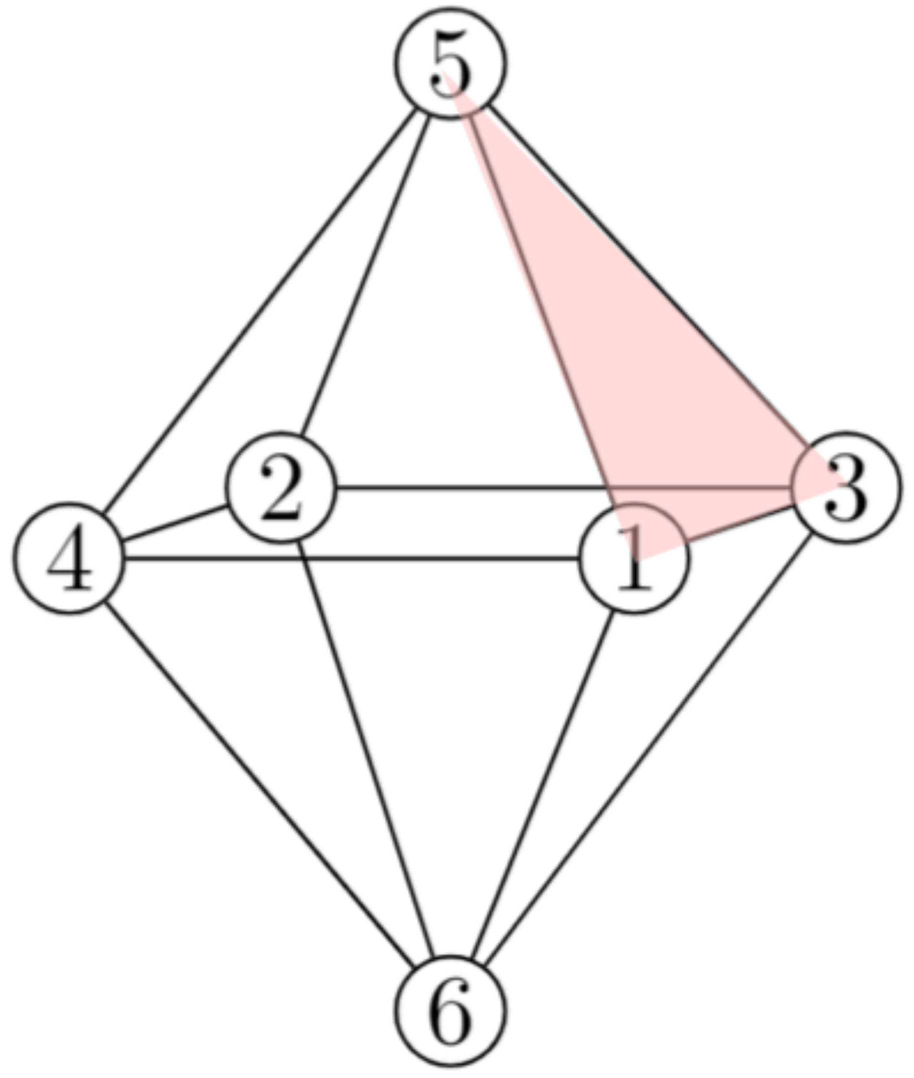
$$U = \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix}$$

1 2 3 4 5 6

$$U^* = \left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

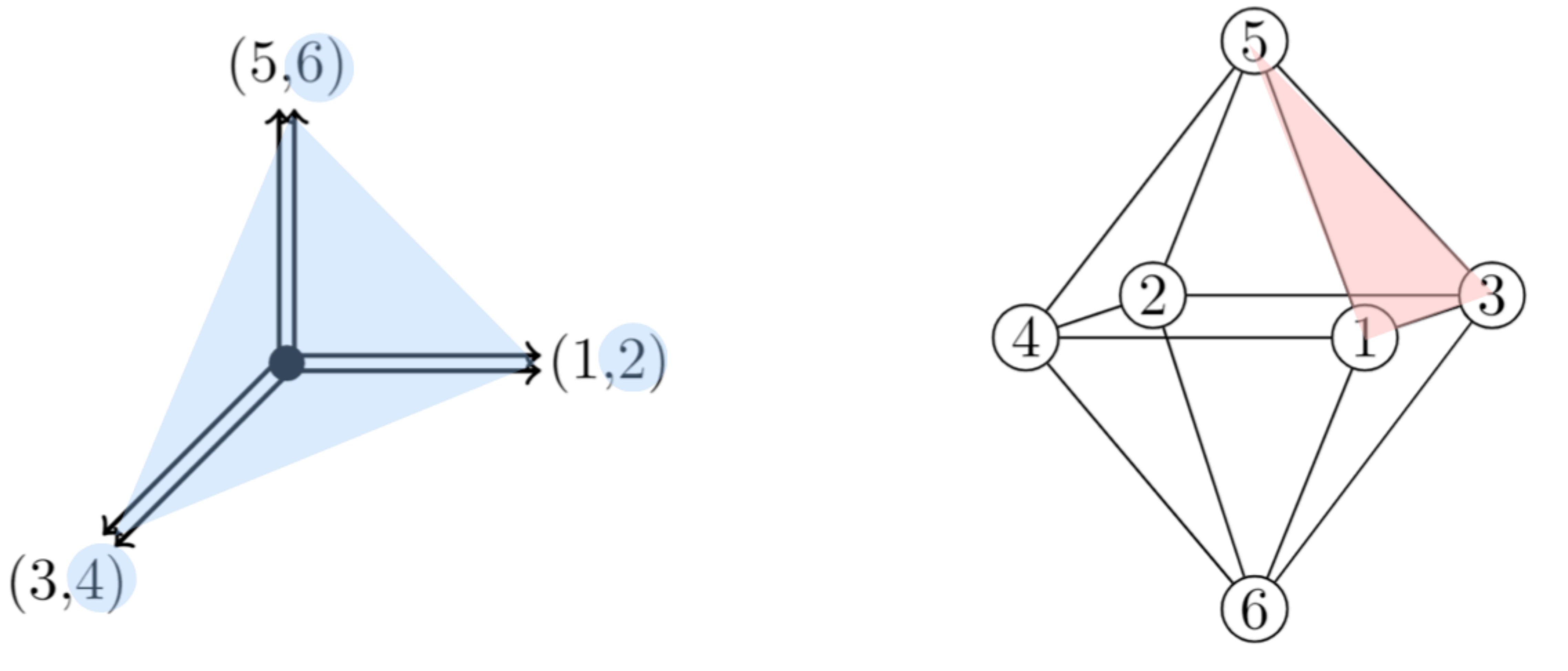
1 2 3 4 5 6

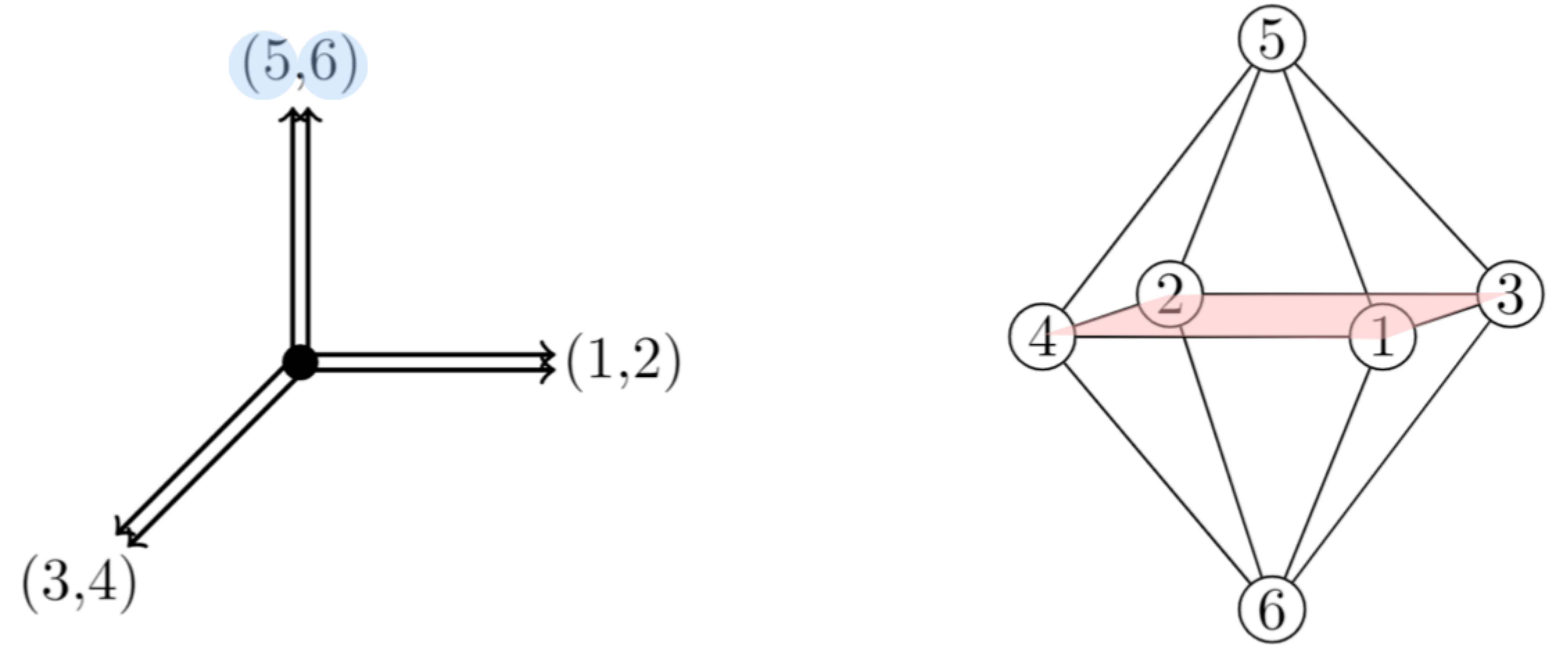




GALE DUALITY

Theorem: For $I \subseteq [n]$, $\mathrm{conv}\{u_i^* : i \in [n] \setminus I\}$ is a face of $\mathrm{conv}(\mathcal{U}^*)$ if and only if 0 is in the relative interior of $\mathrm{conv}\{u_i : i \in I\}$.



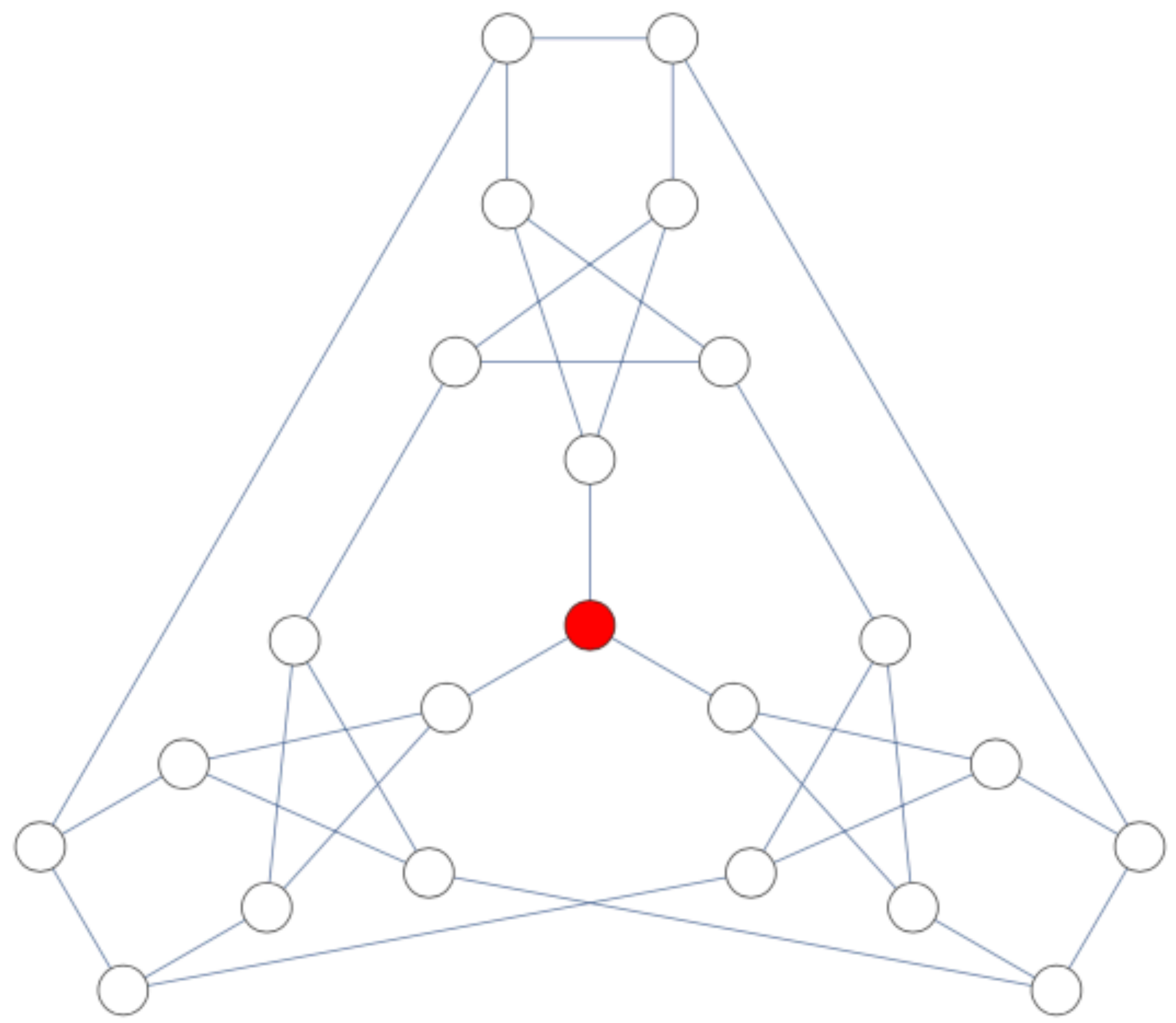


PROOF OF GALE DUALITY

BOUNDS ON SIZE

Theorem: For each $k = 1, \ldots, m-1$ there is a positively weighted k-design of size at most $\sum_{i=1}^{k} \dim \Lambda_i$.

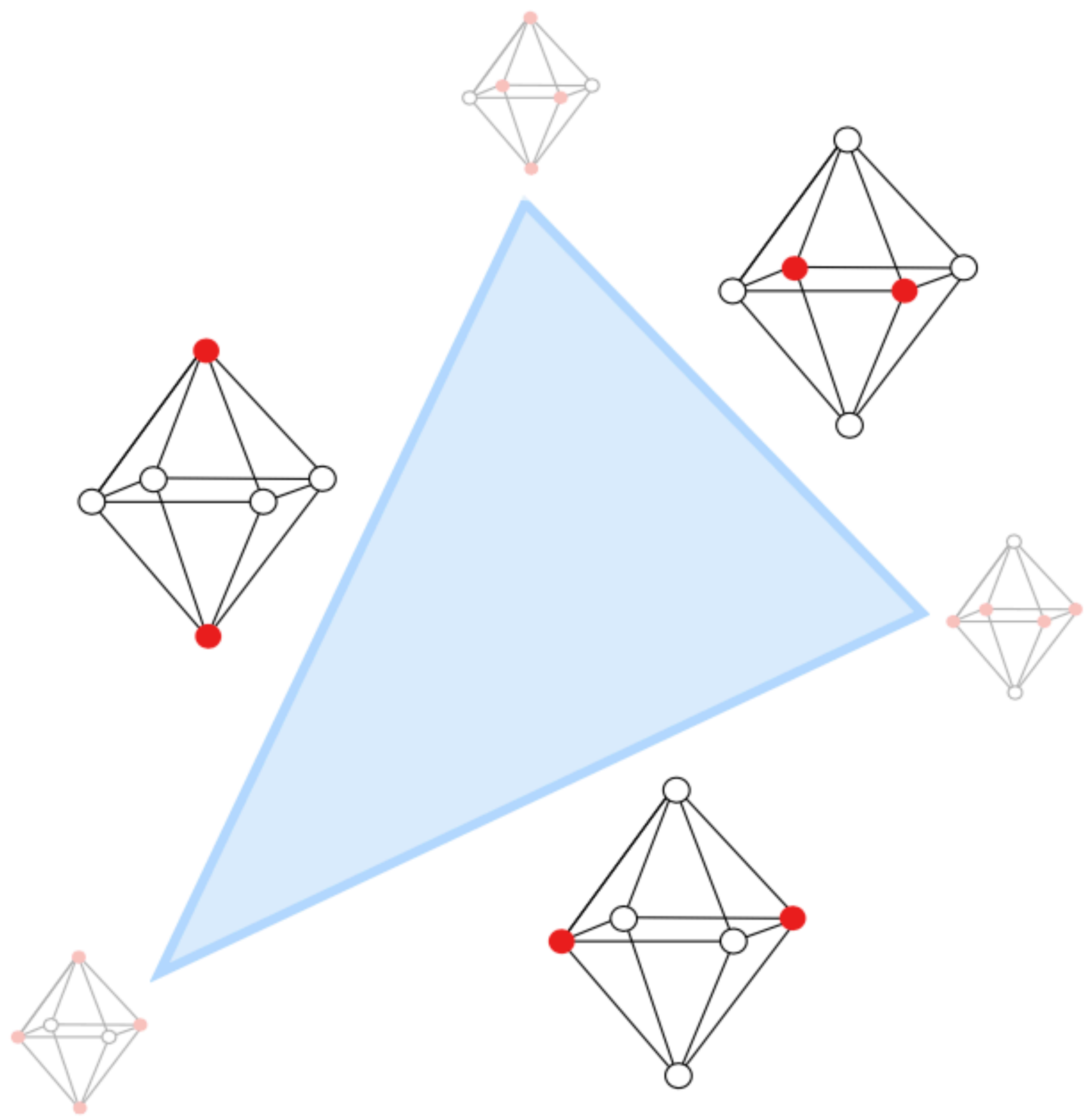
These upper bounds can be tight for every k in a G

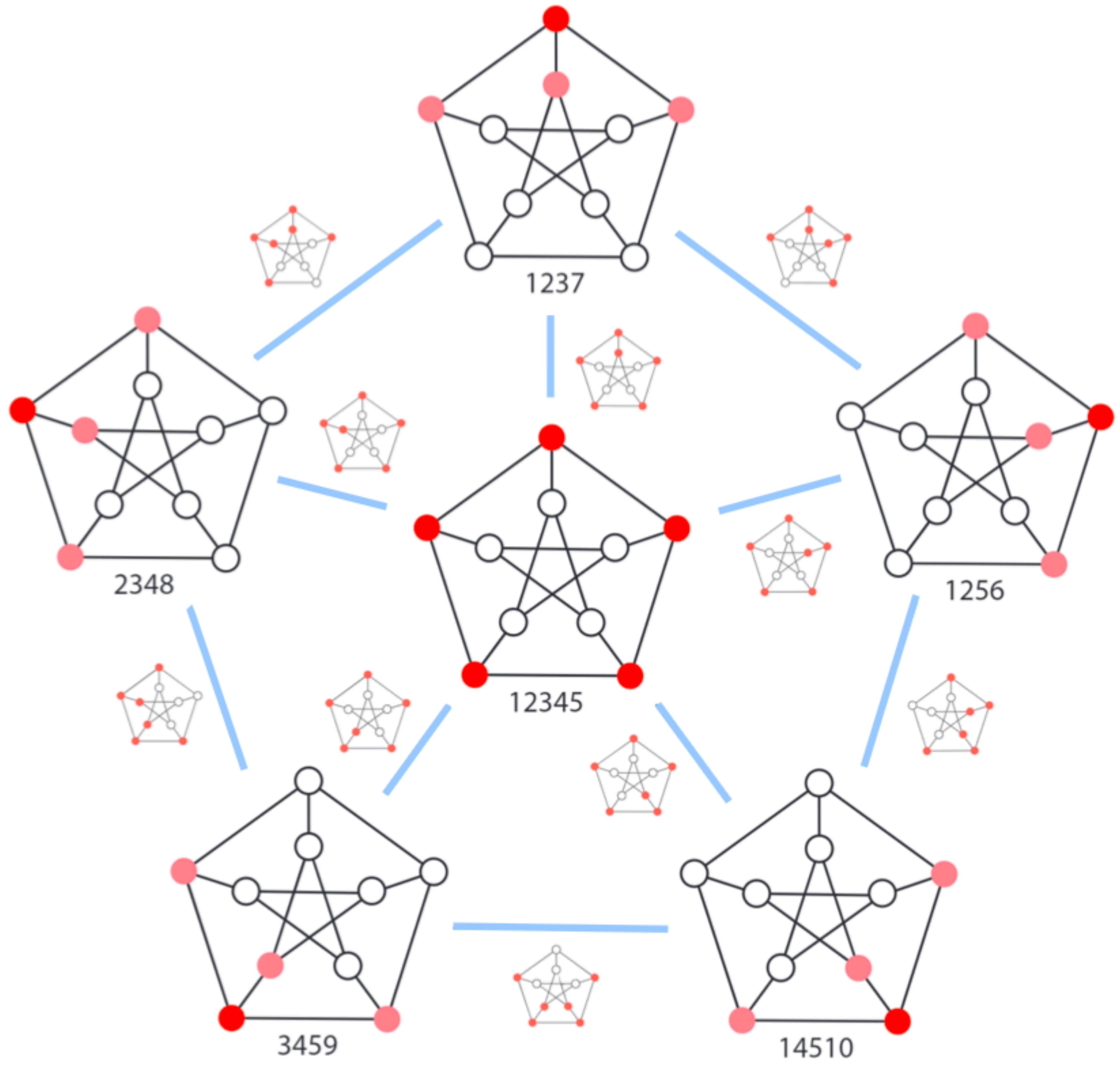


Lower bounds can be trivial.

CONSEQUENCES I

Organization





CONSEQUENCES II

Computation/Optimization

(Babecki-T. 2022)

- Cocktail party graphs
- Cycles
- Graphs of hypercubes

  (uses the theory of linear codes)

CONSEQUENCES III

Random walks & equidistribution

μ_0 – initial probability measure on $G = (V, E)$

Random walk initialized at μ_0 leads to measures

$$\mu_{l+1} = AD^{-1}\mu_l$$

WELL-KNOWN: $$\sum_{v \in V} \left| \mu_l(v) - \frac{1}{n} \right|^2 \leq \lambda_2^{2l}$$

THEOREM (Steinerberger-T. 2022)

$\forall 1 \leq k \leq n-1$ there exists μ_0 supported on at most k vertices, such that

$$\sum_{v \in V} \left| \mu_l(v) - \frac{1}{n} \right|^2 \leq \lambda_{k+1}^{2l}$$

(positively weighted k-design)

THEOREM (Steinerberger-T. 2022)

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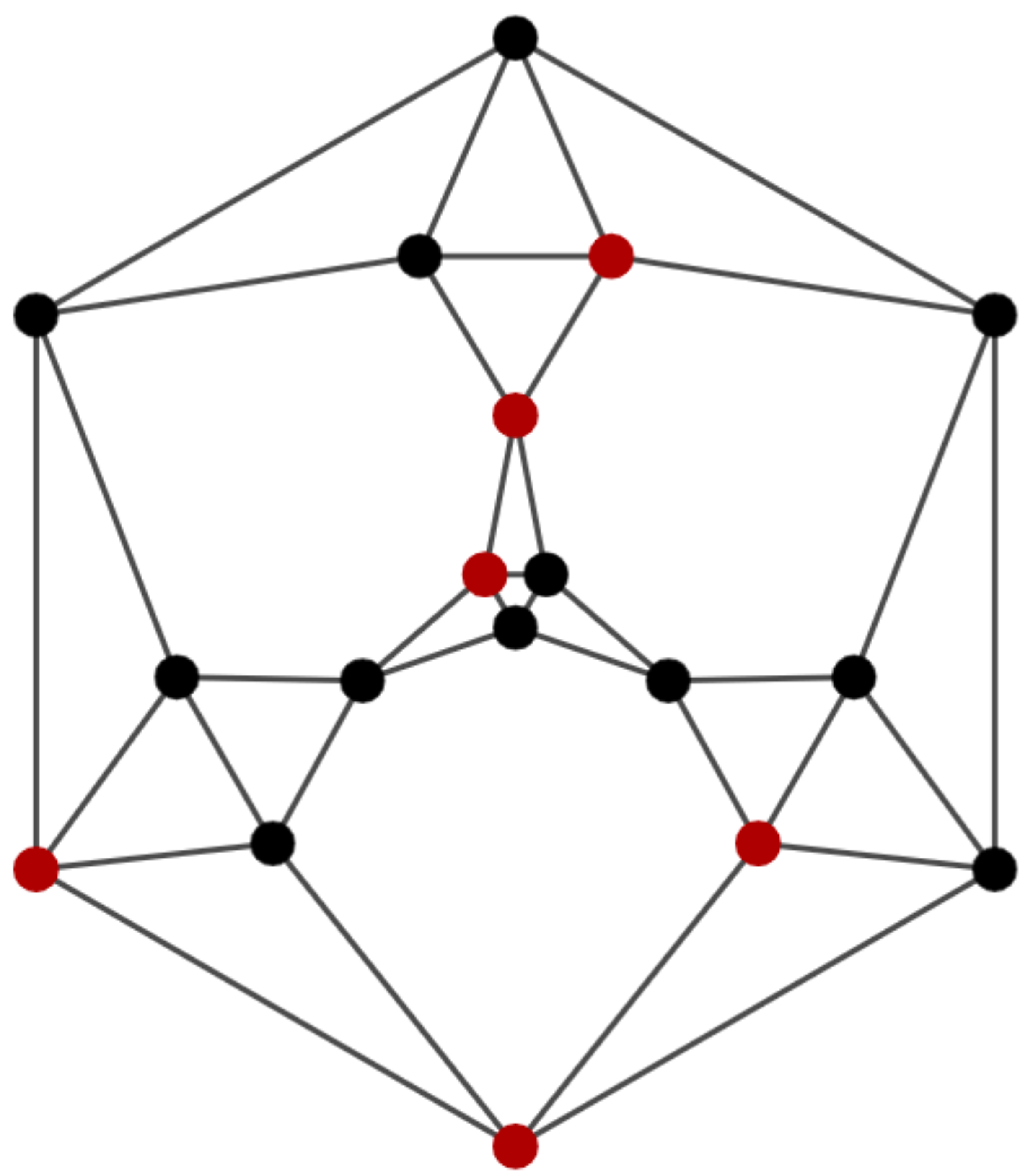
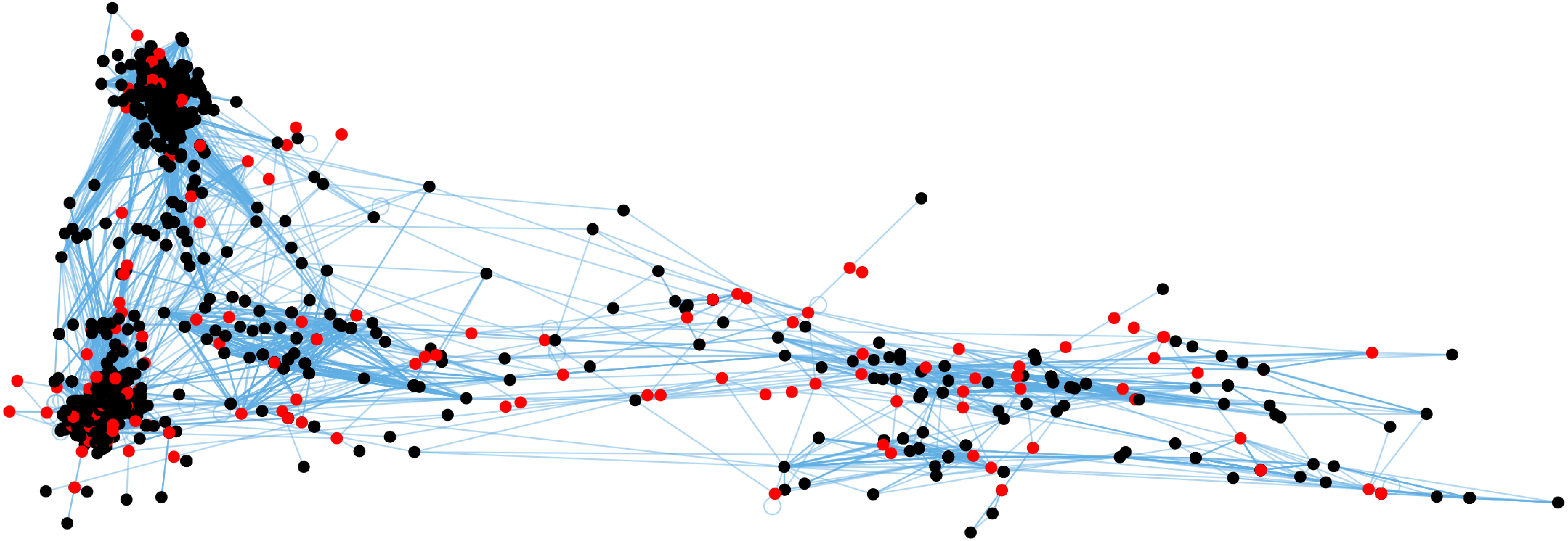


18 vertices $|\lambda_2| = 0.75$

$\exists\ \mu_0$ supported on the red vertices

that decays at rate given by $|\lambda_{11}| \sim 0.25$



The 228 red vertices are a weighted 229-graphical design on this network of 2277 English language Wikipedia pages related to chameleons. Vertices represent pages, and edges join pages that are mutually connected by hyperlinks.

Babecki: “WHAT IS ... a Graphical Design"

(AMS Notices, October 2022)

THANK YOU

Graphical designs and Gale duality (Babecki & Thomas 2022)

Math. Programming (2022)

Random Walks, Equidistribution and Graphical Designs
(Steinerberger & Thomas 2022)

Gallery of graphical designs (Babecki)

https://sites.math.washington.edu/~GraphicalDesigns/