College Algebra

A Partnership between Institutions in the Utah System of Higher Education

Salt Lake Community College
University of Utah
Weber State University

Acknowledgements

The development of this OER textbook was initiated by Salt Lake Community College to provide its students with a low-cost textbook. University of Utah and Weber State University later joined the project. These institutions have cooperatively developed a rigorous text. The body of the text conforms to the College Algebra Learning Outcomes identified by the Utah System of Higher Education.

Salt Lake Community College faculty began the project, with Ruth Trygstad taking the lead and Spencer Bartholomew joining her. Dr. Peter Trapa, seeing the need for a unified approach to College Algebra, brought the University of Utah into the project with Dr. Maggie Cummings representing their perspective. Weber State University, represented by Dr. Afshin Ghoreishi, also joined the project. Rounding out the writing group was Sarah Nicholson, a concurrent enrollment instructor from Kearns High School, Granite School District.

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The content in this text is largely adapted from *College Algebra, 3rd Corrected Edition*, by Carl Stitz and Jeff Zeager. This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License. The team creating this text appreciate the time and effort Carl Stitz and Jeff Zeager invested in creating and sharing their work. You will note there are many places where we have diverged significantly from their original text. We hope our perspective contributes beneficially to the world of OER mathematics material. Additional content, primarily examples and exercises, are from *Algebra and Trigonometry*, by Jay Abramson et al. It is licensed under a Creative Commons Attribution 4.0 International License and can be found at https://openstax.org.

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CHAPTER 1
GETTING STARTED WITH FUNCTIONS

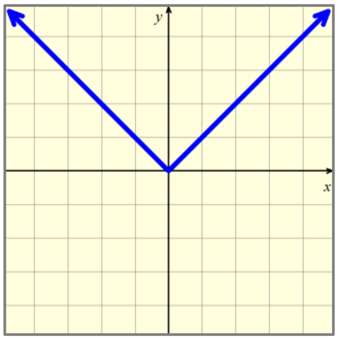


Figure 1.0. 1

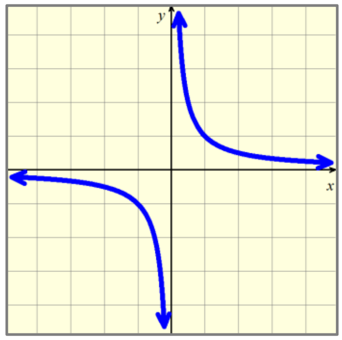


Figure 1.0. 2

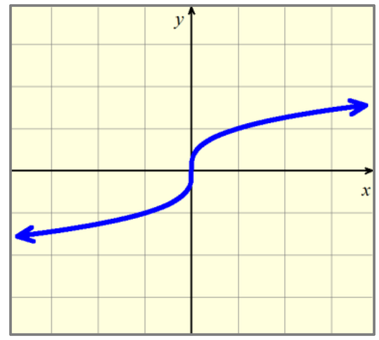


Figure 1.0. 3

Chapter Outline

Introduction

In Chapter 1 we build on understandings of functions from Elementary Algebra. The goal throughout the chapter is to help you a) gain fluidity and flexibility among the various representations of functions (analytic, numeric, and graphic) and b) understand how the various representations are related to one another and might be used in understanding problem situations more thoroughly. By the end of the chapter you should have a firm grasp on how to determine the domain and range of a variety of functions analytically or from a graph, and how and why changes to the formula of a function (such as changing $f(x)=\sqrt{x}$ to $f(x)=\sqrt{x}-a$ or $f(x)=\sqrt{x-a}$) affect the domain and range and thus the graphical representation of the function. We will build on these ideas in future chapters.

Section 1.1 deals with functions primarily from an analytic perspective. We start by reviewing from Elementary/Intermediate Algebra the definition of a function and how to determine if a relation expressed as a set of ordered pairs, an equation, or a graph represents a function. A definition for domain and range are also introduced here and you will learn how to determine both domain and range from a graph, a skill that will be enormously important throughout the entire course. A key idea about domain you should develop in this first section revolves around the notion of 'input restrictions.' You will note that there are no restrictions for inputs to polynomial functions, thus the domain of a polynomial is always all real numbers. However, from your previous work with rational expressions, you know that one cannot divide by zero, hence, the restriction(s) for a rational function will always involve any value that results in a zero in the denominator. Further, you've learned one cannot take the square (or any even root) of a negative number (if the output is to be a real number). Thus, for even radical functions, we restrict the input to values that give us arguments that are greater than or equal to 0.

In Section 1.2 we explore the link between analytic and graphic representations of functions. We start with the introduction of several key 'parent' or 'toolbox' functions (these terms are used interchangeably throughout the text) and explore them numerically, analytically and graphically, with a strong focus on what the domain and range of each is, and how the domain and range are evident in the different representations. We also explore piecewise-defined functions, what it means for a function to be odd or even, symmetry, where functions are increasing/decreasing, and max and min values of functions, all both graphically and analytically.

Section 1.3 deals with transformations of toolbox functions and how the transformations affect the domain and/or range numerically, analytically, and graphically. The primary goal is to understand transformations as more than a set of rules, e.g. to understand why changes to the argument of a function affect the domain and thus result in a horizontal change of the graph; whereas a change outside of the argument may affect the range of the function and thus its graph vertically. By the end of the section, you should be able to graph a variety of transformations of parent graphs and state their domains and ranges.

Section 1.4 deals with operations with functions, namely addition, subtraction, multiplication, division, and composition. A great deal of attention is paid to composing functions and finding the domain of compositions.

Section 1.5 deals with inverse functions: what they are, how to find them, and why they are useful. You will explore all this numerically, analytically and graphically. While you may have worked with inverses in previous courses, we caution you to think carefully about this section as ideas presented here are fundamental to understanding the relationship between logarithms and exponentials which will be explored later in the course. Throughout the section, special attention should be paid to how to find an inverse in each of the representations of a function; numerically by switching the input and output; analytically by expressing the input variable in terms of the output variable; and graphically by reflecting functions over the line $y = x$ and how these methods shed light on the meaning of the inverse. Lastly, you will learn what it means for a function to be one-to-one and how that information may be useful.

1.1 Introduction to Functions

Learning Objectives

- Determine whether a relation represents a function.
- Use the vertical line test to identify graphs of functions.
- Find the domain and range from the graph of a function.
- Find input and output values of a function.
- Find the domain from the equation of a function.

One of the core concepts in College Algebra is the function. We will define a function as a special kind of relation, and thus begin our study of functions by discussing relations.

Relations

Let's get started with the definition of a **relation**.

Definition 1.1. A **relation** is a set of ordered pairs. The set of first components of the ordered pairs is called the **domain** and the set of second components of the ordered pairs is called the **range**.

Consider the following set of ordered pairs.

$$\{(1,2),(2,4),(3,6),(4,8),(5,10)\}$$

In this relation, the domain is $\{1, 2, 3, 4, 5\}$ and the range is $\{2, 4, 6, 8, 10\}$.

Each value in the domain is known as an **input** value and corresponds to at least one value in the range; that corresponding value in the range is known as an **output** value. Any equation in two variables also represents a relation; the ordered pairs are the corresponding input and output values that satisfy the given equation. The input values are assigned to an **independent variable** and the output values are assigned to a **dependent variable**. Unless otherwise stated, for an equation in two variables x and y, x is the independent variable and y is the dependent variable. In addition, any graph in the plane represents a relation; the ordered pairs are the coordinates of the points on the graph.

Functions

As mentioned at the beginning of this section, a **function** is a special kind of relation. To be considered a function, each value in the domain of a relation must be paired with exactly one value in the range.

Equivalently, any two ordered pairs having the same first component must also have the same second component.

Definition 1.2. A **function** is a relation in which any two ordered pairs with the same first component also have the same second component.[1]

We note that the relation $\{(1,2),(2,4),(3,6),(4,8),(5,10)\}$ is a function since no two ordered pairs have the same first component. Likewise, $\{(1,\text{odd}),(2,\text{even}),(3,\text{odd}),(4,\text{even}),(5,\text{odd})\}$, the set of ordered pairs that relates the first five natural numbers to the terms 'even' and 'odd', is a function since each ordered pair has a unique first component.

For an example of a relation that is not a function, consider the set of ordered pairs $\{(\text{odd},1),(\text{even},2),(\text{odd},3),(\text{even},4),(\text{odd},5)\}$ that relates the terms 'even' and 'odd' to the first five natural numbers. Here, the two ordered pairs $(\text{odd},1)$ and $(\text{odd},3)$ have the same first component but have different second components. This violates the definition of a function, so the relation is not a function.

The next three examples include relations defined by sets of ordered pairs, equations and graphs. We will use **Definition 1.2** to determine if these relations are also functions.

Example 1.1.1. Determine if the following relations, represented by sets of ordered pairs, are functions.

1. $R_1 = \{(-2,1),(1,3),(1,4),(3,-1)\}$
2. $R_2 = \{(-2,1),(1,3),(2,3),(3,-1)\}$

Solution.

1. A quick scan of the ordered pairs in R_1 reveals that two ordered pairs have a first component of 1, and that the first component of 1 is matched with two different second components, namely 3 and 4. Hence, R_1 is not a function.
2. Every first component in R_2 occurs only once. Thus, R_2 does represent a function.

□

In the previous example, the relation R_2 contained two ordered pairs with the same second component, namely $(1,3)$ and $(2,3)$. We note that, in order to say that R_2 is a function, we just need to ensure the

[1] You will see other definitions for functions throughout your study of mathematics. It is worth noting that these definitions are simply different ways of identifying this special type of relation.

same first component isn't used with more than one second component. We can similarly test an equation where *x* and *y* represent ordered pairs, with *x* being the independent variable and *y* being the dependent variable.

Example 1.1.2. Determine if the following relations, expressed as equations, are functions.

1. $y = 5x - 2$
2. $x = y^2 + 1$
3. $x^2 + y^3 = 1$

Solution.

1. For the relation $y = 5x - 2$, the first component of each ordered pair is represented by the variable *x* and the second component is represented by *y*. Since, for each value of *x*, we get a different value for *y*, any two ordered pairs with the same first component will have the same second component. Thus, $y = 5x - 2$ is a function.
2. The relation $x = y^2 + 1$, with the first component represented by *x* and the second component represented by *y*, has ordered pairs $(2,1)$ and $(2,-1)$. Since the first component of $x = 2$ corresponds to two different second components, namely $y = 1$ and $y = -1$, the relation $x = y^2 + 1$ is not a function.
3. Here, we solve the equation $x^2 + y^3 = 1$ for *y* to determine whether each choice of *x* results in a single corresponding value for *y*.

$$
\begin{aligned}
x^2 + y^3 &= 1 \\
y^3 &= 1 - x^2 \\
\sqrt[3]{y^3} &= \sqrt[3]{1 - x^2} \\
y &= \sqrt[3]{1 - x^2}
\end{aligned}
$$

For any *x* value we choose, the equation $y = \sqrt[3]{1 - x^2}$ returns only **one** value of *y*. Hence, this equation represents a function.

□

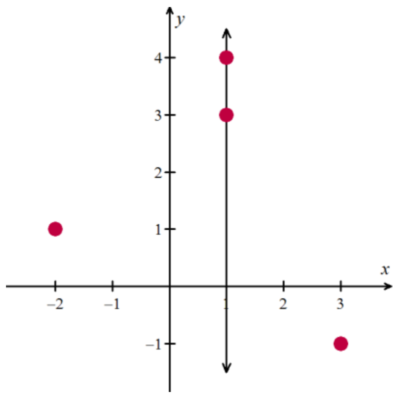
We note that for ordered pairs comprising the graph of the function $y = 5x - 2$, each *x* value is associated with a single *y* value. Since *x* is the independent variable and *y* is the dependent variable, we say the equation represents ***y* as a function of *x***. Similarly, we can say $x^2 + y^3 = 1$ represents *y* as a function of *x*.

We next introduce the vertical line test, which we will use to determine if a graph represents a function.

The Vertical Line Test

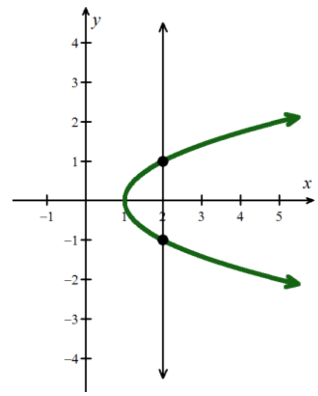
To see what the function concept means geometrically, we graph $R_1 = \{(-2,1),(1,3),(1,4),(3,-1)\}$ and $x = y^2 + 1$; the relations from the previous two examples that were not functions.

Figure 1.1. 1



Graph of R_1 with vertical line $x = 1$

Figure 1.1. 2



Graph of $x = y^2 + 1$ with vertical line $x = 2$

We note that the vertical line $x = 1$ intersects the graph of R_1 at two points: $(1,3)$ and $(1,4)$. From the graph of $x = y^2 + 1$, we see that the vertical line $x = 2$ intersects the graph at two points: $(2,1)$ and $(2,-1)$. Contemplating on these two examples, it is evident that a vertical line passing through a graph more than once would result in two points with the same *x* coordinate but different y coordinates, verifying that the graph does not represent a function.

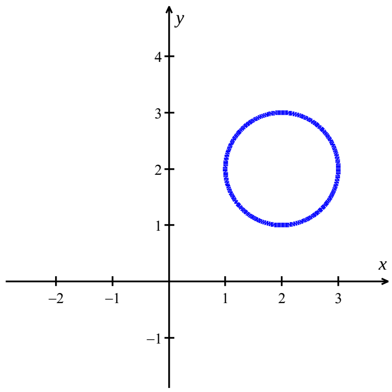
What if no vertical line intersects the graph more than once? Then there would not be two different points with the same *x* coordinate and the graph would represent a function. The vertical line test follows from these observations.

Theorem 1.1. The Vertical Line Test: A graph represents a function if no vertical line intersects it at more than one point.

It is worth taking some time to meditate on the vertical line test to check our understanding of the concept of a function and the concept of a graph.

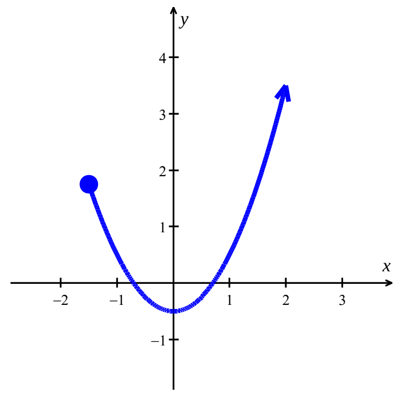
Example 1.1.3. Use the vertical line test to determine which of the following relations represents *y* as a function of *x*.

Figure 1.1. 3



The graph of S

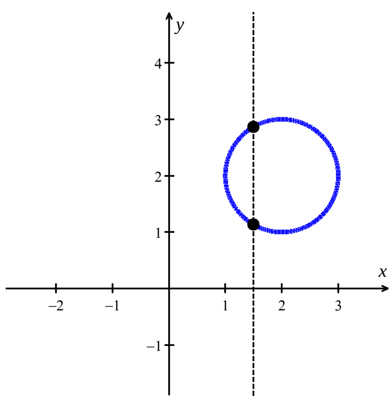
Figure 1.1. 4



The graph of T

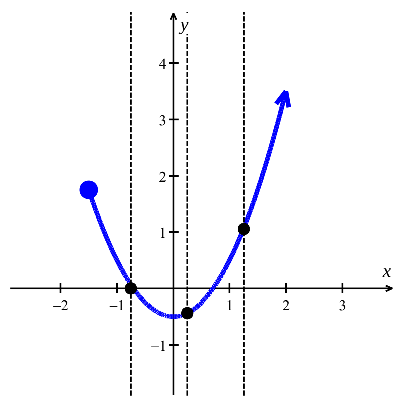
Solution. Looking at the graph of S, we can easily imagine a vertical line crossing the graph more than once. Hence, S does not represent a function. In the graph of T, every vertical line crosses the graph at most once, so T does represent a function.

Figure 1.1. 5



Vertical line passing twice through graph of S

Figure 1.1. 6



Vertical lines passing once through graph of T

□

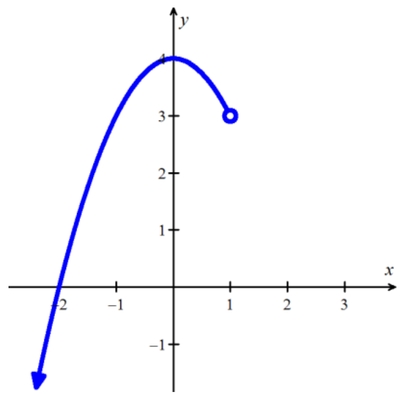
In the previous example, we say the graph of the relation S **fails** the vertical line test whereas the graph of T **passes** the vertical line test. Note that in the graph of S there are infinitely many vertical lines that cross the graph more than once. However, to fail the vertical line test, all that is needed is one vertical line that passes through two points on the graph, as was evidenced in the graph of the relation R_1.

Determining the Domain and Range from a Graph

We next identify the domain and range of functions represented by graphs. Because the domain refers to the set of first components of the ordered pairs, where the ordered pairs are coordinates of points on the graph, the domain consists of all possible *x* values[2]. The range, similarly, is the set of second components and consists of all possible *y* values[3].

Example 1.1.4. Find the domain and range of the function G whose graph is shown below.

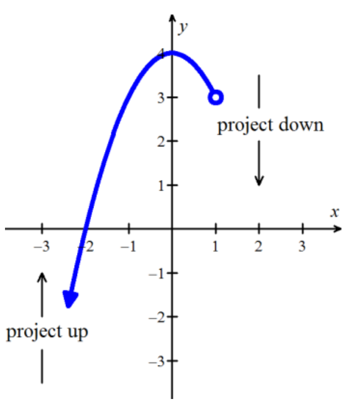
Figure 1.1. 7



The graph of G

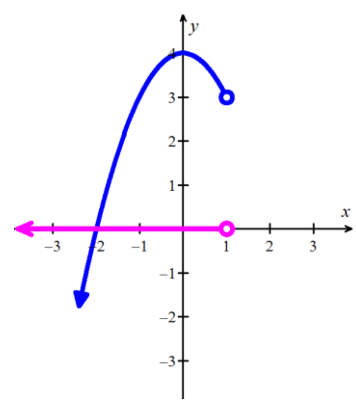
Solution. To find the domain and range of G, we determine which *x* and *y* values occur as coordinates of points on the given graph. Before going further, we need to pay attention to two subtle notations on the graph: the arrowhead on the lower left corner of the graph indicates that the graph continues to curve downward to the left forever more. The open circle at $(1,3)$ indicates that the point $(1,3)$ is not on the graph, but that all points on the curve leading up to that point are.

Figure 1.1. 8



The graph of G

Figure 1.1. 9



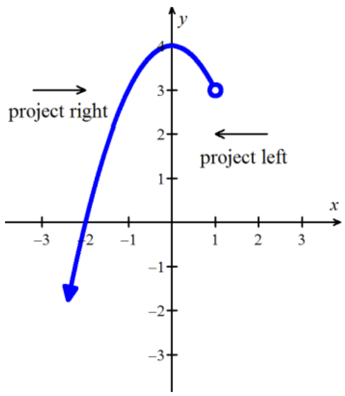
G and its projection to the *x*-axis

To find the domain, it may be helpful to imagine collapsing the curve to the *x*-axis and determining the portion of the *x*-axis that gets covered. This is called **projecting** the curve to the *x*-axis. We see from the

[2] Assuming *x* is the independent variable.
[3] Assuming *y* is the dependent variable.

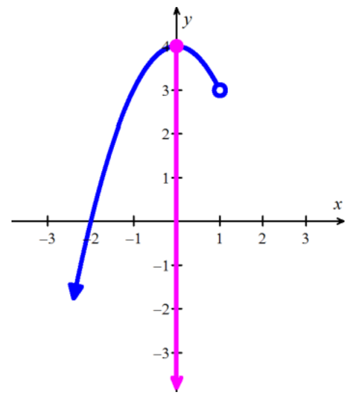
figure that if we project the graph of G to the x-axis, we get all real numbers less than 1. Using interval notation, we write the domain of G as $(-\infty,1)$. To determine the range of G, we project the curve to the y-axis as follows.

Figure 1.1. 10



The graph of G

Figure 1.1. 11



G and its projection to the y-axis

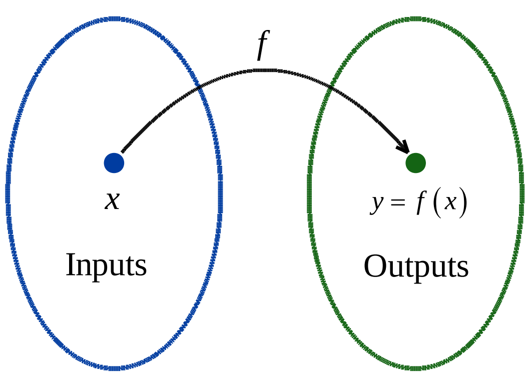
Note that even though there is an open circle at $(1,3)$, we still include the y value of 3 in our range since the point $(-1,3)$ is on the graph of G. We see that the range of G is all real numbers less than or equal to 4 or, in interval notation, $(-\infty,4]$.

□

Function Notation

In **Definition 1.2**, we described a function as a special kind of relation, one in which any two ordered pairs with the same first component also have the same second component. Since we also refer to the first components as **inputs** and the second components as **outputs**, we can think of a function f as a process by which each input is matched with only one output.

Figure 1.1. 12



Additionally, since the independent variable x represents input values and the dependent variable y represents output values, we can think of a function f as a process by which each input value of x is matched with only one output value of y. Since the output is completely determined by the input x and the process f, we symbolize the output with the **function notation** $f(x)$, which is read as 'f of x'. In other words, $f(x)$ is the output that results by applying the process f to the input x. In this case, the parentheses do not indicate multiplication as they do elsewhere in algebra.

Finding Input and Output Function Values

Evaluating formulas using function notation is a key skill for success in this and many other math courses. In the following example, we evaluate output values for the given function, and determine the input value required to result in a given output.

Example 1.1.5. Let $f(x) = -x^2 + 3x + 4$.

1. Find and simplify the following.
   (a) $f(-1)$, $f(0)$, $f(2)$
   (b) $f(a)$
2. Solve $f(x) = 4$.

Solution.

1. (a) To find $f(-1)$ when $f(x) = -x^2 + 3x + 4$, we replace every occurrence of x in the expression $f(x)$ with -1.

$$\begin{aligned} f(-1) &= -(-1)^2 + 3(-1) + 4 \\ &= -(1) + (-3) + 4 \\ &= 0 \end{aligned}$$

Similarly, we find $f(0)$ and $f(2)$.

$$\begin{aligned} f(0) &= -(0)^2 + 3(0) + 4 \\ &= 4 \end{aligned}$$

$$\begin{aligned} f(2) &= -(2)^2 + 3(2) + 4 \\ &= -4 + 6 + 4 \\ &= 6 \end{aligned}$$

(b) To find $f(a)$, we replace every occurrence of x with the quantity a.

$$\begin{aligned} f(a) &= -(a)^2 + 3(a) + 4 \\ &= -a^2 + 3a + 4 \end{aligned}$$

2. Since $f(x) = -x^2 + 3x + 4$, the equation $f(x) = 4$ is equivalent to $-x^2 + 3x + 4 = 4$. Solving, we get

$$\begin{aligned} -x^2 + 3x + 4 &= 4 \\ -x^2 + 3x &= 0 \\ x(-x+3) &= 0 \end{aligned}$$

Setting each factor equal to zero results[4] in the solutions $x = 0$ or $x = 3$, which can be verified by checking that $f(0) = 4$ and $f(3) = 4$.

□

In **Example 1.1.5**, note the practice of using parentheses when substituting algebraic values into functions. This practice is highly recommended, as it will reduce careless errors.

Determining the Domain of a Function from a Formula

Before proceeding with finding the domain of a function defined by a formula, we consider the following definition.

Definition 1.3. The **domain** of a function is the set of all input values for which the function is defined.

To determine the domain of the function $r(x) = \dfrac{2x}{x^2 - 9}$, we look for the set of all *x* values for which the function is defined. At issue are those values of *x* that result in r having a denominator of zero, since division by zero is not allowed.[5] We determine which numbers result in such a transgression by setting the denominator equal to zero and solving:

$$\begin{aligned} x^2 - 9 &= 0 \\ (x-3)(x+3) &= 0 \end{aligned}$$

Setting each factor equal to 0 results in $x = 3$ and $x = -3$. We note that, as long as we substitute numbers other than 3 and -3 for *x*, the expression $r(x)$ is a real number. Hence, we write our domain in interval notation as $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

[4] Our assumption is that you can solve quadratic equations by factoring or applying the Quadratic Formula. A brief review is included in **Section 2.1**, as a foundation for further exploration of quadratic functions.

[5] Take a moment to contemplate why division by zero is undefined, researching if necessary.

When a formula for a function is given, we assume that the function is valid for all real numbers which make arithmetic sense when substituted into the formula. This set of numbers is often called the **implied domain**[6] of the function. At this stage, there are only two mathematical sins we need to avoid: division by 0 and extracting even roots of negative numbers. The following example illustrates these concepts.

Example 1.1.6. Find the domain[7] of the following functions.

1. $f(x)=2x^3-x+3$
2. $g(x)=\dfrac{x^2-2x+3}{2}$
3. $h(x)=\dfrac{5}{x+3}$
4. $I(x)=\sqrt{2-x}$
5. $r(t)=\sqrt[3]{t-2}$
6. $J(x)=\dfrac{x-3}{x^2+2x-3}$

Solution.

1. The function $f(x)=2x^3-x+3$ requires that the number *x* be cubed and then multiplied by 2. From the result of these two operations, the number *x* is subtracted and then the number 3 is added. These operations can be performed on any real number *x* and so the domain of f is all real numbers, or $(-\infty,\infty)$. We note that f is a polynomial[8], and through similar reasoning can conclude that the domain of any polynomial function is all real numbers.

2. The function $g(x)=\dfrac{x^2-2x+3}{2}$ can be rewritten as $g(x)=\dfrac{1}{2}x^2-x+\dfrac{3}{2}$, which we recognize as a polynomial. Our conclusion from part 1, that the domain of a polynomial function is all real numbers, applies here. We identify the domain of g as being $(-\infty,\infty)$.

3. In finding the domain of $h(x)=\dfrac{5}{x+3}$, we notice that we have a denominator containing *x*. Any values of *x* that would result in division by zero must be excluded from the domain. To find those values, we set the denominator equal to 0. We get $x+3=0$, or $x=-3$. In order for a real number *x* to be in the domain of h, $x\neq-3$. In interval notation, the domain is $(-\infty,-3)\cup(-3,\infty)$.

---

[6] Also called **implicit domain**. An **explicit domain** is a domain that is specifically stated, such as $f(x)=x-1$ for $x\geq 2$.
[7] The word ‘implied’ is, well, implied.
[8] You have seen polynomials in prior math classes. A formal introduction to polynomial functions is included in **Section 2.2**.

4. The potential disaster for $I(x)=\sqrt{2-x}$ occurs when the radicand is negative. To avoid this, we set $2-x\geq 0$ and solve for *x*:

$$\begin{aligned}2-x&\geq 0\\-x&\geq -2\\x&\leq 2\end{aligned}$$

What this shows is that as long as $x\leq 2$, the expression $2-x\geq 0$ and the value of $I(x)$ is a real number. Thus, the domain is $(-\infty,2]$.

5. The formula for $r(t)=\sqrt[3]{t-2}$ is hauntingly close to that of $I(x)$ with one key difference. Whereas the expression for $I(x)$ includes an even indexed root (namely a square root), the formula for $r(t)$ involves an odd indexed root (the cube root). Since odd roots of real numbers (including negative real numbers) are real numbers, there is no restriction on the inputs to r. Hence, our domain is $(-\infty,\infty)$.

6. Once again, $J(x)=\dfrac{x-3}{x^2+2x-3}$ has a potential issue with values of *x* resulting in a denominator of zero. To determine those values, we set the denominator equal to zero and solve for *x*.

$$\begin{aligned}x^2+2x-3&=0\\(x+3)(x-1)&=0\end{aligned}$$

We find that $x=-3$ and $x=1$ result in a denominator of zero, so both values must be excluded from the domain. While it is tempting to do something with the numerator of $x-3$, there are no values which must be excluded from the numerator and so, after excluding $x=-3$ and $x=1$, the domain is $(-\infty,-3)\cup(-3,1)\cup(1,\infty)$.

□

It is worth noting the importance of finding the domain of a function before simplifying. As an example, although the function $K(x)=\dfrac{3x^2}{x}$ simplifies to $3x$, the domain excludes $x=0$. It would be inaccurate to write $K(x)=3x$ without adding the stipulation that $x\neq 0$.

1.1 Exercises

1. What is the difference between a relation and a function?

2. Why does the vertical line test tell us whether the graph of a relation represents a function?

In Exercises 3 – 8, determine whether or not the relation represents a function. Find the domain and range of those relations that are functions.

3. $\{(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9)\}$

4. $\{(-3,0),(1,6),(2,-3),(4,2),(-5,6),(4,-9),(6,2)\}$

5. $\{(-3,0),(-7,6),(5,5),(6,4),(4,9),(3,0)\}$

6. $\{(1,2),(4,4),(9,6),(16,8),(25,10),(36,12),\ldots\}$

7. $\{(1,0),(2,1),(4,2),(8,3),(16,4),(32,5),\ldots\}$

8. $\{\ldots,(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9),\ldots\}$

In Exercises 9 – 30, determine whether or not the relation represents *y* as a function of *x*. Find the domain and range of those relations that are functions.

9.

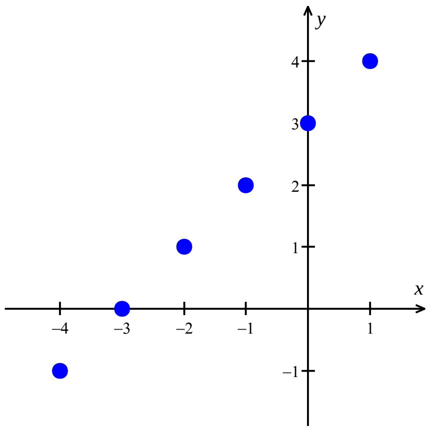


Figure 1.1. 13

10.

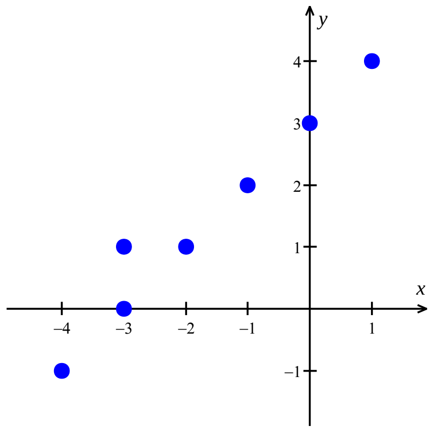


Figure 1.1. 14

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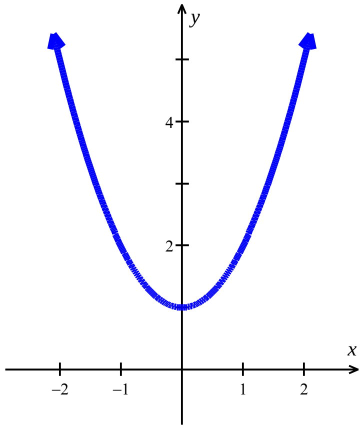

Figure 1.1. 15

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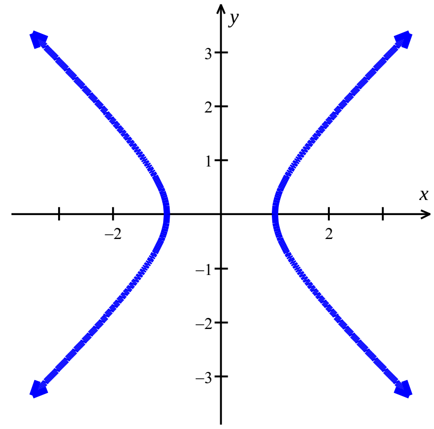

Figure 1.1. 16

13.

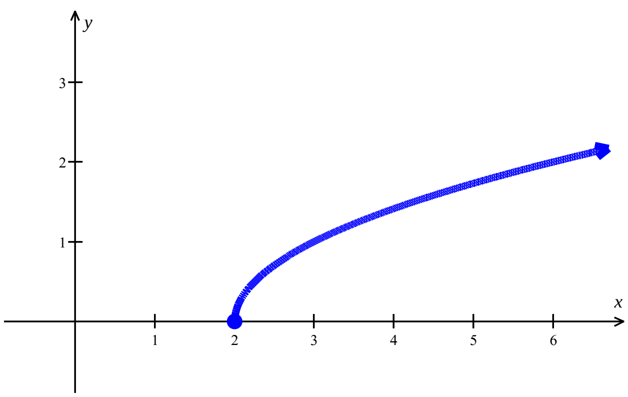

Figure 1.1. 17

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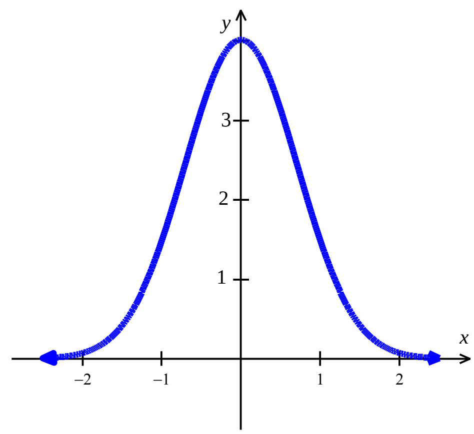

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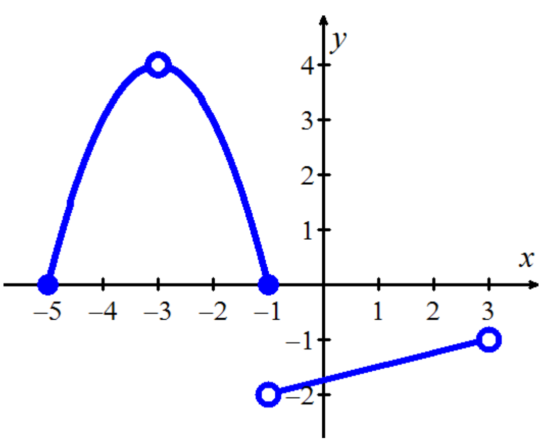

Figure 1.1. 19

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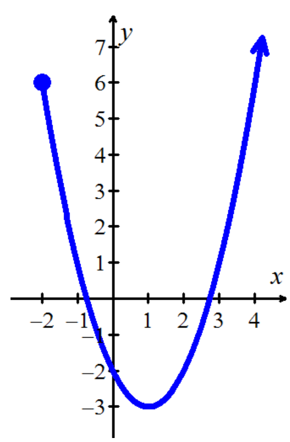

Figure 1.1. 20

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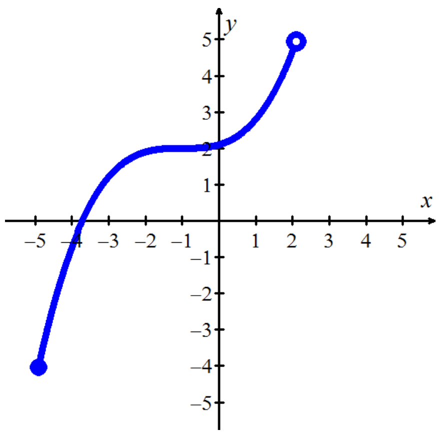

Figure 1.1. 21

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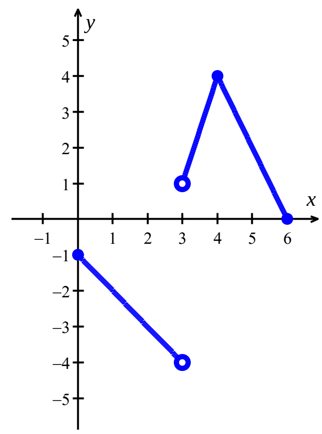

Figure 1.1. 22

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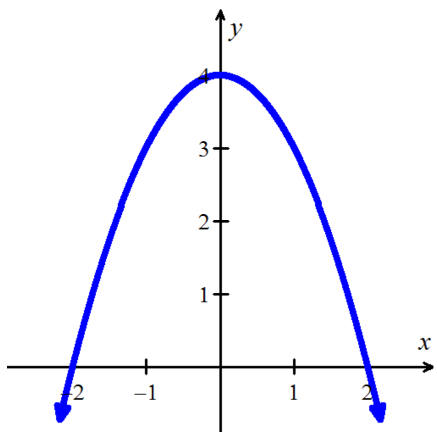

Figure 1.1. 23

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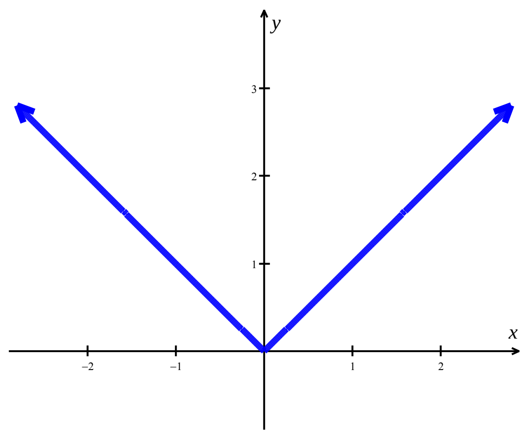

Figure 1.1. 24

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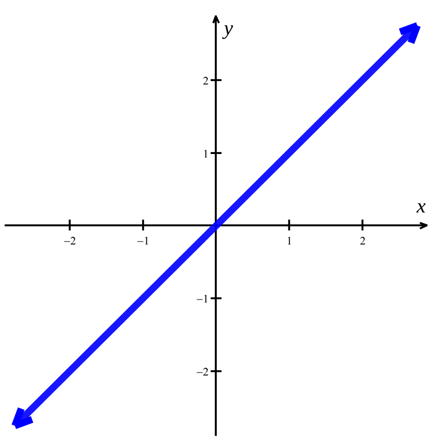

Figure 1.1. 25

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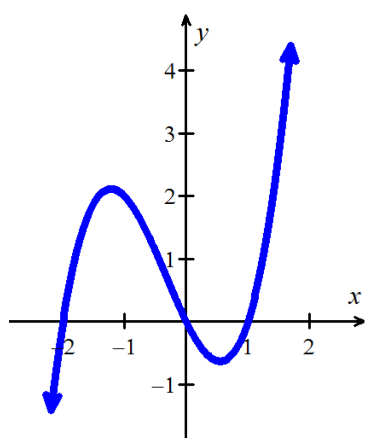

Figure 1.1. 26

23.

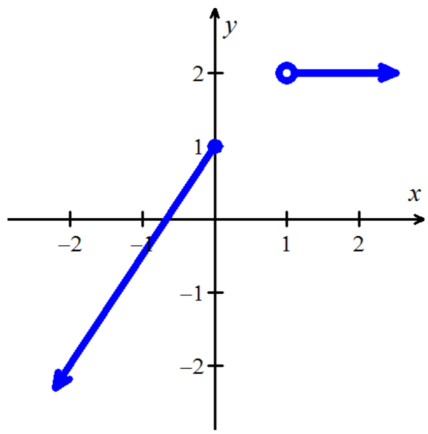

Figure 1.1. 27

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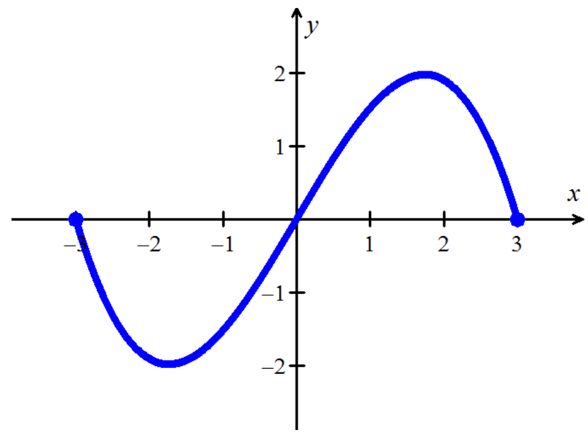

Figure 1.1. 28

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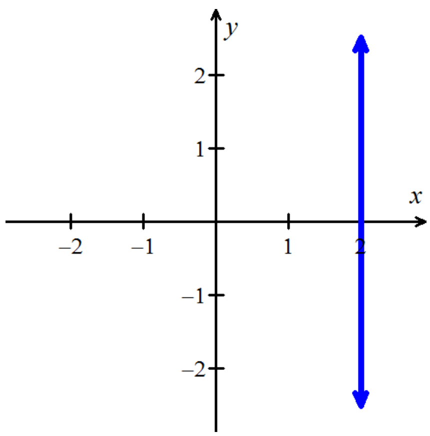


Figure 1.1. 29

26.

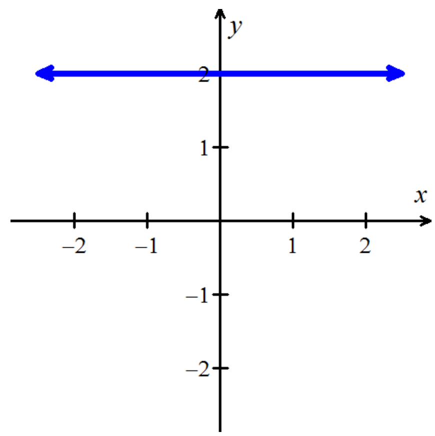


Figure 1.1. 30

27.

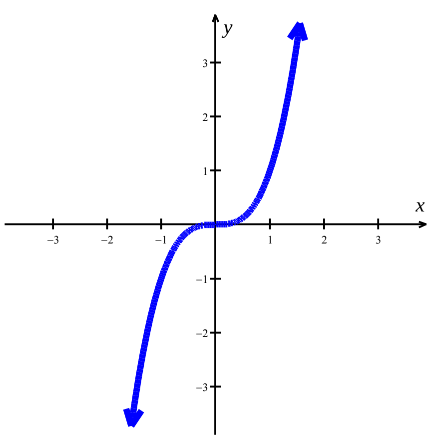


Figure 1.1. 31

28.

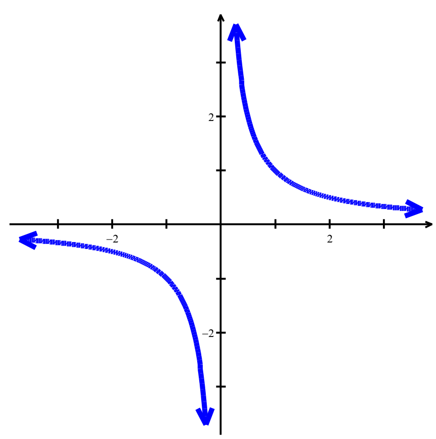


Figure 1.1. 32

29.

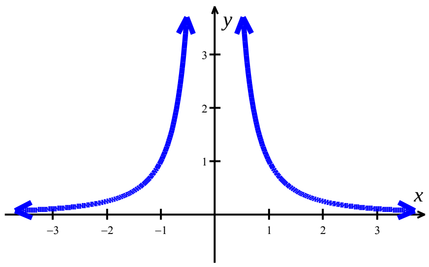


Figure 1.1. 33

30.

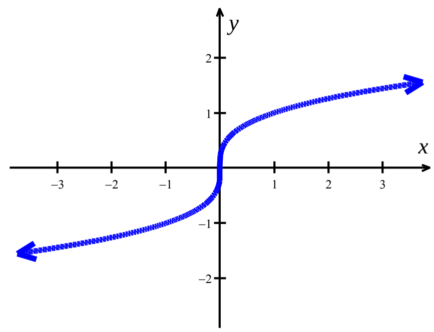


Figure 1.1. 34

In Exercises 31 – 45, determine whether or not the relation, expressed as an equation, is a function. Assume that *x* is the independent variable and *y* is the dependent variable; rewrite the equation as necessary to verify your conclusion that *y* is, or is not, a function of *x*.

31. $y = x^3 - x$

32. $y = \sqrt{x-2}$

33. $x^3 y = -4$

34. $x^2 - y^2 = 1$

35. $y = \dfrac{x}{x^2 - 9}$

36. $x = -6$

37. $x = y^2 + 4$

38. $y = x^2 + 4$

39. $x^2 + y^2 = 4$

40. $y = \sqrt{4 - x^2}$

41. $x^2 - y^2 = 4$

42. $x^3 + y^3 = 4$

43. $2x + 3y = 4$

44. $2xy = 4$

45. $x^2 = y^2$

In Exercises 46 – 55, use the given function f to evaluate, if possible, and simplify the following:

(a) $f(3)$ (b) $f(-1)$ (c) $f\left(\dfrac{3}{2}\right)$ (d) $f(a)$

46. $f(x) = 2x + 1$

47. $f(x) = 3 - 4x$

48. $f(x) = 2 - x^2$

49. $f(x) = x^2 - 3x + 2$

50. $f(x) = \dfrac{x}{x-1}$

51. $f(x) = \dfrac{2}{x^3}$

52. $f(x) = \sqrt{x-2}$

53. $f(x) = \sqrt[3]{x}$

54. $f(x) = 6$

55. $f(x) = 0$

In Exercises 56 – 63, use the given function f to evaluate, if possible, and simplify the following.

(a) $f(2)$ (b) $f(-2)$ (c) $f(h)$ (d) $f(0)$

56. $f(x)=2x-5$
57. $f(x)=5-2x$
58. $f(x)=2x^2-1$
59. $f(x)=3x^2+3x-2$
60. $f(x)=\sqrt{2x+1}$
61. $f(x)=117$
62. $f(x)=\frac{x}{2}$
63. $f(x)=\frac{2}{x}$

In Exercises 64 – 71, use the given function f to find $f(0)$ and to solve $f(x)=0$.

64. $f(x)=2x-1$
65. $f(x)=3-\frac{2}{5}x$
66. $f(x)=2x^2-6$
67. $f(x)=x^2-x-12$
68. $f(x)=\sqrt{x+4}$
69. $f(x)=\sqrt{1-2x}$
70. $f(x)=\frac{3}{4-x}$
71. $f(x)=\frac{3x}{4-x^2}$

In Exercises 72 – 89, find the (implied) domain of the function.

72. $f(x)=x^4-13x^3+56x^2-19$
73. $f(x)=x^4+4$
74. $f(x)=\frac{x-2}{x+1}$
75. $f(x)=\frac{3x}{x^2+x-2}$
76. $f(x)=\frac{2x}{x^2+4}$
77. $f(x)=\frac{2x}{x^2-4}$
78. $f(x)=\frac{x+4}{x^2-36}$
79. $f(x)=\frac{x-2}{x-2}$
80. $f(x)=\sqrt{3-x}$
81. $f(x)=\sqrt{2x+5}$
82. $f(x)=\sqrt{x+3}$
83. $f(x)=\frac{\sqrt{7-x}}{x^2+1}$

84. $f(x) = \sqrt{6x-2}$

85. $f(x) = \dfrac{6}{\sqrt{6x-2}}$

86. $f(x) = \sqrt[3]{6x-2}$

87. $f(x) = \dfrac{\sqrt{6x-2}}{x^2-36}$

88. $s(t) = \dfrac{t}{t-8}$

89. $Q(r) = \dfrac{\sqrt{r}}{r-8}$

1.2 Graphs of Functions

Learning Objectives

- Solve real-world applications of piecewise-defined functions.
- Identify and graph the toolkit/parent functions.
- Graph piecewise-defined functions.
- Determine whether a function is even, odd or neither.
- Determine where a function is increasing, decreasing or constant.
- Determine local maxima and minima.
- Determine absolute maximum and minimum.

Through mathematics, it is possible to predict the high temperature on a given day, determine the hours of daylight on a given day, or predict population trends. In each of these scenarios, functions play an important role. We begin this section by looking at some real-world applications of functions.

Modeling with Functions

It is important to keep in mind that any time mathematics is used to approximate reality, there are always limitations to the model. For example, suppose grapes are on sale at the local market for $1.50 per pound. To develop a formula which relates the cost of buying grapes to the amount of grapes being purchased, we let the variable c denote the cost of the grapes and the variable g denote the amount of grapes purchased. To find the cost of the grapes, we use the formula $c = 1.5g$.

We can think of g as the independent variable and c as the dependent variable, and write our formula in function notation.

$$c(g) = 1.5g$$

With g representing the amount of grapes purchased (in pounds) and $c(g)$ representing the cost (in dollars), we next determine the **applied domain**[9] of this function. Even though, mathematically, $c(g) = 1.5g$ has no domain restrictions (there are no denominators and no even-indexed radicals), there are certain values of g that don't make any physical sense. For example, $g = -1$ corresponds to

[9] Also known as the **explicit domain**.

purchasing −1 pounds of grapes.[10] Also, unless the local market mentioned is the State of California (or some other exporter of grapes), it doesn't make sense for $g = 500{,}000{,}000$. So the reality of the situation limits what g can be, and these limits determine the applied domain. Typically, an applied domain is stated explicitly. In this case it would be common to see something like $c(g) = 1.5g$, $0 \le g \le 100$, meaning the number of pounds of grapes purchased is limited from 0 up to 100.

Example 1.2.1. The height h in feet of a model rocket above the ground t seconds after lift-off is given by

$$h(t) = \begin{cases} -5t^2 + 100t, & \text{if } 0 \le t \le 20 \\ 0, & \text{if } t > 20 \end{cases}$$

1. Find and interpret $h(10)$ and $h(60)$.
2. Solve $h(t) = 375$ and interpret your answers.

Solution.

1. We first note that the independent variable here is t, chosen because it represents time. Secondly, the function is broken up into two rules: one formula for values of t between 0 and 20 inclusive, and another for values of t greater than 20.

   Since $t = 10$ satisfies the inequality $0 \le t \le 20$, we use the first formula listed to find $h(10)$.

$$\begin{aligned} h(t) &= -5t^2 + 100t \\ h(10) &= -5(10)^2 + 100(10) \\ &= 500 \end{aligned}$$

   With t representing the number of seconds since lift-off and $h(t)$ the height above the ground in feet, the equation $h(10) = 500$ means that 10 seconds after lift-off, the model rocket is 500 feet above the ground.

   To find $h(60)$, we note that $t = 60$ satisfies $t > 20$, so we use the formula $h(t) = 0$. This function returns a value of 0 regardless of what value is substituted for t, so $h(60) = 0$. This means that 60 seconds after lift-off, the rocket is 0 feet above the ground; in other words, a minute after lift-off, the rocket has already returned to Earth.

[10] Maybe this means *returning* a pound of grapes?

2. Since the function h is defined in pieces, we need to solve $h(t)=375$ in pieces.

   For $0 \le t \le 20$, $h(t)=-5t^2+100t$, so for these values of *t*, we solve $-5t^2+100t=375$.

$$\begin{aligned} -5t^2+100t &= 375 \\ -5t^2+100t-375 &= 0 \\ -5\left(t^2-20t+75\right) &= 0 \\ -5(t-5)(t-15) &= 0 \end{aligned}$$

   Setting each factor equal to 0, we get $t=5$ and $t=15$. Both of these values of t lie between 0 and 20, so are solutions.

   For $t>20$, $h(t)=0$ and in this case there are no solutions to $0=375$.

   In terms of the model rocket, $h(t)=375$ corresponds to finding when, if ever, the rocket reaches 375 feet above the ground. Our two answers, $t=5$ and $t=15$, correspond to the rocket reaching this altitude twice: once 5 seconds after launch and again 15 seconds after launch.

□

The type of function in the previous example is called a **piecewise-defined function**, or **piecewise function** for short. Many real-world phenomena (income tax formulas, for example) are modeled by piecewise-defined functions. A piecewise-defined function uses more than one formula to define the output, and each formula has its own domain. The domain of the function is the union of all of these smaller domains. Visualizing piecewise functions through graphing is helpful. We will return to piecewise-defined functions and their graphs following the introduction of some basic functions.

The Toolkit Functions

The following basic functions are referred to as **toolkit functions**, or **parent functions**. These functions, combinations of these functions, their graphs and transformations, will be used frequently throughout our study of College Algebra. It is helpful to recognize these functions quickly by name, formula and graph.

We begin with three functions you are already familiar with: $f(x)=c$, where *c* is a constant, $f(x)=x$ and $f(x)=|x|$. Following these first three functions are two polynomial functions: $f(x)=x^2$ and $f(x)=x^3$. Next are two reciprocal functions: $f(x)=\frac{1}{x}$ and $f(x)=\frac{1}{x^2}$. The last two toolkit functions are roots: $f(x)=\sqrt{x}$ and $f(x)=\sqrt[3]{x}$. An identifying name, along with a few sample table values, are included with each graph. The domain, range and intercepts are identified as well.

x	$f(x)=c$
-2	c
-1	c
0	c
1	c
2	c

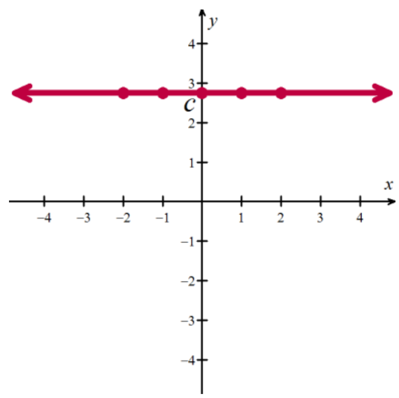


Figure 1.2. 1

Constant Function
$f(x)=c$, where c is a constant

Domain: $(-\infty,\infty)$
Range: $[c,c]$, or $\{c\}$
x-intercepts: if $c\neq 0$, none;
if $c=0$, $\{(x,0)\mid x \text{ is a real number}\}$
y-intercept: $(0,c)$

The domain of the **constant function**, $f(x)=c$, consists of all real numbers; there are no restrictions on the input. The only output value is the constant *c*, so the range is the set $\{c\}$ that contains this single element. To find the *x*-intercepts[11], we set $f(x)=0$, which can happen only when $c=0$, in which case the *x*-intercepts would include all real numbers. To find the *y*-intercept[12], we set $x=0$ to get $y=f(0)=c$.

x	$f(x)=x$
-2	-2
-1	-1
0	0
1	1
2	2

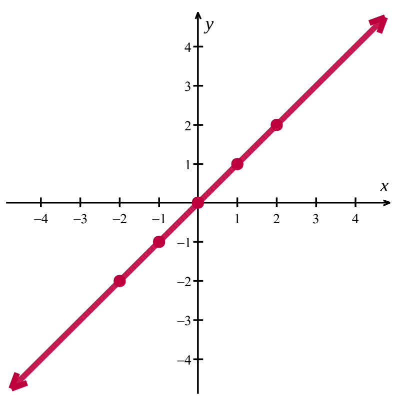


Figure 1.2. 2

Identity Function
$f(x)=x$

Domain: $(-\infty,\infty)$
Range: $(-\infty,\infty)$
x- & *y*-intercept: $(0,0)$

[11] To find *x*-intercepts, set *y* equal to 0 and solve for *x*.
[12] To find *y*-intercepts, set *x* equal to 0 and solve for *y*.

For the **identity function**, $f(x)=x$, there is no restriction on *x*. Both the domain and range are the set of all real numbers. To find the *x*-intercepts, we set $f(x)=0$, which happens only when $x=0$. For the *y*-intercept we set $x=0$ to get $y=f(0)=0$.

x	$f(x)=\lvert x\rvert$
-2	2
-1	1
0	0
1	1
2	2

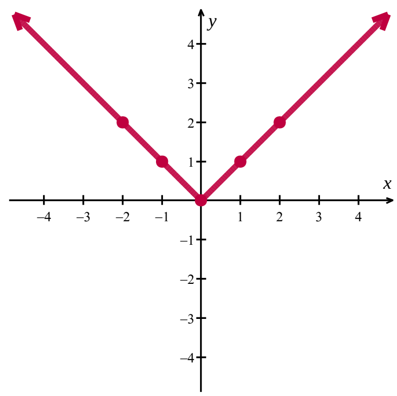


Figure 1.2. 3

Absolute Value Function

$f(x)=\lvert x\rvert$

Domain: $(-\infty,\infty)$

Range: $[0,\infty)$

x- & *y*-intercept: $(0,0)$

There is no restriction on *x* for the **absolute value function**, $f(x)=\lvert x\rvert$. However, the output can only be greater than or equal to 0. For the *x*-intercept, $\lvert x\rvert=0$ occurs only when $x=0$. The *y*-intercept is at $y=\lvert 0\rvert=0$.

We note that $f(x)=\lvert x\rvert$ can be defined as a piecewise-defined function:

$$\lvert x\rvert=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$$

x	$f(x)=x^2$
-2	4
-1	1
0	0
1	1
2	4

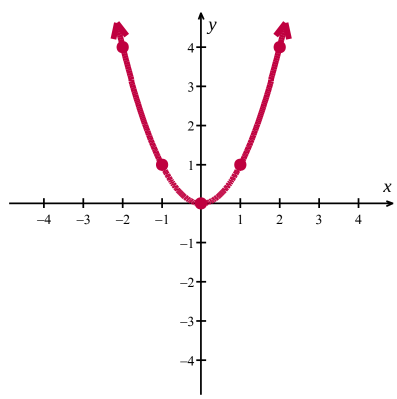


Figure 1.2. 4

Quadratic Function

$f(x)=x^2$

Domain: $(-\infty,\infty)$

Range: $[0,\infty)$

x- & *y*-intercept: $(0,0)$

For the **quadratic function**, $f(x)=x^2$, input values include all real numbers. Because output values are positive or zero, the range includes only nonnegative real numbers and, as seen from the graph, is all nonnegative real numbers. For the x-intercept, we set $x^2 = 0$ and find $x = \pm\sqrt{0} = 0$. To find the y-intercept, we set $x = 0$ to get $y = 0^2 = 0$.

x	$f(x) = x^3$
-2	-8
-1	-1
0	0
1	1
2	8

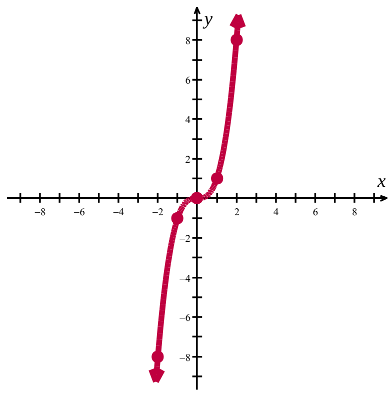


Figure 1.2. 5

Cubic Function

$f(x) = x^3$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

x- & y-intercept: $(0,0)$

For the **cubic function**, $f(x) = x^3$, input values include all real numbers and, as seen from the graph, the range is also all real numbers. For the x-intercept, we set $x^3 = 0$ and find $x = \sqrt[3]{0} = 0$. We set $x = 0$ to find that the y-intercept is $y = 0^3 = 0$.

x	$f(x) = \frac{1}{x}$
-2	$-\frac{1}{2}$
-1	-1
0	undefined
1	1
2	$\frac{1}{2}$

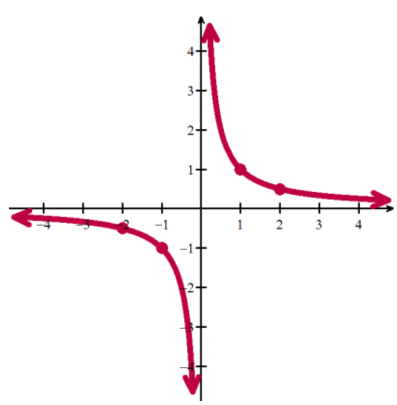


Figure 1.2. 6

Reciprocal Function

$f(x) = \frac{1}{x}$

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

Intercepts: none

For the **reciprocal function**, $f(x) = \frac{1}{x}$, we cannot divide by 0 so we must exclude 0 from the domain. Further, $\frac{1}{x} = 0$ has no solution so the range will exclude 0 and, as seen from the graph, the range is all real numbers except zero. There are no x-intercepts since $\frac{1}{x} = 0$ has no solution. There is no y-intercept since $y = \frac{1}{0}$ is undefined.

x	$f(x)=\frac{1}{x^2}$
-2	$\frac{1}{4}$
-1	1
0	undefined
1	1
2	$\frac{1}{4}$

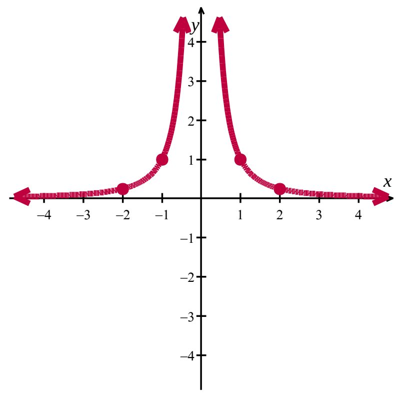


Figure 1.2. 7

Reciprocal Squared Function

$$f(x)=\frac{1}{x^2}$$

Domain: $(-\infty,0)\cup(0,\infty)$

Range: $(0,\infty)$

Intercepts: none

For the **reciprocal squared function**, $f(x)=\frac{1}{x^2}$, we cannot divide by 0 so we must exclude 0 from the domain. There is also no *x* that can give us an output of 0, so 0 is excluded from the range. We note that the output of this function is always positive due to the square in the denominator and, as seen from the graph, the range is the set of all positive numbers. There are no *x*-intercepts since $y=0$ is not in the range. There is no *y*-intercept since $x=0$ is not in the domain.

x	$f(x)=\sqrt{x}$
-4	undefined
-1	undefined
0	0
1	1
4	2

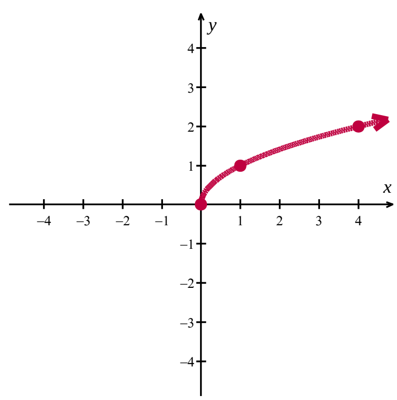


Figure 1.2. 8

Square Root Function

$$f(x)=\sqrt{x}$$

Domain: $[0,\infty)$

Range: $[0,\infty)$

x- & *y*-intercept: $(0,0)$

For the **square root function**, $f(x)=\sqrt{x}$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number *x* is defined to be positive and, as seen from the graph, the range is all nonnegative real numbers. The *x*-intercept occurs when $\sqrt{x}=0$, which happens when $x=0$. We find the *y*-intercept to be $y=\sqrt{0}=0$.

x	$f(x)=\sqrt[3]{x}$
-8	-2
-1	-1
0	0
1	1
8	2

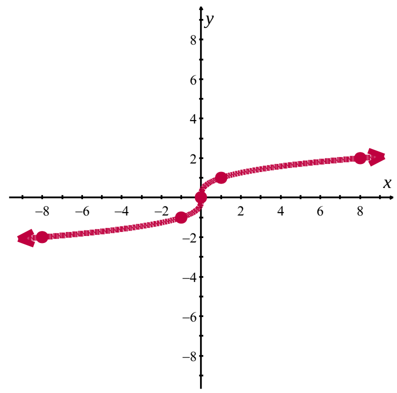


Figure 1.2. 9

Cube Root Function

$f(x)=\sqrt[3]{x}$

Domain: $(-\infty,\infty)$

Range: $(-\infty,\infty)$

x- & *y*-intercept: $(0,0)$

We note that there is no problem taking the cube root of a negative number, and the resulting output is negative. So, for the **cube root function**, $f(x)=\sqrt[3]{x}$, input values include all real numbers and, as seen from the graph, the range is also all real numbers. We find that the *x*-intercept occurs when $\sqrt[3]{x}=0$, so that $x=0^3=0$. The *y*-intercept is $y=\sqrt[3]{0}=0$.

In finding the *x*-intercepts of a function f, we note that the *x*-coordinates are found be solving $f(x)=0$. For this reason, they are called the **zeros** of f. We state the definition before moving on.

Definition 1.4. The **zeros** of a function f are the solutions to the equation $f(x)=0$. In other words, x is a zero of f if and only if $(x,0)$ is an *x*-intercept of the graph of $y=f(x)$.

Graphing Piecewise-Defined Functions

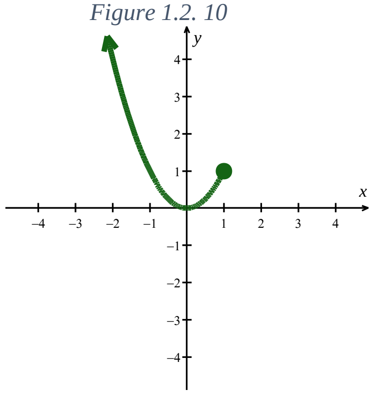
Graphing piecewise-defined functions is a bit of a challenge, but familiarity with the toolkit functions will be helpful.

Example 1.2.2. Sketch a graph of the function.

$$f(x)=\begin{cases} x^2 & \text{if} \quad x\le 1 \\ 3 & \text{if} \quad 1<x\le 2 \\ x & \text{if} \quad x>2 \end{cases}$$

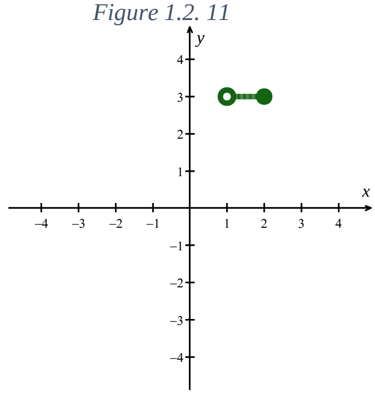
Solution. Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw an open circle to indicate where the endpoint is not included because of a ‘less than’ or ‘greater than’ inequality; we draw a closed circle where the endpoint is included because of a 'less than or equal to' or 'greater than or equal to' inequality. Following are the three components of this piecewise-defined function, each graphed on a separate coordinate system.

Figure 1.2. 10



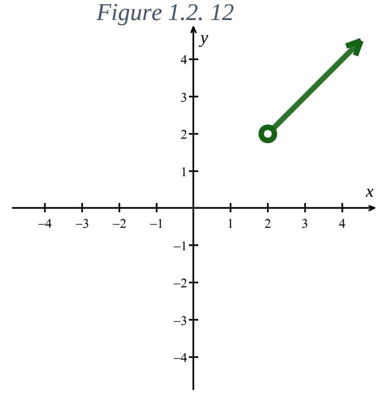
$f(x)=x^2$ if $x\le 1$

Figure 1.2. 11



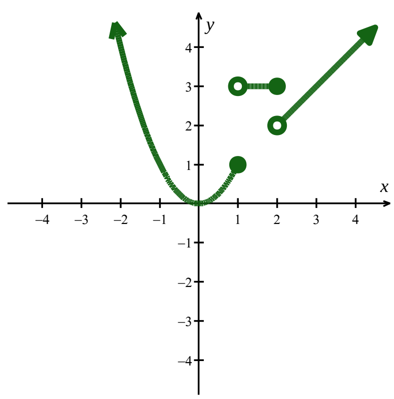
$f(x)=3$ if $1<x\le 2$

Figure 1.2. 12



$f(x)=x$ if $x>2$

Now that we have sketched each piece individually, we combine them in the same coordinate plane.



$$f(x)=\begin{cases} x^2 & \text{if} \quad x\le 1 \\ 3 & \text{if} \quad 1<x\le 2 \\ x & \text{if} \quad x>2 \end{cases}$$

Figure 1.2. 13

□

Even and Odd Functions

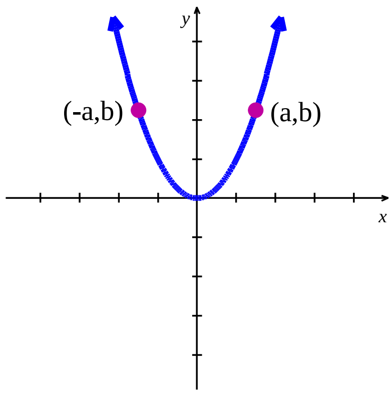
Knowing if the graph of a function is symmetric about the *y*-axis or the origin can be helpful in graphing it. We refer to functions as being 'even' or 'odd' if they possess symmetry about the *y*-axis or origin, respectively.

Definition 1.5. A function f is called

- **even** if its graph is symmetric about the *y*-axis;
- **odd** if its graph is symmetric about the origin.

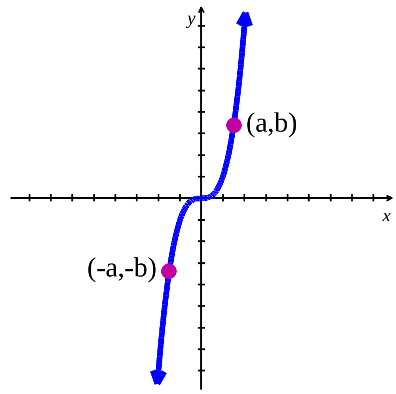
The graphs of the toolkit functions $f(x)=x^2$ and $f(x)=x^3$ are shown below. Note that the graph of $f(x)=x^2$ is symmetric about the *y*-axis and the graph of $f(x)=x^3$ is symmetric about the origin. Thus, the function $f(x)=x^2$ is an even function and the function $f(x)=x^3$ is an odd function.

Figure 1.2. 14



$f(x)=x^2$ is symmetric about the *y*-axis

Figure 1.2. 15



$f(x)=x^3$ is symmetric about the origin

As we see from the above examples, a graph will be symmetric about the *y*-axis if, for every point (a,b) on the graph, the point $(-a,b)$ is also on the graph. For symmetry about the origin, if the point (a,b) is on the graph, then the point $(-a,-b)$ will also be on the graph.

To test if the graph of a function, $y=f(x)$, is symmetric about the *y*-axis, we replace *x* with −*x*, resulting in the equation $y=f(-x)$. For the graph of $y=f(x)$ to be symmetric about the *y*-axis, we must have $f(-x)=f(x)$. In a similar fashion, to test the function $y=f(x)$ for symmetry about the origin, we replace *x* with −*x* and *y* with −*y*. Doing this substitution in the equation $y=f(x)$ results in $-y=f(-x)$. Then, solving for *y* gives $y=-f(-x)$. For the graph of $y=f(x)$ to be symmetric about the origin, we must have $-f(-x)=f(x)$ or, equivalently, $f(-x)=-f(x)$. These results are summarized below.

Testing a Function for Symmetry

The graph of a function f is symmetric

- about the *y*-axis if and only if $f(-x)=f(x)$ for all *x* in the domain of f.
- about the origin if and only if $f(-x)=-f(x)$ for all *x* in the domain of f.

Apart from a very specialized family of functions which are both even and odd, functions fall into one of three distinct categories: even, odd, or neither even nor odd.

Example 1.2.3. Determine analytically if the following functions are even, odd, or neither even nor odd.

1. $f(x) = x^3 + 2x$
2. $g(x) = \dfrac{5}{2 - x^2}$
3. $h(x) = x^2 - \dfrac{x}{100} - 1$

Solution.

1. The first step in determining whether $f(x) = x^3 + 2x$ is even or odd is to replace x with $-x$ and simplify.

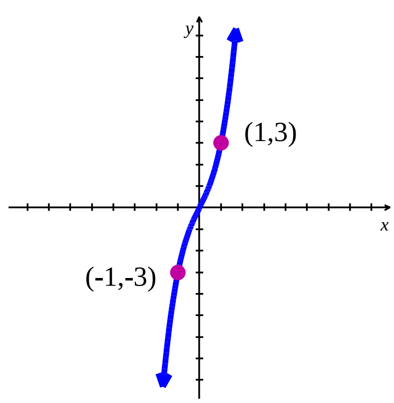
$$f(-x) = (-x)^3 + 2(-x)$$
$$f(-x) = -x^3 - 2x$$

It doesn't appear that $f(x) = f(-x)$. To prove this, we can check with an x value. After some trial and error, we see that $f(1) = 3$ and $f(-1) = -3$. This proves that f is not even, but it doesn't rule out the possibility that f is odd. To check if f is odd, we compare $f(-x)$ with $-f(x)$.

$$-f(x) = -(x^3 + 2x)$$
$$= -x^3 - 2x$$

Because $f(-x) = -f(x)$, f is an odd function. Notice that the graph, shown below, appears to be symmetric about the origin.

Figure 1.2. 16



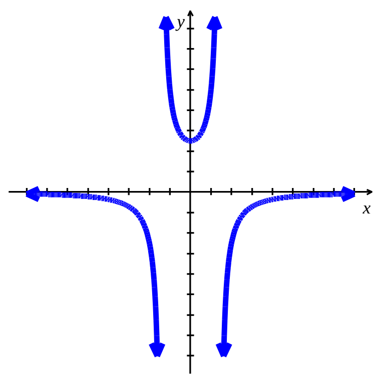
$y = f(x) = x^3 + 2x$

2. To determine if $g(x) = \dfrac{5}{2 - x^2}$ is even or odd, we begin by finding $g(-x)$.

$$g(-x)=\frac{5}{2-(-x)^2}$$
$$g(-x)=\frac{5}{2-x^2}$$

Since $g(-x)=g(x)$, g is even. The following graph of $g(x)$ matches this result in being symmetric about the *y*-axis. However, be cautious! As we will see in part 3 of this example, the appearance of symmetry cannot always be trusted. We must rely on the definition to verify symmetry.

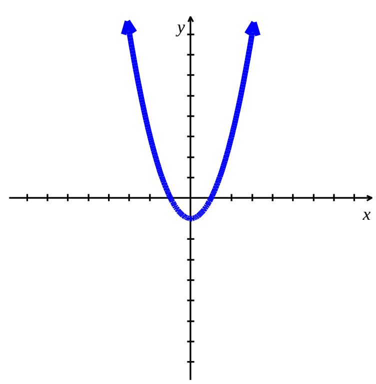
Figure 1.2. 17



$$y=g(x)=\frac{5}{2-x^2}$$

3. For this third example, to demonstrate the need for using the definition to verify symmetry, we begin by showing the graph of $h(x)=x^2-\frac{x}{100}-1$.

Figure 1.2. 18



$$y=h(x)=x^2-\frac{x}{100}-1$$

While the graph appears to represent an even function, we can draw no conclusions without verifying that $h(x)=h(-x)$. However, we can show that h is not even by providing a single value of x for which $h(x) \neq h(-x)$. If we let $x=1$, we get $h(1) = -\frac{1}{100}$ and $h(-1) = \frac{1}{100}$. Thus, $h(1) \neq h(-1)$. This rules out h as being even.

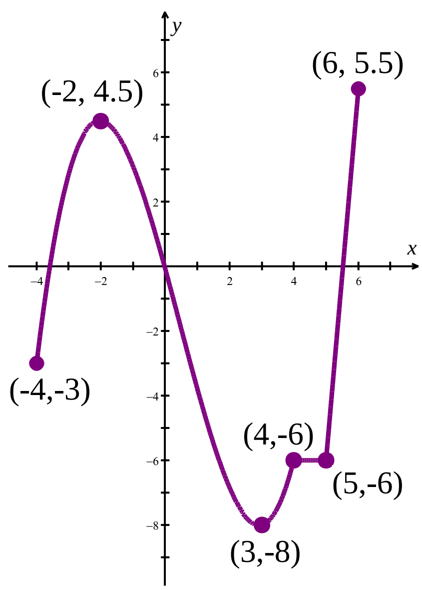
From the graph, it does not appear that h is odd. To verify this, we can again search for a single value of x for which $h(-x) \neq -h(x)$. Here, testing $x=2$ gives $h(-2) = \frac{151}{50}$ and $-h(2) = -\frac{149}{50}$, so h is not odd, either.

□

Determining Where a Function Is Increasing, Decreasing or Constant

Consider the graph of the following function.

Figure 1.2. 19



The graph of $y = f(x)$

Reading from left to right, the graph starts at the point $(-4,-3)$ and ends at the point $(6,\ 5.5)$. If we imagine walking from left to right on the graph then

- between $(-4,-3)$ and $(-2,\ 4.5)$, we are walking uphill;
- between $(-2,\ 4.5)$ and $(3,-8)$, we are walking downhill;
- between $(3,-8)$ and $(4,-6)$, we are walking uphill once more;

- from $(4,-6)$ to $(5,-6)$, we level off;
- from $(5,-6)$ to $(6,\ 5.5)$, we resume walking uphill.

In other words, for the *x* values between -4 and -2, the *y*-coordinates on the graph are getting larger, or **increasing**, as we move from left to right. Since $y=f(x)$, the *y* values on the graph are the function values, and we say that the function f is **increasing** on the interval $(-4,-2)$. Analogously, we way that f is **decreasing** on the interval $(-2,3)$, increasing once more on the interval $(3,4)$, **constant** on the interval $(4,5)$, and finally increasing once again on the interval $(5,6)$.

It is extremely important to notice that the behavior (increasing, decreasing or constant) occurs on an interval on the *x*-axis. When we say that the function f is increasing on $(-4,-2)$, we mean for *x* values between −4 and −2 and we do not mention the actual *y* values along the way. Thus, we report where the behavior occurs, not to what extent the behavior occurs.[13]

We are now ready for the more formal algebraic definitions of what it means for a function to be increasing, decreasing or constant.

Definition 1.6. Suppose f is a function defined on an open interval I. We say f is

- **increasing** on I if and only if, for all real numbers *a* and *b* in I with $a<b$, $f(a)<f(b)$.
- **decreasing** on I if and only if, for all real numbers *a* and *b* in I with $a<b$, $f(a)>f(b)$.
- **constant** on I if and only if, for all real numbers *a* and *b* in I, $f(a)=f(b)$.

It is worth taking some time to see that the algebraic descriptions of increasing, decreasing and constant agree with our graphical descriptions earlier.

Maximum and Minimum Function Values

Now let’s turn our attention to a few of the points on the graph of f. Clearly the point $(-2,\ 4.5)$ does not have the largest *y* value. Indeed, that honor goes to $(6,\ 5.5)$. But $(-2,\ 4.5)$ should get some sort of consolation prize for being at the ‘top of the hill’ between $x=-4$ and $x=3$. We say that the function f has a **local maximum**[14] at the point $(-2,\ 4.5)$ because the *y*-coordinate 4.5 is the largest *y* value (hence,

[13] The notion of how quickly or how slowly a function increases or decreases is explored in Calculus.
[14] Also called a **relative maximum**.

function value) on the curve near $x=-2$. We say that this local maximum value is 4.5. Similarly, the function f has a **local minimum**[15] at the point $(3,-8)$ since the *y*-coordinate -8 is the smallest function value near $x=3$. That local minimum value is -8. As we will see in the next definition, we will not classify the endpoints $(-4,-3)$ and $(6,\ 5.5)$ as local minimum or local maximum points.

Some important terminology to become familiar with is **maxima**, which is the plural of 'maximum', and **minima**, the plural of 'minimum'. **Extrema** is the plural of **extremum**, which combines 'maximum' and 'minimum'. We have one last observation to make before we proceed to the algebraic definitions and look at a fairly tame, yet helpful, example.

If we look at the entire graph of f, we see that the largest *y* value (the largest function value) is 5.5 at $x=6$. In this case, we say the **absolute maximum value**[16] of f is 5.5. Similarly, the **absolute minimum value**[17] of f is -8.

We formalize these concepts in the following definitions.

Definition 1.7. Suppose f is a function with $f(a)=b$.

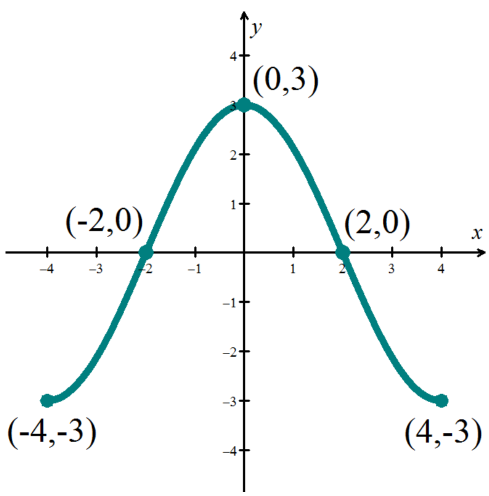
- We say the point (a,b) is a **local maximum** point of f if there is an open interval I containing *a* for which $f(a)\geq f(x)$ for all *x* in I. The value $f(a)=b$ is called a local maximum value of f.
- We say the point (a,b) is a **local minimum** point of f if there is an open interval I containing *a* for which $f(a)\leq f(x)$ for all *x* in I. The value $f(a)=b$ is called a local minimum value of f.
- The value $f(a)=b$ is called the **absolute maximum value** of f if $b\geq f(x)$ for all *x* in the domain of f.
- The value $f(a)=b$ is called the **absolute minimum value** of f if $b\leq f(x)$ for all *x* in the domain of f.

[15] Also called a **relative minimum**.
[16] Absolute maximum is sometimes called **global maximum**.
[17] Absolute miminum is sometimes called **global minimum**.

As mentioned earlier, the above definition does not allow for a local maximum, or local minimum, to occur at the end point of an interval since we require $f(a)$ to be the largest, or smallest, value of the function on an open interval containing *a*. Otherwise stated, $f(a)$ must be the largest, or smallest, value of the function for *x* values on both sides of *a*. However, an endpoint may be the point where an absolute maximum or absolute minimum value occurs. This concept is addressed in the following example, along with a summary of other concepts from this, and the previous, section.

Example 1.2.4. Given the graph of $y = f(x)$ below, answer all of the following questions.

Figure 1.2. 20



1. Find the domain of f.
2. Find the range of f.
3. List the *x*-intercepts, if any exist.
4. List the *y*-intercept, if any exists.
5. Find the zeros of f.
6. Solve $f(x) < 0$.
7. Determine $f(2)$.
8. Solve $f(x) = -3$.
9. Find the number of solutions to $f(x) = 1$
10. Does f appear to be even, odd or neither?
11. List the intervals on which f is increasing.
12. List the intervals on which f is decreasing.
13. List the local maximums, if any exist.
14. List the local minimums, if any exist.
15. Find the absolute maximum, if it exists.
16. Find the absolute minimum, if it exists.

Solution.

1. To find the domain of f, we proceed as in **Section 1.1**. By projecting the graph to the *x*-axis, we see that the portion of the *x*-axis which corresponds to a point on the graph is everything from -4 to 4, inclusive. Hence, the domain is $[-4,4]$.

2. To find the range, we project the graph to the *y*-axis. We see that the *y* values from -3 to 3, inclusive, constitute the range of f. Hence, our answer is $[-3,3]$.

3. The *x*-intercepts are the points on the graph with *y*-coordinate 0, namely $(-2,0)$ and $(2,0)$.

4. The *y*-intercept is the point on the graph with *x*-coordinate 0, namely $(0,3)$.

5. The zeros of f are the *x*-coordinates of the *x*-intercepts on the graph of $y = f(x)$, which are $x = -2$ and $x = 2$.

6. To solve $f(x) < 0$, we look for the *x* values of the points on the graph where the *y*-coordinate is less than 0. Graphically, we are looking for where the graph is below the *x*-axis. This happens at $x = -4$, at $x = 4$, and for *x* values between -4 and -2, then again between 2 and 4. Our answer is $[-4,-2) \cup (2,4]$.

7. Since the graph of f is the graph of the equation $y = f(x)$, $f(2)$ is the *y*-coordinate of the point which corresponds to $x = 2$. Noting that the point $(2,0)$ is on the graph, we have $f(2) = 0$.

8. To solve $f(x) = -3$, we look where $y = f(x) = -3$. We find two points with a *y*-coordinate of -3, namely $(-4,-3)$ and $(4,-3)$. Hence, the solutions to $f(x) = -3$ are $x = \pm 4$.

9. As in the previous problem, to solve $f(x) = 1$, we look for points on the graph where the *y*-coordinate is 1. Even though these points aren't specified, we see that the curve has two points with a *y* value of 1, as shown in the graph below. That means there are two solutions to $f(x) = 1$.

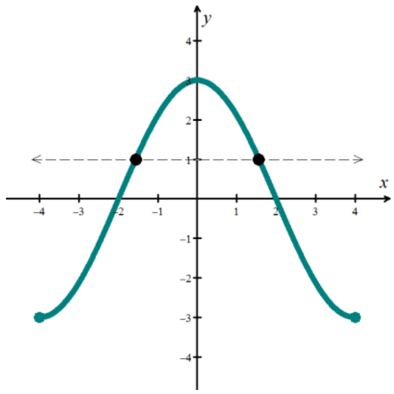


Figure 1.2. 21

10. The graph appears to be symmetric about the *y*-axis. This suggests (but does not prove) that f is even.

11. As we move from left to right, the graph rises from $(-4,-3)$ to $(0,3)$. This means f is increasing on the interval $(-4,0)$. (Remember, the answer here is an interval on the *x*-axis.)

12. As we move from left to right, the graph falls from $(0,3)$ to $(4,-3)$. This means f is decreasing on the interval $(0,4)$. (Remember, the answer here is an interval on the *x*-axis.)

13. The function has its only local maximum at $(0,3)$, so $f(0)=3$ is the local maximum value.

14. There are no local minimums. Why don't $(-4,-3)$ and $(4,-3)$ count? Let's consider the point $(-4,-3)$ for a moment. Recall that, in the definition of local minimum, there needs to be an open interval I which contains $x=-4$ such that $f(-4)\le f(x)$ for all *x* in I. But if we put an open interval around $x=-4$, a portion of that interval will be outside of the domain of f. Because we are unable to fulfill the requirements of the definition for a local minimum, we cannot claim that f has one at $(-4,-3)$. The point $(4,-3)$ fails for the same reason. No open interval around $x=4$ stays within the domain of f.

15. The absolute maximum value of f is the largest *y*-coordinate, which is 3.

16. The absolute minimum value of f is the smallest *y*-coordinate, which is -3.

□

Analyzing the Graphical Behavior of the Toolkit Functions

We end this section by revisiting the toolkit functions. We determine whether these functions are even or odd, where their graphs are increasing or decreasing, and locations of local and absolute extrema.

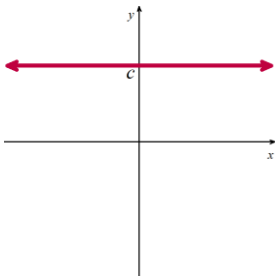


Figure 1.2. 22

Constant Function

$f(x) = c$

Even

Neither increasing nor decreasing

By definition, local and absolute extrema occur at all *x* values. This maximum/minimum value is *c*.

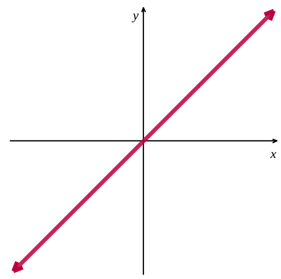


Figure 1.2. 23

Identity Function

$f(x) = x$

Odd

Increasing on $(-\infty, \infty)$

There is no local or absolute extremum.

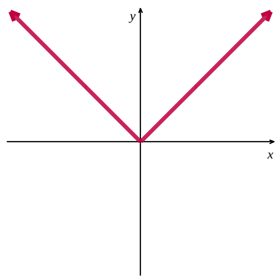


Figure 1.2. 24

Absolute Value Function

$f(x) = |x|$

Even

Increasing on $(0, \infty)$

Decreasing on $(-\infty, 0)$

A local and absolute minimum value of 0 occurs at $x = 0$.

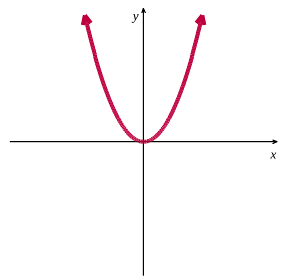


Figure 1.2. 25

Quadratic Function

$f(x) = x^2$

Even

Increasing on $(0, \infty)$

Decreasing on $(-\infty, 0)$

A local and absolute minimum value of 0 occurs at $x = 0$.

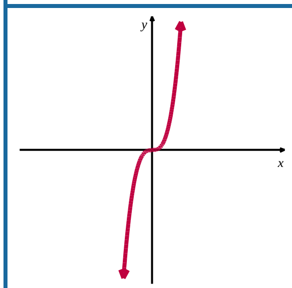


Figure 1.2. 26

Cubic Function

$f(x) = x^3$

Odd

Increasing on $(-\infty, \infty)$

There is no local or absolute extremum.

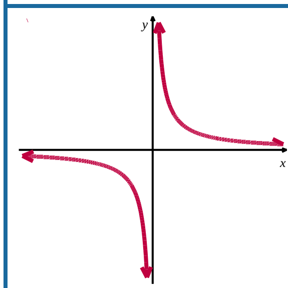


Figure 1.2. 27

Reciprocal Function

$f(x) = \frac{1}{x}$

Odd

Decreasing on $(-\infty, 0) \cup (0, \infty)$

There is no local or absolute extremum.

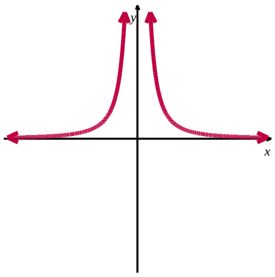


Figure 1.2. 28

Reciprocal Squared Function

$$f(x)=\frac{1}{x^2}$$

Even

Increasing on $(-\infty,0)$

Decreasing on $(0,\infty)$

There is no local or absolute extremum.

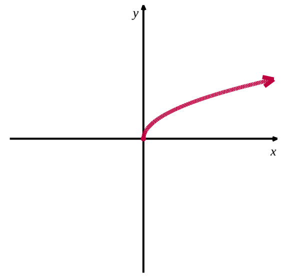


Figure 1.2. 29

Square Root Function

$$f(x)=\sqrt{x}$$

Neither Even nor Odd

Increasing on $(0,\infty)$

There is no local extremum and no absolute maximum. An absolute minimum value of 0 occurs at $x=0$.

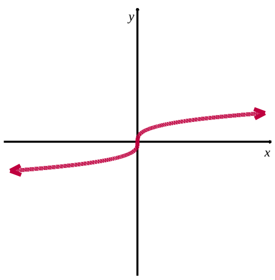


Figure 1.2. 30

Cube Root Function

$$f(x)=\sqrt[3]{x}$$

Odd

Increasing on $(-\infty,\infty)$

There is no local or absolute extremum.

1.2 Exercises

1. How can you determine whether a function is odd or even from the formula for the function?

2. How are the absolute maximum and minimum similar to and different from the local extrema?

3. Compute the following function values for $f(x)=\begin{cases} x+5 & \text{if} \quad x\le -3 \\ \sqrt{9-x^2} & \text{if} \quad -3<x\le 3 \\ -x+5 & \text{if} \quad x>3 \end{cases}$

(a) $f(-4)$ (b) $f(-3)$ (c) $f(3)$

(d) $f(3.001)$ (e) $f(-3.001)$ (f) $f(2)$

4. Compute the following function values for $f(x)=\begin{cases} x^2 & \text{if} \quad x\le -1 \\ \sqrt{1-x^2} & \text{if} \quad -1<x\le 1 \\ x & \text{if} \quad x>1 \end{cases}$

(a) $f(4)$ (b) $f(-3)$ (c) $f(1)$

(d) $f(0)$ (e) $f(-1)$ (f) $f(-0.999)$

In Exercises 5 – 16, sketch the graph of the given piecewise-defined function.

5. $f(x)=\begin{cases} x+1 & \text{if } x<-2 \\ -2x-3 & \text{if } x\ge -2 \end{cases}$

6. $f(x)=\begin{cases} 2x-1 & \text{if } x<1 \\ 1+x & \text{if } x\ge 1 \end{cases}$

7. $f(x)=\begin{cases} x+1 & \text{if } x<0 \\ x-1 & \text{if } x>0 \end{cases}$

8. $f(x)=\begin{cases} 3 & \text{if } x<0 \\ \sqrt{x} & \text{if } x\ge 0 \end{cases}$

9. $f(x)=\begin{cases} x^2 & \text{if } x<0 \\ 1-x & \text{if } x>0 \end{cases}$

10. $f(x)=\begin{cases} x^2 & \text{if } x<0 \\ x+2 & \text{if } x\ge 0 \end{cases}$

11. $f(x)=\begin{cases} x+1 & \text{if } x<1 \\ x^3 & \text{if } x\ge 1 \end{cases}$

12. $f(x)=\begin{cases} |x| & \text{if } x<2 \\ 1 & \text{if } x\ge 2 \end{cases}$

13. $f(x)=\begin{cases} x^2 & \text{if } x\le 0 \\ 2x & \text{if } x>0 \end{cases}$

14. $f(x)=\begin{cases} -3 & \text{if} \quad x<0 \\ 2x-3 & \text{if} \quad 0\le x\le 3 \\ 3 & \text{if} \quad x>3 \end{cases}$

15. $f(x)=\begin{cases} x^2 & \text{if} \quad x \le -2 \\ 3-x & \text{if} \quad -2 < x < 2 \\ 4 & \text{if} \quad x \ge 2 \end{cases}$

16. $f(x)=\begin{cases} \frac{1}{x} & \text{if} \quad -6 < x < -1 \\ x & \text{if} \quad -1 < x < 1 \\ \sqrt{x} & \text{if} \quad 1 < x < 9 \end{cases}$

In Exercises 17 – 37, determine analytically if the following functions are even, odd or neither.

17. $f(x)=7x$

18. $f(x)=7x+2$

19. $f(x)=7$

20. $f(x)=3x^2-4$

21. $f(x)=4-x^2$

22. $f(x)=x^2-x-6$

23. $f(x)=2x^3-x$

24. $f(x)=-x^5+2x^3-x$

25. $f(x)=x^6-x^4+x^2+9$

26. $f(x)=x^3+x^2+x+1$

27. $f(x)=\sqrt{1-x}$

28. $f(x)=\sqrt{1-x^2}$

29. $f(x)=0$

30. $f(x)=\sqrt[3]{x}$

31. $f(x)=\sqrt[3]{x^2}$

32. $f(x)=\frac{3}{x^2}$

33. $f(x)=\frac{2x-1}{x+1}$

34. $f(x)=\frac{3x}{x^2+1}$

35. $f(x)=\frac{x^2-3}{x-4x^3}$

36. $f(x)=\frac{9}{\sqrt{4-x^2}}$

37. $f(x)=\frac{\sqrt[3]{x^3+x}}{5x}$

In Exercises 38 – 41, use the graph of the function to estimate the intervals on which the function is increasing or decreasing.

38.

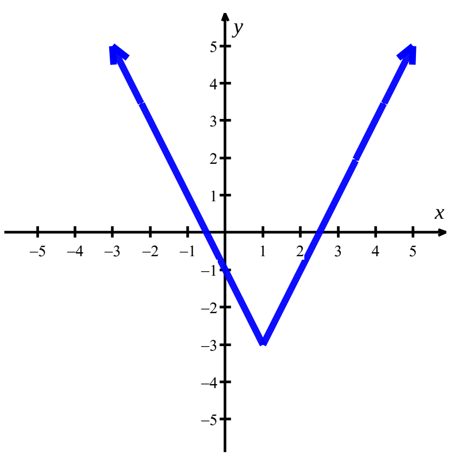


Figure 1.2. 31

39.

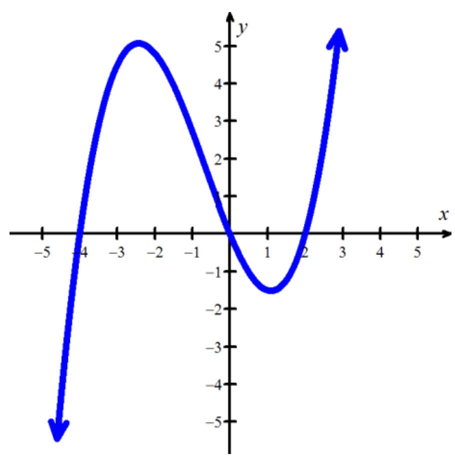


Figure 1.2. 32

40.

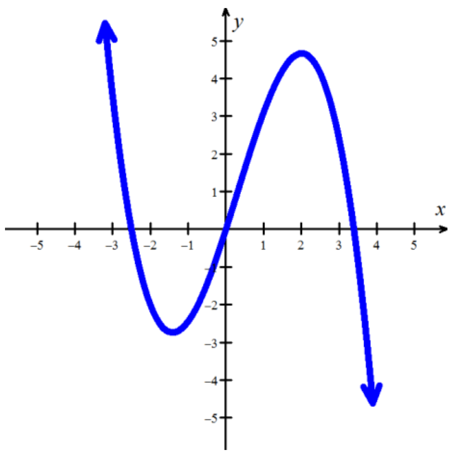


Figure 1.2. 33

41.

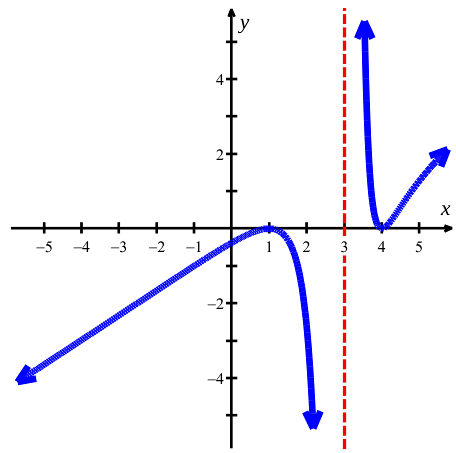
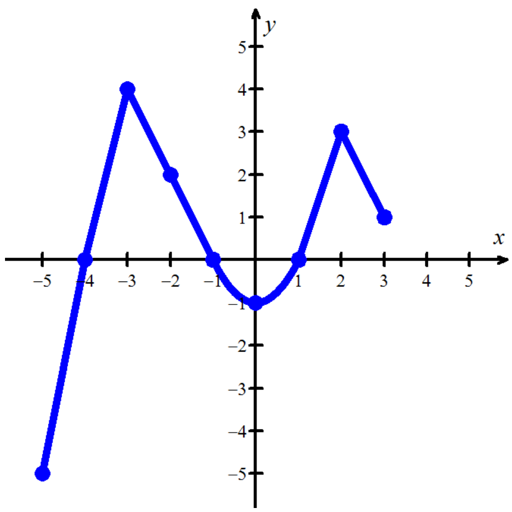


Figure 1.2. 34

In Exercises 42 – 57, use the graph of $y = f(x)$ given below to answer the question.

Figure 1.2. 35



$y = f(x)$

42. Find the domain of f.

43. Find the range of f.

44. Determine $f(-2)$.

45. Solve $f(x) = 4$.

46. List the *x*-intercepts, if any exist.

47. List the *y*-intercept, if any exists.

48. Find the zeros of f.

49. Solve $f(x) \geq 0$.

50. Find the number of solutions to $f(x) = 1$.

51. Does f appear to be even, odd or neither?

52. List the intervals where f is increasing.

53. List the intervals where f is decreasing.

54. List the local maximums, if any exist.

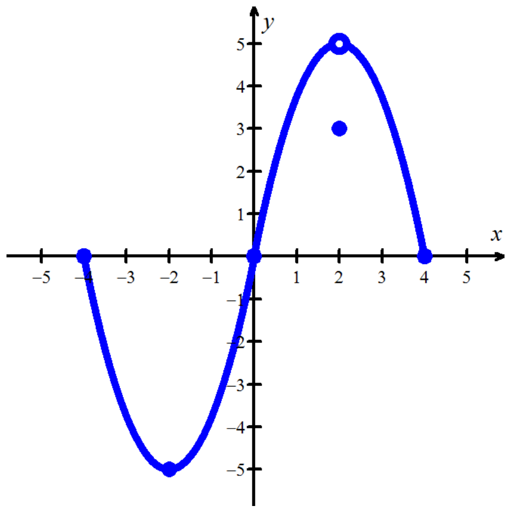
55. List the local minimums, if any exist.

56. Find the absolute maximum, if it exists.

57. Find the absolute minimum, if it exists.

In Exercises 58 – 73, use the graph of $y = f(x)$ given below to answer the question.

Figure 1.2. 36



$y = f(x)$

58. Find the domain of f.

59. Find the range of f.

60. Determine $f(2)$.

61. Solve $f(x) = -5$.

62. List the *x*-intercepts, if any exist.

63. List the *y*-intercept, if any exists.

64. Find the zeros of f.

65. Solve $f(x) \le 0$.

66. Find the number of solutions to $f(x) = 3$.

67. Does f appear to be even, odd or neither?

68. List the intervals where f is increasing.

69. List the intervals where f is decreasing.

70. List the local maximums, if any exist.

71. List the local minimums, if any exist.

72. Find the absolute maximum, if it exists.

73. Find the absolute minimum, if it exists.

74. The area *A* enclosed by a square, in square inches, is a function of the length of one of its sides *x*, when measured in inches. This relation is expressed by the formula $A(x) = x^2$ for $x > 0$. Find $A(3)$ and solve $A(x) = 36$. Interpret your answers to each. Why is *x* restricted to $x > 0$?

75. The area A enclosed by a circle, in square meters, is a function of its radius r, when measured in meters. This relation is expressed by the formula $A(r)=\pi r^2$ for $r>0$. Find $A(2)$ and solve $A(r)=16\pi$. Interpret your answers to each. Why is r restricted to $r>0$?

76. The volume V enclosed by a cube, in cubic centimeters, is a function of the length of one of its sides x, when measured in centimeters. This relation is expressed by the formula $V(x)=x^3$ for $x>0$. Find $V(5)$ and solve $V(x)=27$. Interpret your answers to each. Why is x restricted to $x>0$?

77. The volume V enclosed by a sphere, in cubic feet, is a function of the radius of the sphere r, when measured in feet. This relation is expressed by the formula $V(r)=\frac{4\pi}{3}r^3$ for $r>0$. Find $V(3)$ and solve $V(r)=\frac{32\pi}{3}$. Interpret your answers to each. Why is r restricted to $r>0$?

78. The height of an object dropped from the roof of an eight story building is modeled by $h(t)=-16t^2+64$, $0\le t\le 2$. Here, h is the height of the object off the ground, in feet, t seconds after the object is dropped. Find $h(0)$ and solve $h(t)=0$. Interpret your answers to each. Why is t restricted to $0\le t\le 2$?

79. The temperature T in degrees Fahrenheit t hours after 6 AM is given by $T(t)=-\frac{1}{2}t^2+8t+3$ for $0\le t\le 12$. Find and interpret $T(0)$, $T(6)$ and $T(12)$.

80. The function $C(x)=x^2-10x+27$ models the cost, in hundreds of dollars, to produce x thousand pens. Find and interpret $C(0)$, $C(2)$ and $C(5)$.

81. Using data from the Bureau of Transportation Statistics, the average fuel economy F in miles per gallon for passenger cars in the US can be modeled by $F(t)=-0.0076t^2+0.45t+16$, $0\le t\le 28$, where t is the number of years since 1980. Use your calculator to find $F(0)$, $F(14)$ and $F(28)$. Round your answers to two decimal places and interpret your answers to each.

82. The population of Sasquatch in Portage County can be modeled by the function $P(t)=\frac{150t}{t+15}$, where t represents the number of years since 1803. Find and interpret $P(0)$ and $P(205)$. Discuss with your classmates what the applied domain and range of P should be.

83. For *n* copies of the book *Me and my Sasquatch*, a print on demand company charges $C(n)$ dollars, where $C(n)$ is determined by the formula

$$C(n)=\begin{cases} 15n & \text{if} \quad 1\le n\le 25 \\ 13.50n & \text{if} \quad 25<n\le 50 \\ 12n & \text{if} \quad n>50 \end{cases}$$

(a) Find and interpret $C(20)$.

(b) How much does it cost to order 50 copies of the book? What about 51 copies?

(c) Your answer to part (b) should get you thinking. Suppose a bookstore estimates it will sell 50 copies of the book. How many books can, in fact, be ordered for the same price as those 50 copies? (Round your answer to a whole number of books.)

84. An on-line comic book retailer charges shipping costs according to the following formula

$$S(n)=\begin{cases} 1.5n+2.5 & \text{if} \; 1\le n\le 14 \\ 0 & \text{if} \; n\ge 15 \end{cases}$$

where *n* is the number of comic books purchased and $S(n)$ is the shipping cost in dollars.

(a) What is the cost to ship 10 comic books?

(b) What is the significance of the formula $S(n)=0$ for $n\ge 15$?

85. The cost *C* (in dollars) to talk *m* minutes a month on a mobile phone plan is modeled by

$$C(m)=\begin{cases} 25 & \text{if} \; 0\le m\le 1000 \\ 25+0.1(m-1000) & \text{if} \; \text{m>1000} \end{cases}$$

(a) How much does it cost to talk 750 minutes per month with this plan?

(b) How much does it cost to talk 20 hours a month with this plan?

(c) Explain the terms of the plan in words.

86. We define the set of **integers** as $\mathbb{Z}=\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$.[18] The **greatest integer of *x***, denoted by $\lfloor x \rfloor$, is defined to be the largest integer *k* with $k\le x$.

(a) Find $\lfloor 0.785 \rfloor$, $\lfloor 117 \rfloor$, $\lfloor -2.001 \rfloor$ and $\lfloor \pi+6 \rfloor$.

[18] The use of the letter $\mathbb{Z}$ for the integers is ostensibly because the German word *zahlen* means 'to count'.

(b) Discuss with your classmates how $\lfloor x \rfloor$ may be described as a piecewise defined function.

HINT: There are infinitely many pieces!

(c) Is $\lfloor a+b \rfloor = \lfloor a \rfloor + \lfloor b \rfloor$ always true? What if *a* or *b* is an integer? Test some values, make a conjecture, and explain your results.

87. Let $f(x) = \lfloor x \rfloor$ be the greatest integer function as defined in the last exercise.

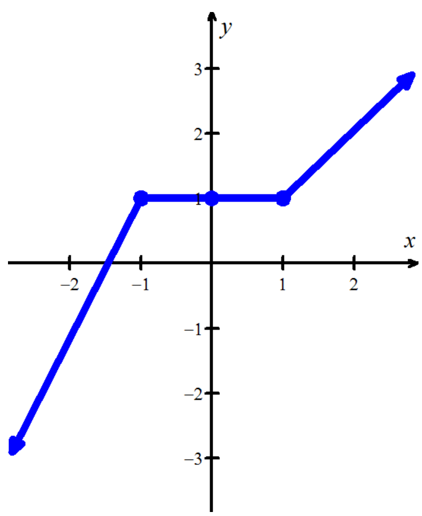
(a) Graph $y = f(x)$. Be careful to correctly describe the behavior of the graph near the integers.

(b) Is f even, odd or neither? Explain.

(c) Discuss with your classmates which points on the graph are local minimums, local maximums, or both. If f ever increasing? Decreasing? Constant?

88. Consider the graph of the function f given below.

Figure 1.2. 37



$y = f(x)$

Refer back to **Definition 1.6** and **Definition 1.7** before answering the following.

(a) Show that f has a local maximum but not a local minimum at the point $(-1,1)$.

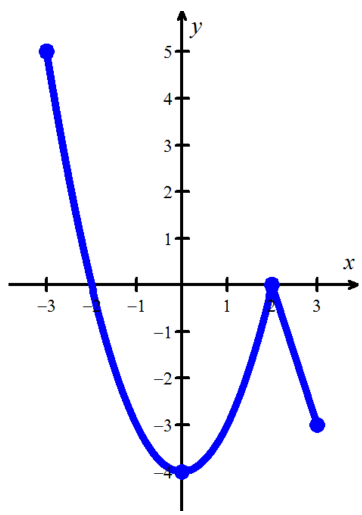
(b) Show that f has a local minimum but not a local maximum at the point $(1,1)$.

(c) Show that f has a local maximum AND a local minimum at the point $(0,1)$.

(d) Show that f is constant on the interval $(-1,1)$ and thus has both a local maximum AND a local minimum at every point $(x, f(x))$ where $-1<x<1$.

89. Using **Example 1.2.4** as a guide, show that the function g whose graph is given below does not have a local maximum at $(-3,5)$; nor does it have a local minimum at $(3,-3)$. Find its extrema, both local and absolute. What's unique about the point $(0,-4)$ on this graph? Also find the intervals on which g is increasing and the intervals on which g is decreasing.

Figure 1.2. 38



$y=g(x)$

1.3 Transformations of Functions

Learning Objectives

- Graph functions using vertical and horizontal shifts.
- Graph functions using reflections about the x-axis and the y-axis.
- Graph functions using vertical and horizontal scalings.
- Graph functions using a combination of transformations.

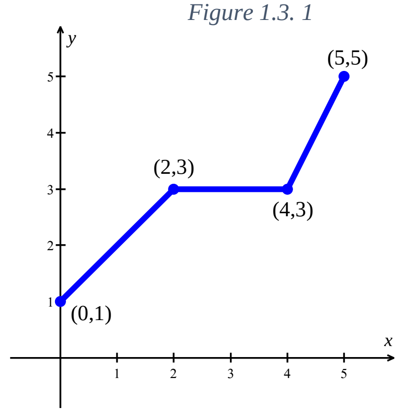
In this section, we study how the graphs of functions change, or **transform**, when certain modifications are made to their inputs or outputs. Transformations fall into two broad categories: **rigid transformations** and **non-rigid transformations**. Rigid transformations, which include **shifts** and **reflections**, do not change the shape of the original graph, only its position and orientation in the plane. Non-rigid transformations, which include scalings, change the shape of the graph. Both types of transformations may affect the domain and/or range of the function.

Adding, subtracting or multiplying the inputs or outputs of a function by a constant are the types of transformations we will discuss here. We will start with these changes to the output.

Vertical Shifts

Example 1.3.1. Use the function $y = f(x)$, graphed below, to graph the function $y = f(x) + 2$.

Figure 1.3. 1



$y = f(x)$

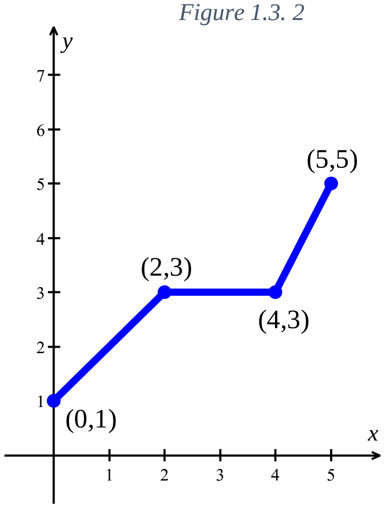
Solution. To evaluate $y = f(x) + 2$ at each x value, we take the output of the original function, $f(x)$, and add 2 to it. The following table shows what's happening.

Input x	Output $y=f(x)$	Coordinates on graph of $y=f(x)$
0	1	$(0,1)$
2	3	$(2,3)$
4	3	$(4,3)$
5	5	$(5,5)$

Input x	Output $y=f(x)+2$	Coordinates on graph of $y=f(x)+2$
0	$1+2=3$	$(0,3)$
2	$3+2=5$	$(2,5)$
4	$3+2=5$	$(4,5)$
5	$5+2=7$	$(5,7)$

The *y*-coordinate of each point on the graph of $y=f(x)+2$ is two units more than the *y*-coordinate of $y=f(x)$. Geometrically, adding 2 to the *y*-coordinate of a point moves the point 2 units above its previous location.

Figure 1.3. 2



$y=f(x)$

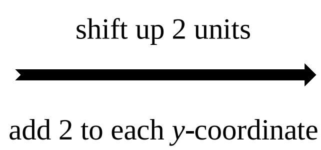
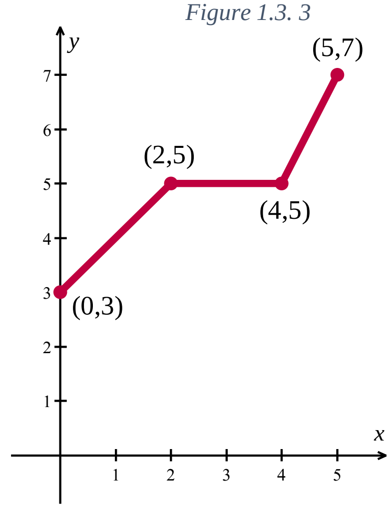


Figure 1.3. 3



$y=f(x)+2$

□

Adding 2 to the *y*-coordinate of every point on a graph is usually described as 'shifting the graph up 2 units'. As seen from the graphs, this transformation does not affect the domain, but it does affect the range; the range of $y=f(x)$ is $[1,5]$ and the range of $y=f(x)+2$ is $[3,7]$. Since the two graphs have the same shape, this is a rigid transformation.

The same logic explains the transformation that is required to obtain the graph of $y=f(x)-2$ from the graph of $y=f(x)$. Instead of adding 2 to the *y*-coordinates on the graph of $y=f(x)$, we'd be subtracting 2. Geometrically, we would be moving the graph down 2 units and the new range would be $[-1,3]$. What we have discussed is generalized below.

Vertical Shift

Suppose f is a function and D is a constant. To graph $y=f(x)+D$, shift the graph of $y=f(x)$ vertically by adding D to the *y*-coordinates of the points on the graph of f. If D is positive, the graph will shift up. If D is negative, the graph will shift down.

A vertical shift is a rigid transformation that only affects the range.

Vertical Reflections

Example 1.3.2. Use the function $y=f(x)$ from **Example 1.3.1** to graph $y=-f(x)$.

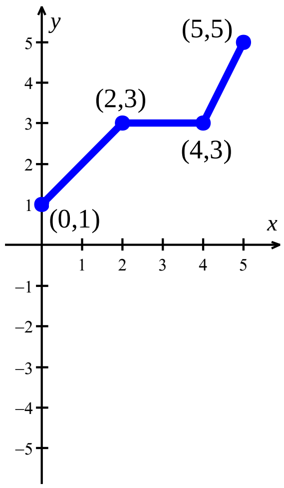
Solution. To evaluate $y=-f(x)$ at each *x* value, we take the output of the original function, $f(x)$, and multiply it by -1. The following table shows what's happening.

Input x	Output $y=f(x)$	Coordinates on graph of $y=f(x)$
0	1	$(0,1)$
2	3	$(2,3)$
4	3	$(4,3)$
5	5	$(5,5)$

Input x	Output $y=-f(x)$	Coordinates on graph of $y=-f(x)$
0	-1	$(0,-1)$
2	-3	$(2,-3)$
4	-3	$(4,-3)$
5	-5	$(5,-5)$

For every value of *x* in the domain, $y=-f(x)$ is opposite in sign from $y=f(x)$. Geometrically, multiplying the *y*-coordinate of a point by -1 reflects that point across the *x*-axis.

Figure 1.3. 4

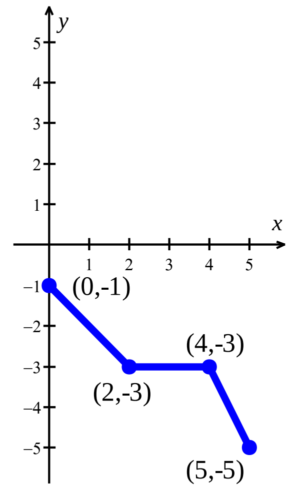


$y = f(x)$

reflect across x-axis

multiply each y-coordinate by -1

Figure 1.3. 5



$y = -f(x)$

□

Multiplying the y-coordinate of every point on a graph by -1 is usually described as 'reflecting the graph across the x-axis'. As seen from the graphs, this transformation does not affect the domain, but it does affect the range; the range of $y = f(x)$ is $[1,5]$ and the range of $y = -f(x)$ is $[-5,-1]$. The two graphs have the same shape, verifying that this is a rigid transformation.

Vertical Reflections

Suppose f is a function. To graph $y = -f(x)$, reflect the graph of $y = f(x)$ across the x-axis by multiplying the y-coordinates of the points on the graph of $y = f(x)$ by -1.

A vertical reflection is a rigid transformation that only affects the range.

Vertical Scalings

Example 1.3.3. Use the function $y = f(x)$ from **Example 1.3.1** to graph $y = 2f(x)$.

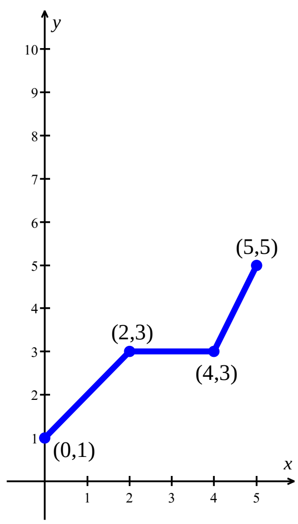
Solution. To evaluate $y = 2f(x)$ at each x value, we take the output of the original function $f(x)$ and multiply it by 2. The following table shows what is happening.

Input x	Output $y = f(x)$	Coordinates on graph of $y = f(x)$
0	1	$(0,1)$
2	3	$(2,3)$
4	3	$(4,3)$
5	5	$(5,5)$

Input x	Output $y = 2f(x)$	Coordinates on graph of $y = 2f(x)$
0	$2 \cdot 1 = 2$	$(0,2)$
2	$2 \cdot 3 = 6$	$(2,6)$
4	$2 \cdot 3 = 6$	$(4,6)$
5	$2 \cdot 5 = 10$	$(5,10)$

For each value of *x* in the domain, $y = 2f(x)$ is twice as large as $y = f(x)$. Geometrically, multiplying the *y*-coordinate of a point by the positive number 2 stretches the graph in the *y* direction. This is known as a 'vertical scaling by a factor of 2' and the result is shown below.

Figure 1.3. 6

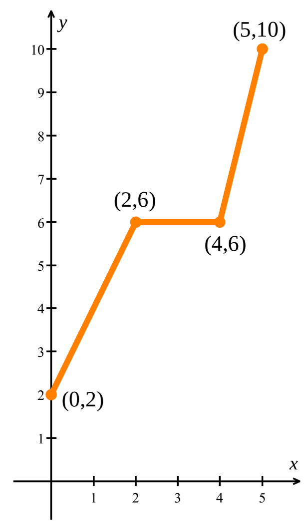


$y = f(x)$

vertical scaling by a factor of 2

multiply each *y*-coordinate by 2

Figure 1.3. 7



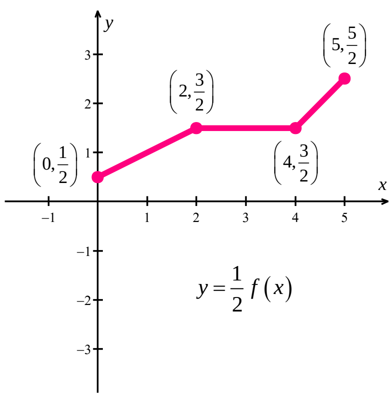
$y = 2f(x)$

□

As seen from the graphs, this transformation does not affect the domain, but it does affect the range; the range of $y = f(x)$ is $[1,5]$ and the range of $y = 2f(x)$ is $[2,10]$. We note that the two graphs do not have the same shape, so this is a non-rigid transformation.

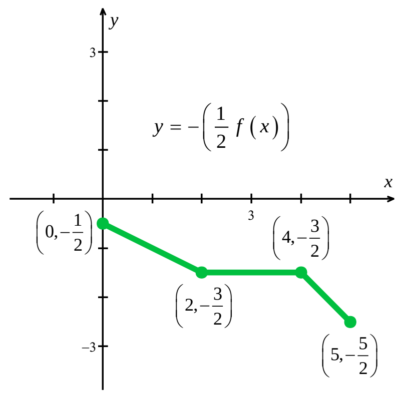
To graph $y = \frac{1}{2} f(x)$, we would multiply all of the *y*-coordinates of the points on the graph of f by $\frac{1}{2}$, resulting in a compression in the *y* direction, or a vertical scaling by a factor of $\frac{1}{2}$. To graph $y = -\frac{1}{2} f(x)$, we multiply all of the *y*-coordinates of the points on the graph of f by $-\frac{1}{2}$. We can rewrite $y=-\frac{1}{2}f(x)$ as $y=-\left(\frac{1}{2}f(x)\right)$. Geometrically, we think of this as two operations: first perform a vertical scaling by a factor of $\frac{1}{2}$ and then reflect the resulting graph across the *x*-axis.

Figure 1.3. 8



vertical scaling by factor of $\frac{1}{2}$

Figure 1.3. 9



reflection across *x*-axis

These results are generalized below.

Vertical Scalings

Suppose f is a function and A is a nonzero constant. To graph $y=Af(x)$, multiply all of the *y*-coordinates of the points on the graph of $y=f(x)$ by A.

- If $A>0$, we say the graph of f has been vertically scaled (stretched if $A>1$ and compressed if $0<A<1$) by a factor of A.
- If $A<0$, the graph of f is both vertically scaled and reflected across the *x*-axis.

A vertical scaling is a non-rigid transformation that only affects the range.

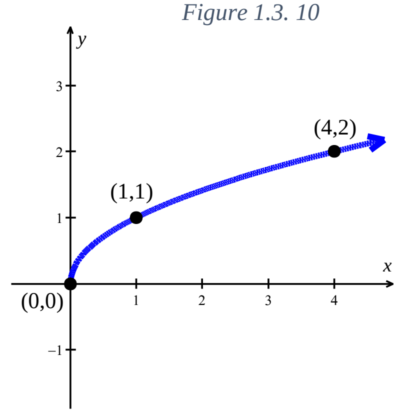
It is sometimes necessary to apply multiple transformations, as in the following example. Here, we limit our transformations to those affecting the output of a function f.

Example 1.3.4. Graph $f(x)=\sqrt{x}$ and plot at least three points. Use transformations to graph $g(x)=2\sqrt{x}-1$. State the domain and range of g.

Solution. Owing to the square root, the domain of $f(x)=\sqrt{x}$ is $x\geq 0$, or $[0,\infty)$. We choose perfect squares to build our table and graph below.[19]

x	$f(x)=\sqrt{x}$	$(x, f(x))$
0	0	$(0,0)$
1	1	$(1,1)$
4	2	$(4,2)$

Figure 1.3. 10



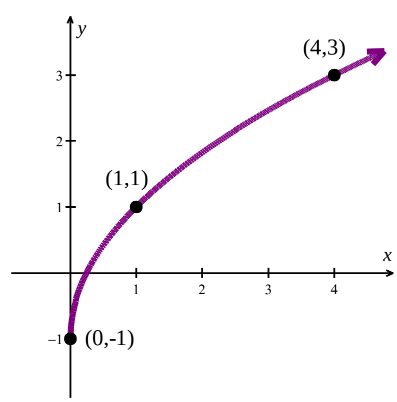
$y=f(x)=\sqrt{x}$

From the graph, we can verify that the domain of f is $[0,\infty)$ and the range of f is also $[0,\infty)$.

There are two transformations of f in the function $g(x)=2\sqrt{x}-1$. The order we perform the two transformations in is simply the order of operations in calculating $g(x)=2f(x)-1$. First, we multiply $f(x)=\sqrt{x}$ by 2 and then subtract 1. Multiplication by 2 results in a vertical stretch while subtracting 1 results in a shift down by one unit.

x	$f(x)$	$g(x)=2f(x)-1$	$(x, g(x))$
0	0	$2(0)-1=-1$	$(0,-1)$
1	1	$2(1)-1=1$	$(1,1)$
4	2	$2(2)-1=3$	$(4,3)$

Figure 1.3. 11



$y=g(x)=2\sqrt{x}-1$

The domain of g is the same as f, $[0,\infty)$, but the range of g is $[-1,\infty)$, which is different than the range of f.

□

[19] Recall that we graphed this function earlier as one of the toolkit functions.

Notice that the order of transformations does matter, since $2\left(\sqrt{x}-1\right) \neq 2\sqrt{x}-1$. For correct order of transformations, simply follow the order of arithmetic operations.

Horizontal Shifts

Example 1.3.5. Use the function $y = f\left(x\right)$ from **Example 1.3.1**, to graph $y = f\left(x+2\right)$.

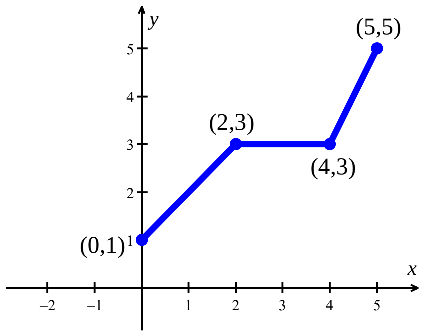
Solution. Both functions, $y = f\left(x\right)$ and $y = f\left(x+2\right)$, have the same range. However, to obtain the same *y* value we must input different *x* values. In the following table, to the left, you see several inputs and outputs for $y = f\left(x\right)$, along with their corresponding points on the graph. In the table to the right, for $y = f\left(x+2\right)$, we keep the outputs the same as $y = f\left(x\right)$, but look for new inputs that, when 2 is added, will give us those outputs.

Output $y = f\left(x\right)$	Input x	Coordinates on graph of $y = f\left(x\right)$
1	0	$\left(0,1\right)$
3	2	$\left(2,3\right)$
3	4	$\left(4,3\right)$
5	5	$\left(5,5\right)$

Output $y = f\left(x+2\right)$	Input x	Coordinates on graph of $y = f\left(x+2\right)$
1	need $x+2=0$ so $x=-2$	$\left(-2,1\right)$
3	need $x+2=2$ so $x=0$	$\left(0,3\right)$
3	need $x+2=4$ so $x=2$	$\left(2,3\right)$
5	need $x+2=5$ so $x=3$	$\left(3,5\right)$

The tables show us that, to get $y = f\left(x+2\right)$ to have the same output value as $y = f\left(x\right)$, the input value to $y = f\left(x+2\right)$ must be $x-2$: $y = f\left(\left(x-2\right)+2\right) = f\left(x\right)$. As illustrated in the following side-by-side graphs, this results in a horizontal shift of 2 units to the left.

Figure 1.3. 12

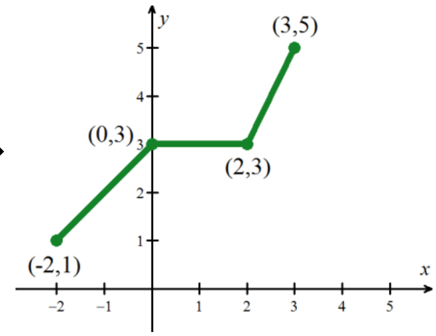


$y = f(x)$

shift left 2 units

subtract 2 from each
x-coordinate

Figure 1.3. 13



$y = f(x+2)$

□

As seen from the graphs, this transformation does not affect the range, but it does affect the domain; the domain of $y = f(x)$ is $[0,5]$ while the domain of $y = f(x+2)$ is $[-2,3]$. Since the two graphs have the same shape, this is a rigid transformation.

If we represent a general horizontal shift of $y = f(x)$ by $y = f(x-C)$, then the value of C determines both the amount and direction of the horizontal shift, as shown below.

- In comparing $y = f(x+2)$ with $y = f(x-C)$, we see that $C = -2$. The size of C determines a shift of 2 units, and its sign indicates a shift toward the negative side of the *x*-axis, or to the left.
- Another way to determine C is to set the argument of the function $y = f(x+2)$ equal to zero, giving us $x+2=0$, which we then solve to get $x=-2$. This solution is the value of C.

The same logic explains the needed transformation to the graph of $y = f(x)$ to obtain the graph of $y = f(x-2)$. Instead of subtracting 2 from the *x*-coordinates on the graph of $y = f(x)$, we'd be adding 2. Geometrically, we would be moving the graph of f right 2 units and the new domain would be $[2,7]$. What we have discussed is generalized below.

Horizontal Shift

Suppose f is a function and C is a constant. To graph $y = f(x-C)$, shift the graph of $y = f(x)$ horizontally by adding C to the *x*-coordinates of the points on the graph of $y = f(x)$. If C is positive, the graph will shift to the right. If C is negative, the graph will shift to the left.

A horizontal shift is a rigid transformation that only affects the domain.

Horizontal Reflections

Example 1.3.6. Use the function $y = f(x)$ from **Example 1.3.1** to graph $y = f(-x)$.

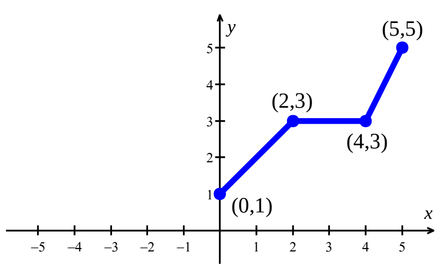
Solution. Both functions, $y = f(x)$ and $y = f(-x)$, have the same range, but to obtain the same *y* values we must input different *x* values. In order for $y = f(-x)$ to have the same output as $y = f(x)$, we must input the negative of each *x* value: $y = f(-(-x)) = f(x)$. This is demonstrated in the following tables.

Output $y = f(x)$	Input x	Coordinates on graph of $y = f(x)$
1	0	$(0,1)$
3	2	$(2,3)$
3	4	$(4,3)$
5	5	$(5,5)$

Output $y = f(-x)$	Input $-x$	Coordinates on graph of $y = f(-x)$
1	need $-x = 0$ so $x = 0$	$(0,1)$
3	need $-x = 2$ so $x = -2$	$(-2,3)$
3	need $-x = 4$ so $x = -4$	$(-4,3)$
5	need $-x = 5$ so $x = -5$	$(-5,5)$

The tables show us that, to get $y = f(-x)$ to have the same output as $y = f(x)$, the input value to $y = f(-x)$ must be the negative of the input value to $y = f(x)$. Geometrically, multiplying the *x*-coordinate of a point by −1 reflects the point across the *y*-axis.

Figure 1.3. 14

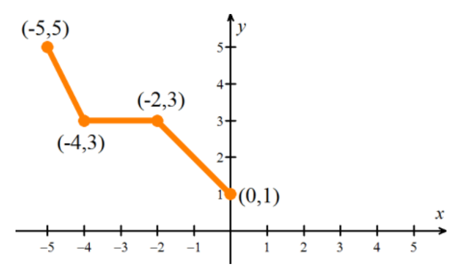


$y = f(x)$

reflect across *y*-axis

multiply each *x*-coordinate by −1

Figure 1.3. 15



$y = f(-x)$

□

Multiplying the *x*-coordinate of every point on a graph by −1 is usually described as ‘reflecting the graph across the *y*-axis’. As seen from the graphs, this transformation does not affect the range, but it does affect the domain. The domain of $y = f(x)$ is $[0,5]$ while the domain of $y = f(-x)$ is $[-5,0]$. The two graphs have the same shape, verifying that this is a rigid transformation.

Horizontal Reflections

Suppose f is a function. To graph $y = f(-x)$, reflect the graph of $y = f(x)$ across the *y*-axis by multiplying the *x*-coordinates of the points on the graph of $y = f(x)$ by -1.

A horizontal reflection is a rigid transformation that only affects the domain.

Horizontal Scalings

Example 1.3.7. Use the function $y = f(x)$ from **Example 1.3.1** to graph $y = f(2x)$.

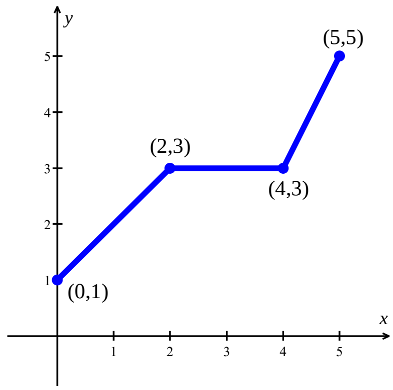
Solution. The functions $y = f(x)$ and $y = f(2x)$ have the same range, but to obtain the same *y* values we must input different *x* values. In order for $y = f(2x)$ to have the same output as $y = f(x)$, we must halve the *y* values: $y = f\left(2\left(\frac{x}{2}\right)\right) = f(x)$. This is demonstrated in the following table.

Output $y = f(x)$	Input x	Coordinates on graph of $y = f(x)$
1	0	$(0,1)$
3	2	$(2,3)$
3	4	$(4,3)$
5	5	$(5,5)$

Output $y = f(2x)$	Input x	Coordinates on graph of $y = f(2x)$
1	need $2x = 0$ so $x = 0$	$(0,1)$
3	need $2x = 2$ so $x = 1$	$(1,3)$
3	need $2x = 4$ so $x = 2$	$(2,3)$
5	need $2x = 5$ so $x = 5/2$	$(5/2,\ 5)$

From the tables, we see that to get $y = f(2x)$ to have the same output as $y = f(x)$, the input value to $y = f(2x)$ must be half of the input value to $y = f(x)$. Geometrically, multiplying the *x*-coordinate of a point by the positive number $\frac{1}{2}$ compresses the graph in the *x* direction.

Figure 1.3. 16

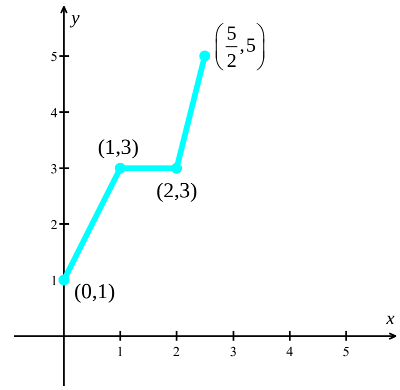


$y = f(x)$

horizontal scaling by a factor of $\frac{1}{2}$

multiply each x-coordinate by $\frac{1}{2}$

Figure 1.3. 17



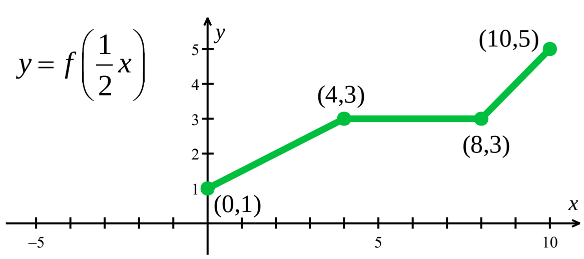
$y = f(2x)$

□

Multiplying the *x* coordinate of every point on a graph by $\frac{1}{2}$ is usually described as a 'horizontal scaling by a factor of $\frac{1}{2}$'. As seen from the graphs, this transformation did not affect the range, but it did affect the domain; the domain of $y = f(x)$ is $[0,5]$ while the domain of $y = f(2x)$ is $\left[0,\frac{5}{2}\right]$. Since the two graphs do not have the same shape, this is a non-rigid transformation.

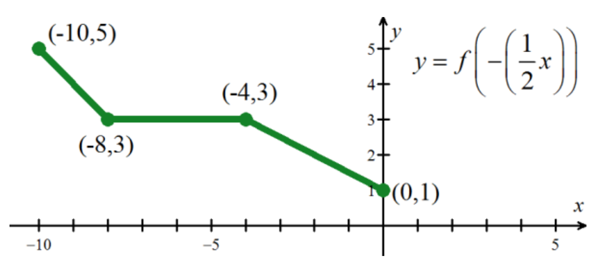
If we wish to graph $y = f\left(\frac{1}{2}x\right)$, we multiply all of the *x*-coordinates of the points on the graph of f by 2 which results in a horizontal scaling by a factor of 2. On the other hand, if we wish to graph $y = f\left(-\frac{1}{2}x\right)$, we multiply all of the *x*-coordinates of the points on the graph by −2. Geometrically, we can think of two operations. Since $f\left(-\frac{1}{2}x\right) = f\left(-\left(\frac{1}{2}x\right)\right)$, we first perform a horizontal scaling by a factor of 2 and then reflect the resulting graph across the *y*-axis.

Figure 1.3. 18



horizontal scaling by factor of 2

Figure 1.3. 19



reflection across *y*-axis

We summarize these results as follows.

Horizontal Scalings

Suppose f is a function and B is a nonzero constant. To graph $y = f(Bx)$, multiply all of the x-coordinates of points on the graph of $y = f(x)$ by $\frac{1}{B}$.

- If $B > 0$, we say the graph of f has been horizontally scaled (stretched[20] if $0 < B < 1$ and compressed[21] if $B > 1$) by a factor of $\frac{1}{B}$.
- If $B < 0$, the graph of f is both horizontally scaled and reflected across the y-axis.

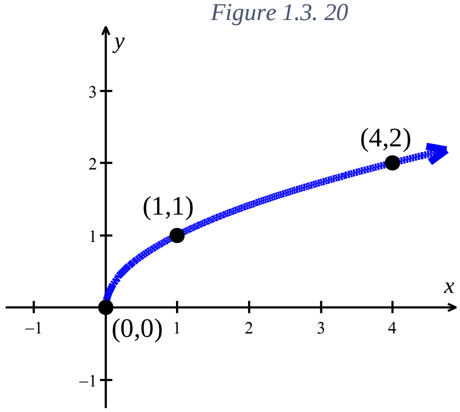
A horizontal scaling is a non-rigid transformation that only affects the domain.

In the next example, we apply multiple transformations to the input to a function.

Example 1.3.8. Use the graph of $f(x) = \sqrt{x}$, shown in **Example 1.3.4**, to sketch the graph of $g(x) = \sqrt{2x+1}$. State the domain and range of g.

Solution. Below is the graph of f.

Figure 1.3. 20



$y = f(x) = \sqrt{x}$

There are two transformations of f in the function g. We consider the input value to g that results in the same output value of $y = f(x)$.

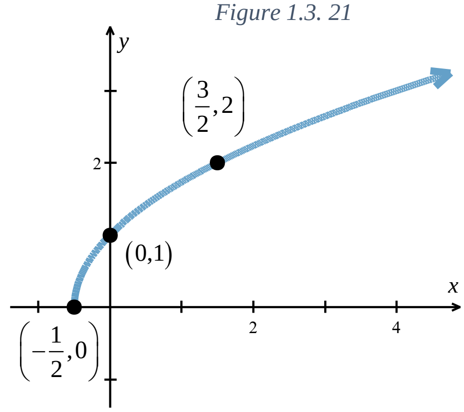
[20] Also called a horizontal expansion or a horizontal dilation.
[21] Also called a horizontal shrinking or a horizontal contraction.

x	$y=f(x)=\sqrt{x}$	$(x,f(x))$
0	0	$(0,0)$
1	1	$(1,1)$
4	2	$(4,2)$

x	$y=g(x)=\sqrt{2x+1}$	$(x,g(x))$
need $2x+1=0$ $\Rightarrow x=-\frac{1}{2}$	0	$\left(-\frac{1}{2},0\right)$
need $2x+1=1$ $\Rightarrow x=0$	1	$(0,1)$
need $2x+1=4$ $\Rightarrow x=\frac{3}{2}$	2	$\left(\frac{3}{2},2\right)$

Notice that, in every instance, we first subtract 1 from the *x*-values, resulting in a horizontal shift to the left of 1 unit, then divide the new *x* values by 2, which is a horizontal scaling. The order we perform the two transformations in is simply the order of operations in calculating the input to g that gives the same output as $y=f(x)$.

Figure 1.3. 21



$y=g(x)=\sqrt{2x+1}$

The domain of g is $\left[-\frac{1}{2},\infty\right)$, while its range is $[0,\infty)$, the same as the range of f.

□

A general technique for finding the required transformations is to consider the input value of $y=g(x)=\sqrt{2x+1}$ that gives the same output as $y=f(x)=\sqrt{x}$. By solving $2\times?+1=x$, we get

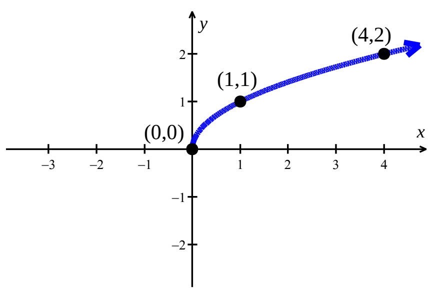
$? = \frac{x-1}{2}$ and are thus reminded to follow the order of operations: first subtract 1 from the *x* value and then divide the result by 2. Notice that the order of transformations does matter since $\frac{x}{2} - 1 \neq \frac{x-1}{2}$.

A Combination of Transformations

Example 1.3.9. Use the graph of $f(x) = \sqrt{x}$, shown in **Example 1.3.4**, to sketch the graph of $g(x) = \sqrt{x+3} - 2$. State the domain and range of g.

Solution. To graph the function g, we use two transformations of f, one that affects the input value to f and another that affects the output value from f. For the transformation affecting the input, since $x + 3 = x - (-3)$, we need to subtract 3 units from each *x* value, or shift the graph of f left 3 units. The transformation affecting the output requires us to subtract 2 units from each *y* value, or to shift the graph down 2 units. In either order we will get the same final graph.

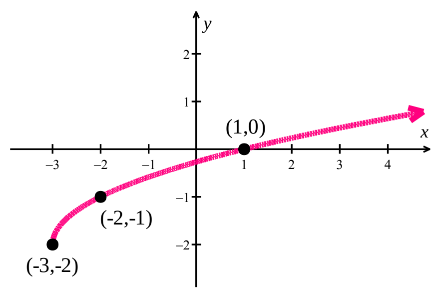
Figure 1.3. 22



$y = f(x) = \sqrt{x}$

shift left 3 units
and down 2 units

Figure 1.3. 23



$y = g(x) = \sqrt{x+3} - 2$

We can check our work by finding a specific point on the curve: $g(-3) = \sqrt{-3+3} - 2 = -2$. The domain of g is $[-3, \infty)$ and its range is $[-2, \infty)$.

□

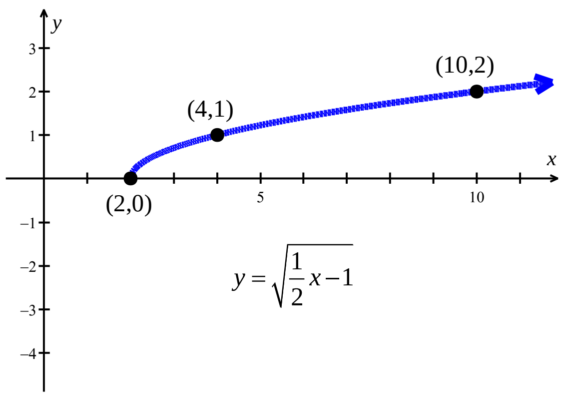
Example 1.3.10. Use the graph of $f(x) = \sqrt{x}$, shown in **Example 1.3.4**, to sketch the graph of $g(x) = -\sqrt{\frac{1}{2}x - 1} - 2$. State the domain and range of g.

Solution. There are two required transformations affecting the input to f and two required transformations affecting the output from f. We can perform the input transformations first and then the output transformations, or vice-versa.

We choose to begin with the input, and look for the input values to $y=\sqrt{\frac{1}{2}x-1}$ that result in the same output values as $y=f(x)=\sqrt{x}$. Solving $\frac{1}{2}\times ?-1=x$ is equivalent to $?=2(x+1)$. Following this order of operations, we first add 1 to each x value, or shift the graph to the right by one unit. We then multiply these new values by 2, which is a horizontal scaling. These two operations give us the graph of $y=\sqrt{\frac{1}{2}x-1}$.

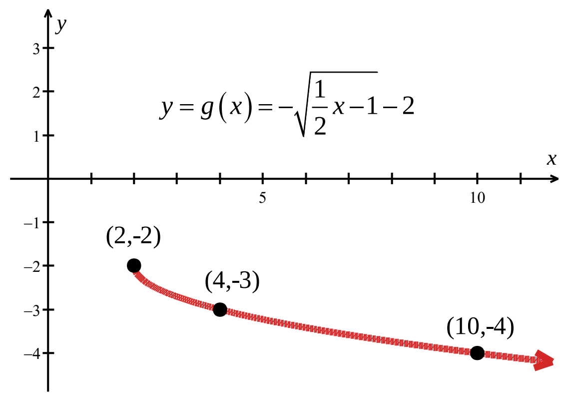
The order of operations for calculating the y values in $y=g(x)=-\sqrt{\frac{1}{2}x-1}-2$ requires first multiplying $y=\sqrt{\frac{1}{2}x-1}$ by −1, which is a vertical reflection, and then subtracting 2 from all y values to shift the graph down 2 units.

Figure 1.3. 24



after horizontal transformations of $y=\sqrt{x}$

Figure 1.3. 25



after vertical transformations of $y=\sqrt{\frac{1}{2}x-1}$

We can check our work by finding a specific point on the curve: $g(2)=-\sqrt{\frac{1}{2}(2)-1}-2=-2$. The domain of g is $[2,\infty)$ and its range is $(-\infty,-2]$.

□

We can apply these ideas in general to obtain the graph of $y=g(x)=A\,f(B\,x-C)+D$ by transforming the graph of $y=f(x)$. We can perform either the input or output transformations first. In the following summary, we begin with the input transformations.

- For the input transformations, consider the input value which results in the same output value as $y = f(x)$. You can think of this as solving $B \times ? - C = x$ to get $? = \frac{x+C}{B}$. To go from the input values for f to the input values for g, first add C to the input values for f, and then divide the result by B. The resulting graph is $y = f(Bx - C)$.

- For the output transformations, consider the order of operations for calculating the *y* values in $y = g(x) = Af(Bx - C) + D$ from $y = f(Bx - C)$: first multiply each *y* value by A and then add D to them.

A summary of these results follows.

Graphing Transformations of a Function *f*

Suppose f is a function. If $A \neq 0$ and $B \neq 0$, then to graph $y = A\,f(Bx - C) + D$

1. Add C to each *x*-coordinate of the graph of $y = f(x)$.

   This results in a horizontal shift. If C is positive, the graph will shift to the right. If C is negative, the graph will shift to the left.

2. Multiply the *x*-coordinates of the graph obtained in step 1 by $\frac{1}{B}$.

   If $B > 0$, this results in a horizontal scaling. If $B < 0$, this results in a horizontal scaling and a reflection about the *y*-axis.

3. Multiply the *y*-coordinates of the graph obtained in step 2 by A.

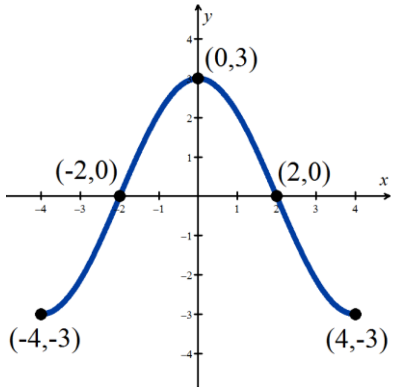
   If $A > 0$, this results in a vertical scaling. If $A < 0$, this results in a vertical scaling and a reflection across the *x*-axis.

4. Add D to the *y*-coordinates of the graph obtained in step 3.

   This results in a vertical shift. If D is positive, the graph will shift up. If D is negative, the graph will shift down.

Example 1.3.11. Below is a complete graph of $y = f(x)$. Use it to graph $g(x) = \frac{4-3f(1-2x)}{2}$.

Figure 1.3. 26

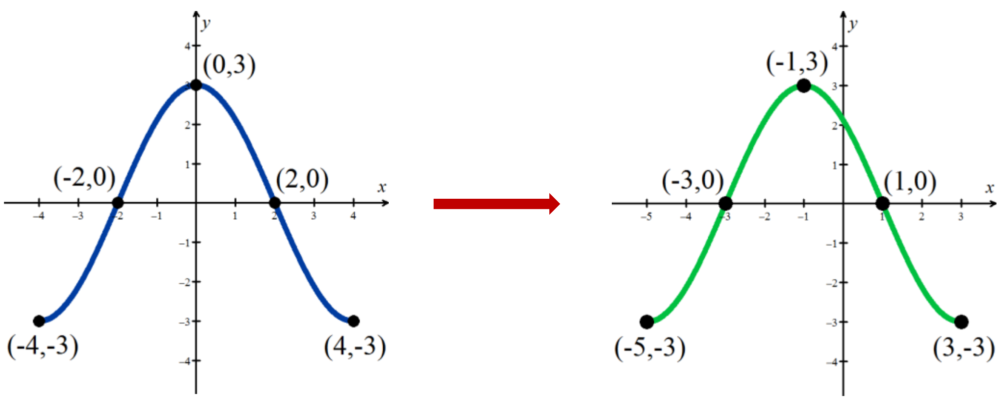


$y = f(x)$

Solution. We first rewrite g in the form $g(x) = -\frac{3}{2}f(-2x+1)+2$, and then follow the four steps as outlined above.

1. Since $C = -1$, our first step is to add -1 to each of the x-coordinates of the points on the graph of $y = f(x)$.

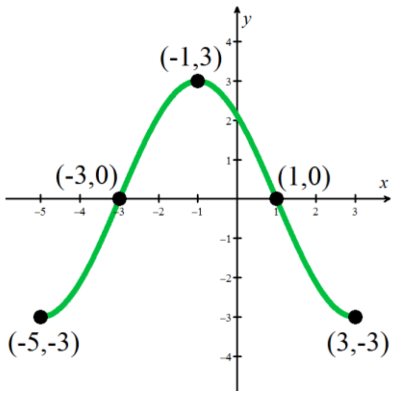
Figure 1.3. 27 *Figure 1.3. 28*



$y = f(x)$ $y = f(x+1)$

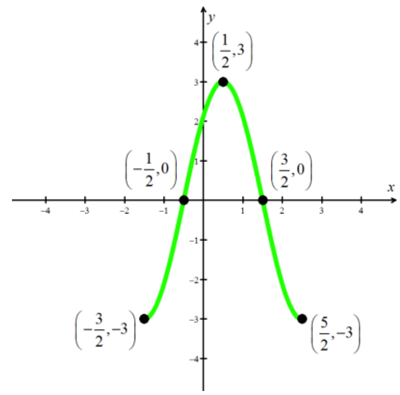
2. With $B = -2$, we next multiply the x-coordinates of the points on the graph of $y = f(x+1)$ by $\frac{1}{-2}$.

Figure 1.3. 29



$y = f(x+1)$

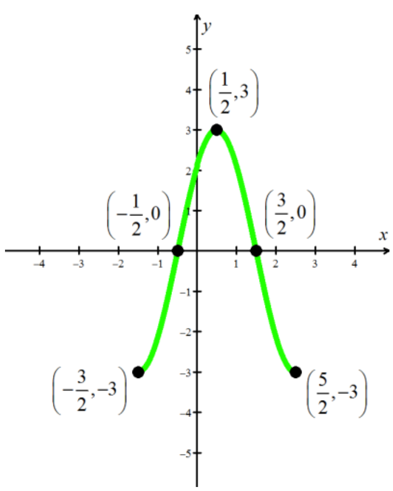
Figure 1.3. 30



$y = f(-2x+1)$

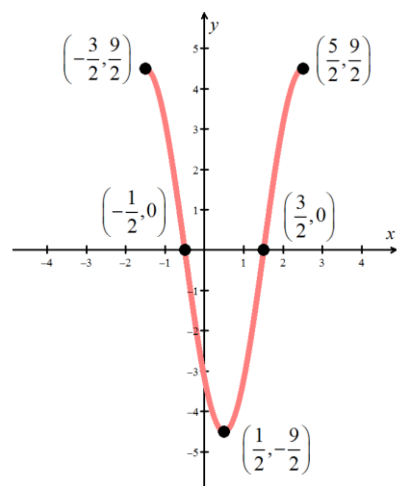
3. Our third step is to multiply each of the y-coordinates of the points on the graph of $y = f(-2x+1)$ by A, which is $-\frac{3}{2}$.

Figure 1.3. 31



$y = f(-2x+1)$

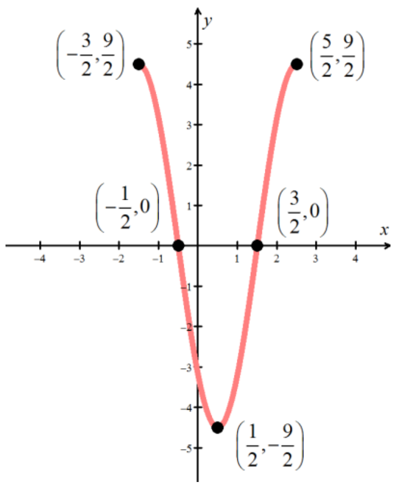
Figure 1.3. 32



$y = -\frac{3}{2} f(-2x+1)$

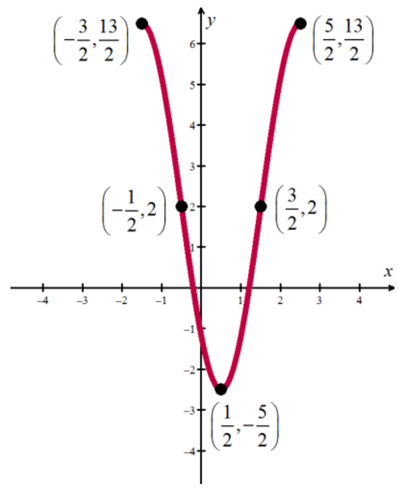
4. In our last step, we add $D = 2$ to each of the y-coordinates of the points on the graph of $y = -\frac{3}{2} f(-2x+1)$.

Figure 1.3. 33



$$y = -\frac{3}{2} f(-2x+1)$$

Figure 1.3. 34



$$y = -\frac{3}{2} f(-2x+1) + 2 = g(x)$$

□

Our last example turns the tables and asks for the formula of a function given a desired sequence of transformations.

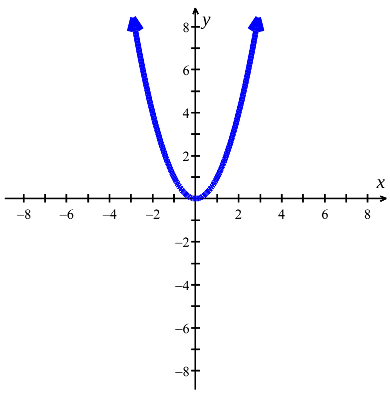
Example 1.3.12. Let $f(x) = x^2$. Find and simplify the formula of the function $g(x)$ whose graph is the result of f undergoing the following sequence of transformations.

1. Vertical shift up 2 units.
2. Reflection across the x-axis.
3. Horizontal shift right 1 unit.
4. Horizontal stretching by a factor of 2.

Solution. We build up to a formula for $g(x)$, beginning with the transformation in part 1.

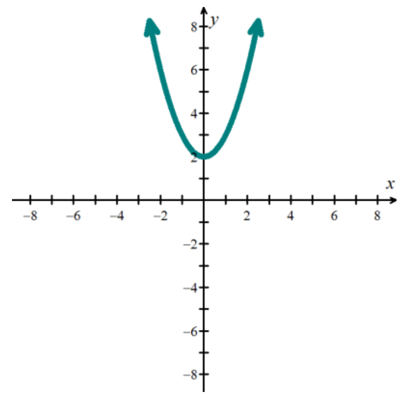
1. To achieve a vertical shift up 2 units, we add 2 to $f(x) = x^2$ and have $y = f(x) + 2 = x^2 + 2$.

Figure 1.3. 35



$y = f(x) = x^2$

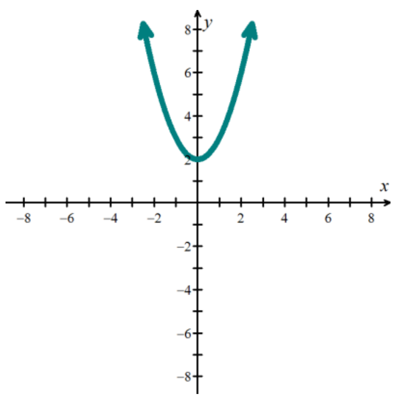
Figure 1.3. 36



$y = f(x) + 2 = x^2 + 2$

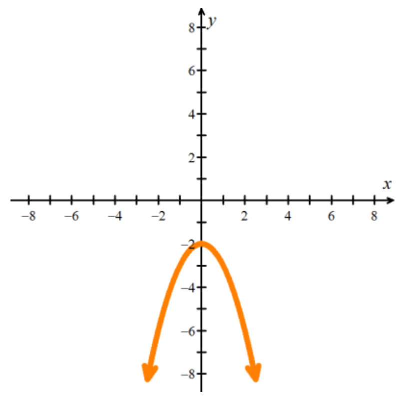
2. Next, we reflect the graph of $y = x^2 + 2$ about the *x*-axis to get $y = -(x^2 + 2) = -x^2 - 2$.

Figure 1.3. 37



$y = x^2 + 2$

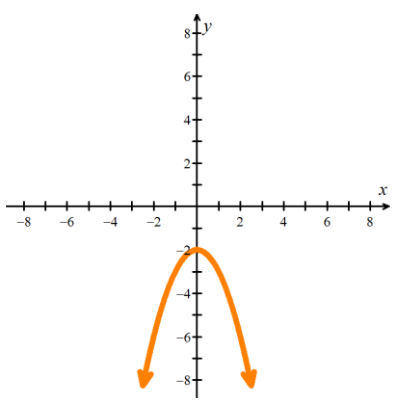
Figure 1.3. 38



$y = -(x^2 + 2) = -x^2 - 2$

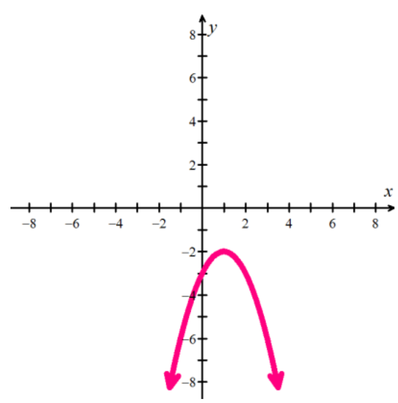
3. The third step is to shift the graph of $y = -x^2 - 2$ right by one unit, so we get $y = -(x-1)^2 - 2$.

Figure 1.3. 39



$y = -x^2 - 2$

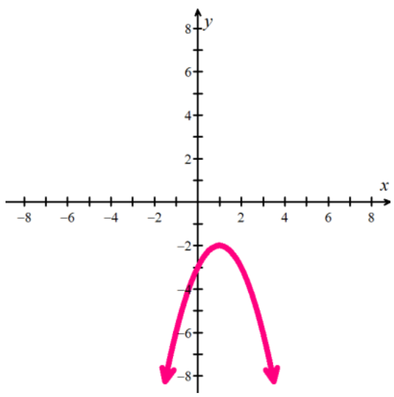
Figure 1.3. 40



$y = -(x-1)^2 - 2$

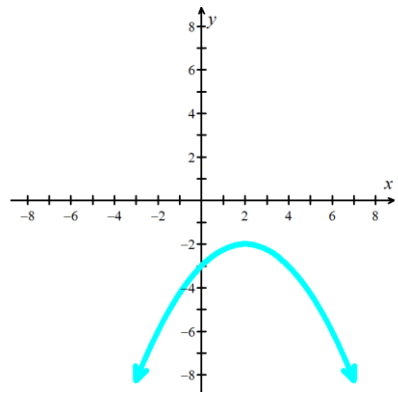
4. Finally, we stretch the graph of $y=-(x-1)^2-2$ horizontally by a factor of 2 as follows: $y=-\left(\left(\frac{1}{2}x\right)-1\right)^2-2$ which, after simplifying, yields $y=-\frac{1}{4}x^2+x-3$. This is the function $g(x)$ that we have been seeking.

Figure 1.3. 41



$y=-(x-1)^2-2$

Figure 1.3. 42



$y=g(x)=-\frac{1}{4}x^2+x-3$

□

1.3 Exercises

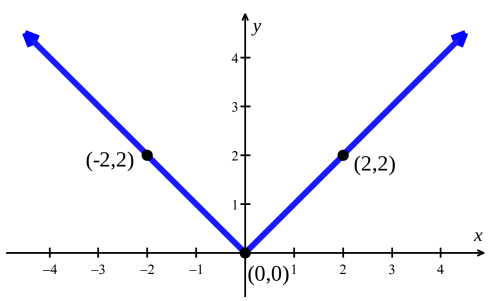
1. When examining the formula of a function that is the result of multiple transformations, how can you distinguish between a horizontal shift and a vertical shift.
2. When examining the formula of a function that is the result of multiple transformations, how can you distinguish between a reflection across the *x*-axis and a reflection across the *y*-axis.

Suppose $(2,-3)$ is on the graph of $y=f(x)$. In Exercises 3 – 20, use the point $(2,-3)$ to find a point on the graph of the given transformed function.

3. $y=f(x)+3$
4. $y=f(x+3)$
5. $y=f(x)-1$
6. $y=f(x-1)$
7. $y=3f(x)$
8. $y=f(3x)$
9. $y=-f(x)$
10. $y=f(-x)$
11. $y=f(x-3)+1$
12. $y=2f(x+1)$
13. $y=10-f(x)$
14. $y=3f(2x)-1$
15. $y=\frac{1}{2}f(4-x)$
16. $y=5f(2x+1)+3$
17. $y=2f(1-x)-1$
18. $y=f\left(\frac{7-2x}{4}\right)$
19. $y=\frac{f(3x)-1}{2}$
20. $y=\frac{4-f(3x-1)}{7}$

The complete graph of $y=f(x)$ is given below. In Exercises 21 – 29, use it to sketch a graph of the given transformed function.

Figure 1.3. 43



The graph of $y=f(x)$ for Exercises 21 – 29

21. $y=f(x)+1$
22. $y=f(x)-2$
23. $y=f(x+1)$
24. $y=f(x-2)$
25. $y=2f(x)$
26. $y=f(2x)$

27. $y = 2 - f(x)$

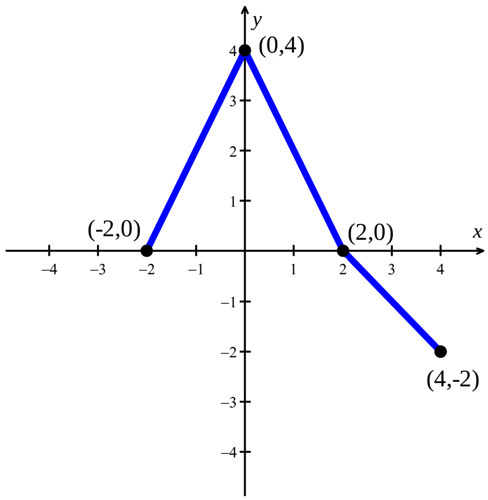
28. $y = f(2 - x)$

29. $y = 2 - f(2 - x)$

30. Some of the answers to Exercises 21 – 29 above should be the same. Which ones match up? What properties of the graph of $y = f(x)$ contribute to the duplication?

The complete graph of $y = f(x)$ is given below. In Exercises 31 – 39, use it to sketch a graph of the given transformed function.

Figure 1.3. 44



The graph of $y = f(x)$ for Exercises 31 – 39

31. $y = f(x) - 1$

32. $y = f(x + 1)$

33. $y = \frac{1}{2} f(x)$

34. $y = f(2x)$

35. $y = -f(x)$

36. $y = f(-x)$

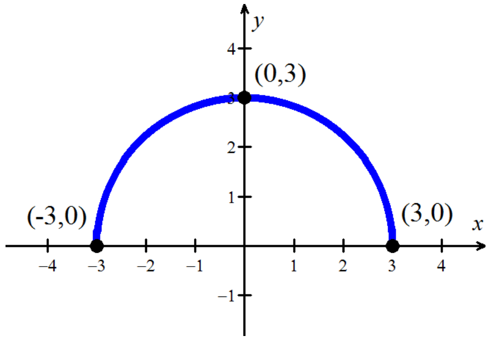
37. $y = f(x + 1) - 1$

38. $y = 1 - f(x)$

39. $y = \frac{1}{2} f(x + 1) - 1$

The complete graph of $y = f(x)$ is given below. In Exercises 40 – 51, use it to sketch a graph of the given transformed function.

Figure 1.3. 45



The graph of $y = f(x)$ for Exercises 40 – 51

40. $g(x) = f(x) + 3$

41. $h(x) = f(x) - \frac{1}{2}$

42. $j(x) = f\left(x - \frac{2}{3}\right)$

43. $a(x) = f(x+4)$

44. $b(x) = f(x+1) - 1$

45. $c(x) = \frac{3}{5} f(x)$

46. $d(x) = -2f(x)$

47. $k(x) = f\left(\frac{2}{3}x\right)$

48. $m(x) = -\frac{1}{4} f(3x)$

49. $n(x) = 4f(x-3) - 6$

50. $p(x) = 4 + f(1-2x)$

51. $q(x) = -\frac{1}{2} f\left(\frac{x+4}{2}\right) - 3$

52. Write a formula for a function *g* whose graph is obtained from $f(x) = \sqrt{x}$ after the sequence of transformations: (1) shift right 2 units; (2) shift down 3 units.

53. Write a formula for a function *g* whose graph is obtained from $f(x) = \sqrt{x}$ after the sequence of transformations: (1) shift down 3 units; (2) shift right 2 units.

54. Write a formula for a function *g* whose graph is obtained from $f(x) = \sqrt{x}$ after the sequence of transformations: (1) reflect across the *x*-axis; (2) shift up 1 unit.

55. Write a formula for a function *g* whose graph is obtained from $f(x) = \sqrt{x}$ after the sequence of transformations: (1) shift up 1 unit; (2) reflect across the *x*-axis.

56. Write a formula for a function *g* whose graph is obtained from $f(x) = \sqrt{x}$ after the sequence of transformations: (1) shift left 1 unit; (2) reflect across the *y*-axis; (3) shift up 2 units.

57. Write a formula for a function *g* whose graph is obtained from $f(x)=\sqrt{x}$ after the sequence of transformations: (1) reflect across the *y*-axis; (2) shift left 1 unit; (3) shift up 2 units.

58. Write a formula for a function *g* whose graph is obtained from $f(x)=\sqrt{x}$ after the sequence of transformations: (1) shift left 3 units; (2) scale vertically by a factor of 2; (3) shift down 4 units.

59. Write a formula for a function *g* whose graph is obtained from $f(x)=\sqrt{x}$ after the sequence of transformations: (1) shift left 3 units; (2) shift down 4 units; (3) scale vertically by a factor of 2.

60. Write a formula for a function *g* whose graph is obtained from $f(x)=\sqrt{x}$ after the sequence of transformations: (1) shift right 3 units; (2) scale horizontally by a factor of $\frac{1}{2}$; (3) shift up 1 unit.

61. Write a formula for a function *g* whose graph is obtained from $f(x)=\sqrt{x}$ after the sequence of transformations: (1) scale horizontally by a factor of $\frac{1}{2}$; (2) shift right 3 units; (3) shift up 1 unit.

62. Write a formula for a function *g* whose graph is obtained from $f(x)=|x|$ after the sequence of transformations: (1) shift down 3 units; (2) shift right 1 unit.

63. Write a formula for a function *g* whose graph is obtained from $f(x)=\frac{1}{x}$ after the sequence of transformations: (1) shift down 4 units; (2) shift right 3 units.

64. Write a formula for a function *g* whose graph is obtained from $f(x)=\frac{1}{x^2}$ after the sequence of transformations: (1) shift up 2 units; (2) shift left 4 units.

65. Write a formula for a function *g* whose graph is obtained from $f(x)=|x|$ after the sequence of transformations: (1) reflect across the *y*-axis; (2) scale horizontally by a factor of $\frac{1}{4}$.

66. Write a formula for a function *g* whose graph is obtained from $f(x)=\frac{1}{x^2}$ after the sequence of transformations: (1) scale vertically by a factor of $\frac{1}{3}$; (2) shift left 2 units; (3) shift down 3 units.

67. Write a formula for a function *g* whose graph is obtained from $f(x)=\frac{1}{x}$ after the sequence of transformations: (1) scale vertically by a factor of 8; (2) shift right 4 units; (3) shift up 2 units.

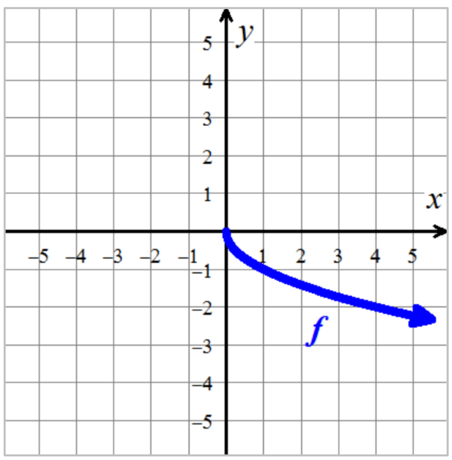
68. Write a formula for a function g whose graph is obtained from $f(x)=x^2$ after the sequence of transformations: (1) scale vertically by a factor of $\frac{1}{2}$; (2) shift right 5 units; (3) shift up 1 unit.

69. Write a formula for a function g whose graph is obtained from $f(x)=x^2$ after the sequence of transformations: (1) scale horizontally by a factor of $\frac{1}{3}$; (2) shift left 4 units; (3) shift down 3 units.

In Exercises 70 – 75, use the graphs of the transformed toolkit functions to write a formula for each of the resulting functions.

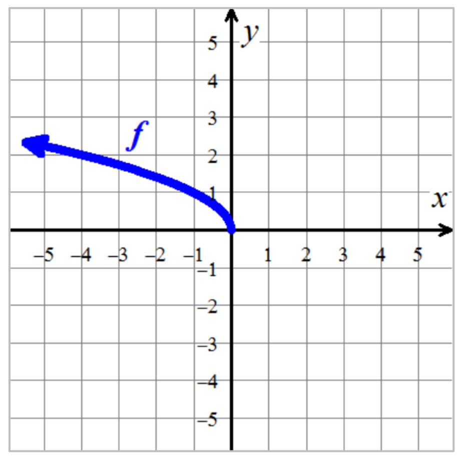
70.

Figure 1.3. 46



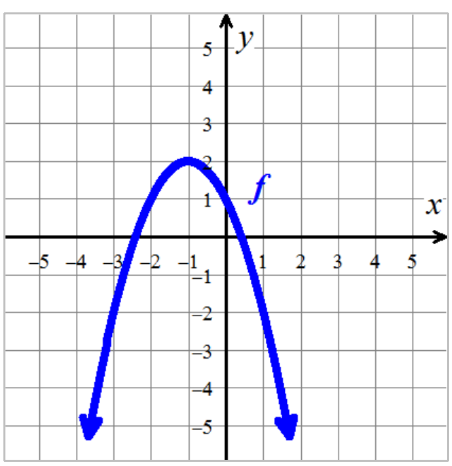
71.

Figure 1.3. 47



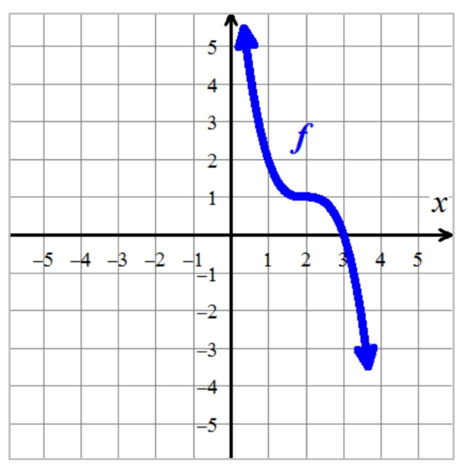
72.

Figure 1.3. 48



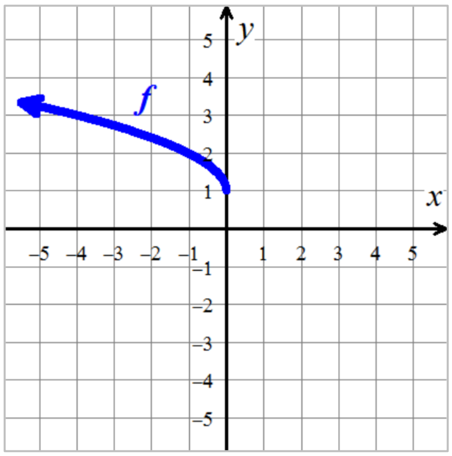
73.

Figure 1.3. 49



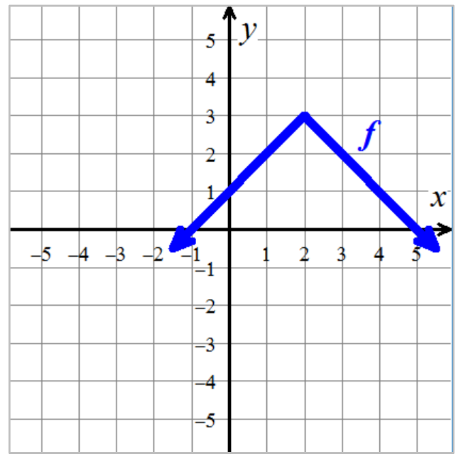
74.

Figure 1.3. 50



75.

Figure 1.3. 51



In Exercises 76 – 88, sketch a graph of the function as a transformation of the graph of one of the toolkit functions.

76. $f(x)=(x+1)^2-3$

77. $h(x)=|x-1|+4$

78. $k(x)=(x-2)^3-1$

79. $m(x)=3+\sqrt{x+2}$

80. $g(x)=4(x+1)^2-5$

81. $g(x)=5(x+3)^2-2$

82. $h(x)=-2|x-4|+3$

83. $k(x)=-3\sqrt{x}-1$

84. $m(x)=\frac{1}{2}x^3$

85. $n(x)=\frac{1}{3}|x-2|$

86. $p(x)=\left(\frac{1}{3}x\right)^3-3$

87. $q(x)=\left(\frac{1}{4}x\right)^3+1$

88. $a(x)=\sqrt{-x+4}$

89. For many common functions, the properties of algebra make a horizontal scaling the same as a vertical scaling by (possibly) a different factor. For example, we stated earlier that $\sqrt{9x}=3\sqrt{x}$. With the help of your classmates, find the equivalent vertical scaling produced by the horizontal scalings $y=(2x)^3$, $y=|5x|$, $y=\sqrt[3]{27x}$ and $y=\left(\frac{1}{2}x\right)^2$. What about $y=(-2x)^3$, $y=|-5x|$, $y=\sqrt[3]{-27x}$ and $y=\left(-\frac{1}{2}x\right)^2$?

90. As mentioned earlier in the section, in general, the order in which transformations are applied matters. Yet, in one of our examples with two transformations, the order did not matter. With the help of your classmates, determine the situations in which order does matter and those in which it does not.

91. What happens if you reflect an even function across the *y*-axis?

92. What happens if you reflect an odd function across the *y*-axis?

93. What happens if you reflect an even function across the *x*-axis?

94. What happens if you reflect an odd function across the *x*-axis?

95. How would you describe symmetry about the origin in terms of reflections?

1.4. Combinations of Functions

Learning Objectives

- Find and simplify functions involving arithmetic expressions.
- Combine functions through addition, subtraction, multiplication and division.
- Determine the domain of a function resulting from an arithmetic operation.
- Find the difference quotient of a function.
- Create a new function through composition of functions.
- Find the domain of a composite function.
- Find values of composite functions.
- Decompose a composite function into its component functions.

We begin this section by again evaluating functions, but now add expressions to the numerical values we have focused on up to this point. We then move on to combining functions using the four basic arithmetic operations, and later introduce function composition, providing us with yet another way to combine functions.

Functions Involving Arithmetic Expressions

Through the following example, we begin finding, and simplifying, functions of expressions and expressions of functions, in preparation for arithmetic operations involving functions.

Example 1.4.1. Let $f(x) = -x^2 + 3x + 4$. Find and simplify the following.

1. $f(2x)$, $2f(x)$
2. $f(x+2)$, $f(x)+2$, $f(x)+f(2)$

Solution.

1. To find $f(2x)$, we replace every occurrence of x with the quantity $2x$.

$$\begin{aligned} f(2x) &= -(2x)^2 + 3(2x) + 4 \\ &= -(4x^2) + (6x) + 4 \\ &= -4x^2 + 6x + 4 \end{aligned}$$

The expression $2f(x)$ means we multiply the expression $f(x)$ by 2.

$$\begin{aligned} 2f(x) &= 2(-x^2+3x+4) \\ &= -2x^2+6x+8 \end{aligned}$$

2. To find $f(x+2)$, we replace every occurrence of x with the quantity $x+2$.

$$\begin{aligned} f(x+2) &= -(x+2)^2+3(x+2)+4 \\ &= -(x^2+4x+4)+(3x+6)+4 \\ &= -x^2-4x-4+3x+6+4 \\ &= -x^2-x+6 \end{aligned}$$

To find $f(x)+2$, we add 2 to the expression for $f(x)$.

$$\begin{aligned} f(x)+2 &= (-x^2+3x+4)+2 \\ &= -x^2+3x+6 \end{aligned}$$

For $f(x)+f(2)$, we evaluate $f(x)$ and $f(2)$ separately and then add the results.

$$\begin{aligned} f(x)+f(2) &= (-x^2+3x+4)+(-(2)^2+3(2)+4) \\ &= -x^2+3x+4+6 \\ &= -x^2+3x+10 \end{aligned}$$

□

A couple notes about **Example 1.4.1** are in order.

1. First, note the difference between the answers for $f(2x)$ and $2f(x)$. For $f(2x)$, we are multiplying the input by 2; for $2f(x)$, we are multiplying the output by 2. As we see, we get entirely different results.

2. Also note that $f(x+2)$, $f(x)+2$ and $f(x)+f(2)$ are three different expressions. Even though function notation uses parentheses, as does multiplication, there is no general distributive property of function notation.

3. Observe the use of parentheses when substituting one algebraic expression into another, an important practice in evaluating functions.

Arithmetic Operations

We have used function notation to make sense of expressions such as $f(x)+2$ and $2f(x)$ for a given function f. It would seem natural then that functions should have their own arithmetic that is consistent with the arithmetic of real numbers. The following definitions allow us to add, subtract, multiply and divide functions using the arithmetic we already know for real numbers.

Function Arithmetic

Suppose f and g are functions and x is in both the domain of f and the domain of g.[22]

- The **sum** of f and g, denoted $f+g$, is the function defined by the formula

$$(f+g)(x)=f(x)+g(x)$$

- The **difference** of f and g, denoted $f-g$, is the function defined by the formula

$$(f-g)(x)=f(x)-g(x)$$

- The **product** of f and g, denoted $f\cdot g$, is the function defined by the formula

$$(f\cdot g)(x)=f(x)\cdot g(x)$$

- The **quotient** of f and g, denoted $\frac{f}{g}$, is the function defined by the formula

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}, \text{ provided } g(x)\neq 0.$$

In other words, to add two functions, we add their outputs; to subtract two functions, we subtract their outputs, and so on. Note that while the formula $(f+g)(x)=f(x)+g(x)$ looks suspiciously like some kind of distributive property, it is nothing of the sort. The addition on the left hand side of the equation is function addition, and we are using this equation to define the output of the new function $f+g$ as the sum of the real number outputs f and g.

Example 1.4.2. Let $f(x)=6x^2-2x$ and $g(x)=3-\frac{1}{x}$. Find the following.

1. $(f+g)(-1)$
2. $(f\cdot g)(2)$

Solution.

1. To find $(f+g)(-1)$ we first find $f(-1)=8$ and $g(-1)=4$. By definition, we have

$$\begin{aligned}(f+g)(-1)&=f(-1)+g(-1)\\&=8+4\\&=12\end{aligned}$$

[22] Thus x is an element of the intersection of the two domains.

2. For $(f \cdot g)(2)$, we need $f(2)$ and $g(2)$. Since $f(2)=20$ and $g(2)=\frac{5}{2}$, our formula yields

$$\begin{aligned}(f \cdot g)(2) &= f(2)\cdot g(2)\\ &= (20)\left(\frac{5}{2}\right)\\ &= 50\end{aligned}$$

□

Example 1.4.3. As in **Example 1.4.2**, let $f(x)=6x^2-2x$ and $g(x)=3-\frac{1}{x}$. Find and simplify $(g-f)(x)$. Determine the domain of the resulting function.

Solution. To find $(g-f)(x)$, we begin with the definition and proceed with simplifying the resulting formula.

$$\begin{aligned}(g-f)(x) &= g(x)-f(x)\\ &= \left(3-\frac{1}{x}\right)-\left(6x^2-2x\right)\\ &= 3-\frac{1}{x}-6x^2+2x\\ &= \frac{3x}{x}-\frac{1}{x}-\frac{6x^3}{x}+\frac{2x^2}{x} \quad \text{obtain common denominator}\\ &= \frac{3x-1-6x^3+2x^2}{x}\\ &= \frac{-6x^3+2x^2+3x-1}{x}\end{aligned}$$

To find the domain of $g-f$, we find the domains of g and f separately, and then determine the intersection of these two sets. Owing to the denominator in the expression $g(x)=3-\frac{1}{x}$, we get that the domain of g is $(-\infty,0)\cup(0,\infty)$. Since $f(x)=6x^2-2x$ is valid for all real numbers, we have no further restrictions. Thus, the domain of $g-f$ matches the domain of g, namely $(-\infty,0)\cup(0,\infty)$.

□

Example 1.4.4. As in the previous two examples, let $f(x)=6x^2-2x$ and $g(x)=3-\frac{1}{x}$. Find and simplify $\left(\frac{g}{f}\right)(x)$. Determine the domain of the resulting function.

Solution. We begin with the definition for the quotient of f and g and simplify the resulting formula.

$$\begin{aligned}\left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)} \\ &= \frac{3-\frac{1}{x}}{6x^2-2x} \\ &= \frac{3-\frac{1}{x}}{6x^2-2x}\cdot\frac{x}{x} \quad \text{simplify compound fraction} \\ &= \frac{3x-1}{6x^3-2x^2} \\ &= \frac{(1)(3x-1)}{(2x^2)(3x-1)} \quad \text{factor} \\ &= \frac{1}{2x^2}\end{aligned}$$

To find the domain of $\frac{g}{f}$, we start by identifying the domains of f and g separately. In **Example 1.4.3**, we found that the domain of g is $(-\infty,0)\cup(0,\infty)$ and the domain of f is $(-\infty,\infty)$. Thus, as in **Example 1.4.3**, we exclude $x=0$ from the domain. Additionally, with a quotient of functions, we must guard against the denominator being 0. In this case, for $\left(\frac{g}{f}\right)(x)=\frac{g(x)}{f(x)}$, we must guarantee that $f(x)$ is not equal to 0. Setting $f(x)=0$ gives $6x^2-2x=0$, which occurs when $x=0$ or $x=\frac{1}{3}$. As a result, the domain of $\frac{g}{f}$ is all real numbers except $x=0$ and $x=\frac{1}{3}$, or $(-\infty,0)\cup\left(0,\frac{1}{3}\right)\cup\left(\frac{1}{3},\infty\right)$.

□

Please note the importance of finding the domain of a function before simplifying its expression. In **Example 1.4.4**, had we waited to find the domain of $\frac{g}{f}$ until after simplifying, we'd just have the expression $\frac{1}{2x^2}$ to go by and we would (incorrectly!) state the domain as $(-\infty,0)\cup(0,\infty)$, since the other troublesome number, $x=\frac{1}{3}$, was cancelled away.[23]

[23] We'll see what this means geometrically in **Chapter 3**.

Next, we turn our attention to the **difference quotient** of a function.

The Difference Quotient

Definition 1.8. Given a function f, the **difference quotient** of f is the expression

$$\frac{f(x+h)-f(x)}{h}$$

For reasons that will become clear in Calculus, simplifying a difference quotient is an important skill. Graphically, we may interpret the difference quotient as being the slope of a line connecting two points on a curve. For the following graph of $y=f(x)$, the slope of the line connecting the points $(x, f(x))$ and $(x+h, f(x+h))$ can be found by taking 'rise over run', which is $\dfrac{f(x+h)-f(x)}{h}$.

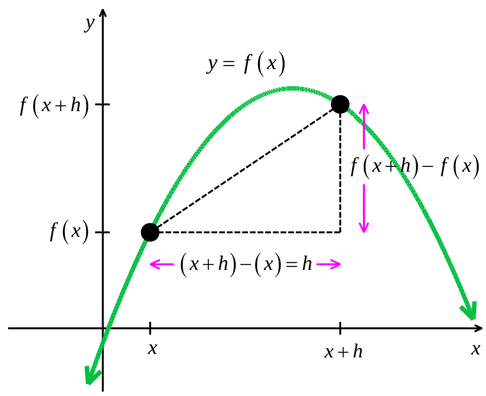


Figure 1.4. 1

Example 1.4.5. Find and simplify the difference quotients for the following functions.

1. $f(x)=x^2-x-2$
2. $g(x)=\dfrac{3}{2x+1}$

Solution.

1. To find $f(x+h)$, we replace every occurrence of x in the formula $f(x)=x^2-x-2$ with the quantity $(x+h)$ to get

$$\begin{aligned} f(x+h) &= (x+h)^2-(x+h)-2 \\ &= x^2+2xh+h^2-x-h-2 \end{aligned}$$

So the difference quotient is

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\left(x^2+2xh+h^2-x-h-2\right)-\left(x^2-x-2\right)}{h}\\ &= \frac{x^2+2xh+h^2-x-h-2-x^2+x+2}{h}\\ &= \frac{2xh+h^2-h}{h}\\ &= \frac{h(2x+h-1)}{h(1)}\\ &= 2x+h-1\end{aligned}$$

2. To find $g(x+h)$, we replace every occurrence of x in the formula $g(x)=\frac{3}{2x+1}$ with the quantity $(x+h)$ to get

$$\begin{aligned}g(x+h) &= \frac{3}{2(x+h)+1}\\ &= \frac{3}{2x+2h+1}\end{aligned}$$

This yields

$$\begin{aligned}\frac{g(x+h)-g(x)}{h} &= \frac{\frac{3}{2x+2h+1}-\frac{3}{2x+1}}{h}\\ &= \frac{\frac{3}{2x+2h+1}-\frac{3}{2x+1}}{h}\cdot\frac{(2x+2h+1)(2x+1)}{(2x+2h+1)(2x+1)}\\ &= \frac{3(2x+1)-3(2x+2h+1)}{h(2x+2h+1)(2x+1)}\\ &= \frac{6x+3-6x-6h-3}{h(2x+2h+1)(2x+1)}\\ &= \frac{-6h}{h(2x+2h+1)(2x+1)}\\ &= \frac{-6}{(2x+2h+1)(2x+1)}\end{aligned}$$

□

We next introduce composition of functions as a method for combining functions.

Function Composition

Definition 1.9. Suppose f and g are two functions. The **composite** of g with f, denoted $g \circ f$, is defined by the formula $(g \circ f)(x) = g(f(x))$, provided x is in the domain of f and $f(x)$ is in the domain of g.

The quantity $g \circ f$ is also read 'g composed with f' or, more simply, 'g of f'. At its most basic level, **Definition 1.9** tells us that to obtain the formula for $(g \circ f)(x)$, we replace every occurrence of x in the formula for $g(x)$ with the formula we have for $f(x)$.

If we take a step back and look at this from a procedural 'inputs and outputs' perspective, **Definifion 1.9** tells us the output from $g \circ f$ is found by taking the output from f, $f(x)$, and then making that the input to g. The result, $g(f(x))$, is the output from $g \circ f$. From this perspective, we see $g \circ f$ as a two step process taking an input x and first applying the procedure f, then applying the procedure g, as illustrated below.

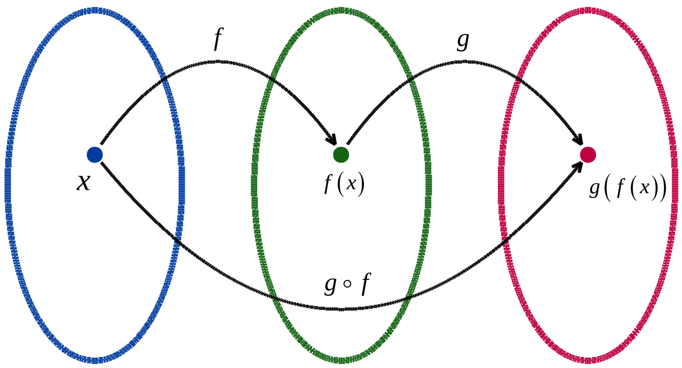


Figure 1.4. 2

In the expression $g(f(x))$, the function f is often called the **inside function** while g is called the **outside function**. We proceed with an example in which we determine formulas for composite functions from three given functions: f, g and h.

Example 1.4.6. Let $f(x) = 2x+3$, $g(x) = \frac{1}{x-1}$ and $h(x) = \sqrt{x+4}$. Find and simplify expressions for the following functions.

1. $(g \circ f)(x)$
2. $(f \circ g)(x)$
3. $(h \circ f)(x)$

4. $(f \circ f)(x)$ 5. $(g \circ g)(x)$ 6. $(h \circ g \circ f)(x)$

Solution.

1. By **Definition 1.9**, $(g \circ f)(x) = g(f(x))$, and we have

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(2x+3) && \textbf{since } f(x) = 2x+3 \\ &= \frac{1}{(2x+3)-1} && \textbf{substitute } (2x+3) \textbf{ for } x \textbf{ in } g(x) = \frac{1}{x-1} \\ &= \frac{1}{2x+2} \end{aligned}$$

2. After swapping the g and f in part 1, we find an expression for $(f \circ g)(x)$.

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) && \textbf{by definition} \\ &= f\left(\frac{1}{x-1}\right) && \textbf{since } g(x) = \frac{1}{x-1} \\ &= 2\left(\frac{1}{x-1}\right)+3 && \textbf{substitute } \left(\frac{1}{x-1}\right) \textbf{ for } x \textbf{ in } f(x) = 2x+3 \\ &= \frac{2}{x-1} + 3 \cdot \frac{x-1}{x-1} && \textbf{obtain common denominator} \\ &= \frac{3x-1}{x-1} \end{aligned}$$

3. We next find an expression for $(h \circ f)(x)$.

$$\begin{aligned} (h \circ f)(x) &= h(f(x)) && \textbf{by definition} \\ &= h(2x+3) && \textbf{since } f(x) = 2x+3 \\ &= \sqrt{(2x+3)+4} && \textbf{substitute } (2x+3) \textbf{ for } x \textbf{ in } h(x) = \sqrt{x+4} \\ &= \sqrt{2x+7} \end{aligned}$$

4. To find $(f \circ f)(x)$, we substitute the function f into itself.

$$\begin{aligned} (f \circ f)(x) &= f(f(x)) && \textbf{by definition} \\ &= f(2x+3) && \textbf{since } f(x) = 2x+3 \\ &= 2(2x+3)+3 && \textbf{substitute } (2x+3) \textbf{ for } x \textbf{ in } f(x) = 2x+3 \\ &= 4x+9 \end{aligned}$$

5. Our next function, $(g \circ g)(x)$, is defined as $g(g(x))$ from which we get

$$\begin{aligned} g(g(x)) &= g\left(\frac{1}{x-1}\right) && \textbf{since } g(x)=\frac{1}{x-1} \\ &= \frac{1}{\left(\frac{1}{x-1}\right)-1} && \textbf{substitute } \left(\frac{1}{x-1}\right) \textbf{ for } x \textbf{ in } g(x)=\frac{1}{x-1} \\ &= \frac{1}{\left(\frac{1}{x-1}-1\right)}\cdot\frac{(x-1)}{(x-1)} && \textbf{simplify compound fraction} \\ &= \frac{x-1}{1-(x-1)} \\ &= \frac{x-1}{2-x} \end{aligned}$$

6. We find an expression for $(h\circ g\circ f)(x)$ by rewriting the function as $(h\circ(g\circ f))(x)$[24] to get

$$\begin{aligned} (h\circ g\circ f)(x) &= (h\circ(g\circ f))(x) \\ &= h((g\circ f)(x)) && \textbf{by definition} \\ &= h\left(\frac{1}{2x+2}\right) && \textbf{from part 1} \\ &= \sqrt{\left(\frac{1}{2x+2}\right)+4} && \textbf{substitute } \left(\frac{1}{2x-2}\right) \textbf{ for } x \textbf{ in } h(x)=\sqrt{x+4} \\ &= \sqrt{\frac{1}{2x+2}+\frac{4\cdot(2x+2)}{2x+2}} && \textbf{obtain common denominator} \\ &= \sqrt{\frac{8x+9}{2x+2}} \end{aligned}$$

□

In the previous example, we found that $(g\circ f)(x)=\frac{1}{2x+2}$ while $(f\circ g)(x)=\frac{3x-1}{x-1}$. In general, when we compose two functions, the order matters. Another observation from this example is the composition of a function with itself. Composing a function with itself is called 'iterating' the function.

Domains of Composite Functions

We continue with composite functions and their domains. From **Definifion 1.9**, we write the composite function $f\circ g$, with an input *x*, as $f(g(x))$, and we can see right away that *x* must be in the domain of g in order to evaluate the inner function. We can also see that $g(x)$ must be in the domain of f;

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[24] This is equivalent to $((h\circ g)\circ f)(x)$. Try it!

otherwise the evaluation of the outer function cannot be completed. Thus, the domain of $f \circ g$ consists of only those inputs in the domain of g that produce outputs belonging to the domain of f. We can say that the domain of f composed with g is the set of all *x* such that *x* is in the domain of g and $g(x)$ is in the domain of f.

Example 1.4.7. Find the domain of $(f \circ g)(x)$ where $f(x) = \dfrac{5}{x-1}$ and $g(x) = \dfrac{4}{3x-2}$.

Solution. To determine the domain of $(f \circ g)(x) = f(g(x))$, we begin with the domain of the inner function, $g(x) = \dfrac{4}{3x-2}$. The domain of $g(x)$ consists of all real numbers except $x = \dfrac{2}{3}$, since that input value would cause us to divide by 0.

By writing the outer function as $f(g(x)) = \dfrac{5}{g(x)-1}$, we can see that we also need to exclude from the domain any values of *x* for which $g(x) = 1$:

$$\begin{aligned} g(x) &= 1 \\ \frac{4}{3x-2} &= 1 \\ 4 &= 3x-2 \\ 6 &= 3x \\ x &= 2 \end{aligned}$$

So the domain of $f \circ g$ is the set of all real numbers except $x = \dfrac{2}{3}$ and $x = 2$. We can write this in interval notation as $\left(-\infty, \dfrac{2}{3}\right) \cup \left(\dfrac{2}{3}, 2\right) \cup (2, \infty)$.

□

Example 1.4.8. Find the domain of $(g \circ f)(x)$ where $f(x) = \sqrt{3-x}$ and $g(x) = \sqrt{x+2}$.

Solution. In this example, since $(g \circ f)(x) = g(f(x))$, the inner function is $f(x)$. To determine the domain of f, since we cannot take the square root of a negative number, we look for values of *x* for which $3 - x \ge 0$.

$$\begin{aligned} 3 - x &\ge 0 \\ 3 &\ge x \\ x &\le 3 \end{aligned}$$

So the domain of f includes all real numbers less than or equal to 3.

Next, we find values of x for which $f(x)$ is in the domain of g. Since $g(f(x))=\sqrt{f(x)+2}$, we must have

$$\begin{aligned} f(x)+2 &\geq 0 \\ \left(\sqrt{3-x}\right)+2 &\geq 0 \\ \sqrt{3-x} &\geq -2 \end{aligned}$$

Knowing that $\sqrt{3-x}\geq -2$ for any value of x, our only restriction on the domain of $g\circ f$ comes from the domain of f. This means the domain of $g\circ f$ is the same as the domain of f, which is $(-\infty,3]$.

□

While our choice of examples for domains of composite functions is somewhat limited until we solve quadratic and rational inequalities in **Chapters 2** and **3**, the previous two examples should provide a basic understanding of the procedure for determining domains of composite functions. We move on to determining composite function values.

Evaluating Composite Functions

Example 1.4.9. Let $f(x)=x^2-4x$ and $g(x)=2-\sqrt{x+3}$. Find the indicated function value.

1. $(g\circ f)(1)$
2. $(f\circ g)(1)$
3. $(g\circ g)(6)$

Solution.

1. Using **Definition 1.9**, $(g\circ f)(1)=g(f(1))$. We find $f(1)=-3$, so

$$\begin{aligned} (g\circ f)(1) &= g(f(1)) \\ &= g(-3) \\ &= 2-\sqrt{(-3)+3} \\ &= 2 \end{aligned}$$

2. As before, we use **Definition 1.9** to write $(f\circ g)(1)=f(g(1))$. We find $g(1)=0$, so

$$\begin{aligned} (f\circ g)(1) &= f(g(1)) \\ &= f(0) \\ &= (0)^2-4(0) \\ &= 0 \end{aligned}$$

3. Once more, **Definition 1.9** tells us $(g\circ g)(6)=g(g(6))$. That is, we evaluate g at 6, then plug that result back into g. Since $g(6)=-1$,

$$\begin{aligned}(g\circ g)(6) &= g(g(6))\\ &= g(-1)\\ &= 2-\sqrt{(-1)+3}\\ &= 2-\sqrt{2}\end{aligned}$$

□

Function composition is often used to relate two quantities that may not be directly related, but have a variable in common, as illustrated in our next example.

Example 1.4.10. The surface area *S* of a sphere is a function of its radius *r*, and is given by the formula $S(r)=4\pi r^2$. Suppose the sphere is being inflated so that the radius of the sphere is increasing according to the formula $r(t)=3t^2$, where *t* is measured in seconds, $t\geq 0$, and *r* is measured in inches. Find and interpret $(S\circ r)(t)$.

Solution. If we look at the functions $S(r)$ and $r(t)$ individually, we see the former gives the surface area of a sphere of a given radius while the latter gives the radius at a given time. So, given a specific time, *t*, we could find the radius at that time, $r(t)$, and feed the radius, *r*, into $S(r)$ to find the surface area at time *t*. From this we see that the surface area *S* is ultimately a function of time *t* and we find

$$\begin{aligned}(S\circ r)(t) &= S(r(t))\\ &= 4\pi(r(t))^2\\ &= 4\pi(3t^2)^2\\ &= 36\pi t^4\end{aligned}$$

The formula $(S\circ r)(t)=36\pi t^4$ allows us to compute the surface area directly given the time without going through the 'middle man' *r*.

□

Decomposing a Composite Function

A useful skill in Calculus is to be able to take a complicated function and break it down into a composition of easier functions as our last example illustrates.

Example 1.4.11. Write each of the following functions as a composition of two or more (non-identity) functions. Check your answer by performing the function composition.

1. $F(x)=|3x-1|$
2. $G(x)=\dfrac{2}{x^2+1}$

Solution. There are many approaches to this kind of problem, and we showcase a different methodology in each of the solutions below.

1. Our goal is to express the function $F(x)=|3x-1|$ as $F=g\circ f$ for functions f and g. From **Definition 1.9**, we know $F(x)=g(f(x))$, where $f(x)$ is the inside function and $g(x)$ is the outside function. Looking at $F(x)=|3x-1|$ from an inside versus outside perspective, we can think of $3x-1$ as being inside the absolute value symbols. Taking this cue, we define $f(x)=3x-1$. At this point, we have $F(x)=|f(x)|$. What is the outside function? The function which takes the absolute value of its input, $g(x)=|x|$. We check our answers for *f* and *g*.

$$\begin{aligned}(g\circ f)(x)&=g(f(x))\\&=|f(x)|\\&=|3x-1|\\&=F(x)\end{aligned}$$

This verifies the solution $F=g\circ f$ where $f(x)=3x-1$ and $g(x)=|x|$.

2. We attack deconstructing $G(x)=\dfrac{2}{x^2+1}$ from an operational approach. Given an input *x*, the first step is to square *x*, then add 1, then divide the result into 2. We will assign each of these steps a function so as to write *G* as a composite of three functions: f, g and h. Our first function, f, is the function that squares its input, $f(x)=x^2$. The next function is the function that adds 1 to its input, $g(x)=x+1$. Our last function takes its input and divides it into 2, $h(x)=\dfrac{2}{x}$. The claim is that $G=h\circ g\circ f$. We test the claim as follows.

$$\begin{aligned}(h\circ g\circ f)(x)&=h(g(f(x)))\\&=h(g(x^2))\\&=h(x^2+1)\\&=\frac{2}{x^2+1}\\&=G(x)\end{aligned}$$

We have verified our solution of $G=h\circ g\circ f$ for $f(x)=x^2$, $g(x)=x+1$ and $h(x)=\dfrac{2}{x}$.

□

1.4 Exercises

1. If the order is reversed when composing two functions, can the result ever be the same as the answer in the original order of the composition? If yes, give an example. If no, explain why not.

2. How do you find the domain for the composition of two functions, $f \circ g$?

In Exercises 3 – 8, use the given function f to find and simplify the following:

(a) $f(4x)$ (b) $4f(x)$ (c) $f(-x)$

(d) $f(x-4)$ (e) $f(x)-4$ (f) $f(x^2)$

3. $f(x)=2x+1$
4. $f(x)=3-4x$
5. $f(x)=2-x^2$
6. $f(x)=x^2-3x+2$
7. $f(x)=\dfrac{x}{x-1}$
8. $f(x)=\dfrac{2}{x^3}$

In Exercises 9 – 16, use the given function f to find and simplify the following:

(a) $f(2a)$ (b) $2f(a)$ (c) $f\left(\dfrac{2}{a}\right)$

(d) $\dfrac{f(a)}{2}$ (e) $f(a+h)$ (f) $f(a)+f(h)$

9. $f(x)=2x-5$
10. $f(x)=5-2x$
11. $f(x)=2x^2-1$
12. $f(x)=3x^2+3x-2$
13. $f(x)=\sqrt{2x+1}$
14. $f(x)=117$
15. $f(x)=\dfrac{x}{2}$
16. $f(x)=\dfrac{2}{x}$

In Exercises 17 – 26, use the pair of functions f and g to find the following values, if they exist.

(a) $(f+g)(2)$ (b) $(f-g)(-1)$ (c) $(g-f)(1)$

(d) $(f\cdot g)\left(\dfrac{1}{2}\right)$ (e) $\left(\dfrac{f}{g}\right)(0)$ (f) $\left(\dfrac{g}{f}\right)(-2)$

17. $f(x)=3x+1$, $g(x)=4-x$

18. $f(x)=x^2$, $g(x)=-2x+1$

19. $f(x)=x^2-x$, $g(x)=12-x^2$

20. $f(x)=2x^3$, $g(x)=-x^2-2x-3$

21. $f(x)=\sqrt{x+3}$, $g(x)=2x-1$

22. $f(x)=\sqrt{4-x}$, $g(x)=\sqrt{x+2}$

23. $f(x)=2x$, $g(x)=\dfrac{1}{2x+1}$

24. $f(x)=x^2$, $g(x)=\dfrac{3}{2x-3}$

25. $f(x)=x^2$, $g(x)=\dfrac{1}{x^2}$

26. $f(x)=x^2+1$, $g(x)=\dfrac{1}{x^2+1}$

In Exercises 27 – 36, use the pair of functions f and g to find and simplify an expression for the indicated function. Determine the domain.

(a) $(f+g)(x)$ (b) $(f-g)(x)$ (c) $(f\cdot g)(x)$ (d) $\left(\dfrac{f}{g}\right)(x)$

27. $f(x)=2x+1$, $g(x)=x-2$

28. $f(x)=1-4x$, $g(x)=2x-1$

29. $f(x)=x^2$, $g(x)=3x-1$

30. $f(x)=x^2-x$, $g(x)=7x$

31. $f(x)=x^2-4$, $g(x)=3x+6$

32. $f(x)=-x^2+x+6$, $g(x)=x^2-9$

33. $f(x)=\dfrac{x}{2}$, $g(x)=\dfrac{2}{x}$

34. $f(x)=x-1$, $g(x)=\dfrac{1}{x-1}$

35. $f(x)=x$, $g(x)=\sqrt{x+1}$

36. $f(x)=\sqrt{x-5}$, $g(x)=f(x)=\sqrt{x-5}$

In Exercises 37 – 54, find and simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ for the given function.

37. $f(x)=2x-5$

38. $f(x)=-3x+5$

39. $f(x)=6$

40. $f(x)=3x^2-x$

41. $f(x)=-x^2+2x-1$

42. $f(x)=4x^2$

43. $f(x)=x-x^2$

44. $f(x)=x^3+1$

45. $f(x)=mx+b$ where $m\neq 0$

46. $f(x)=ax^2+bx+c$ where $a\neq 0$

47. $f(x)=\frac{2}{x}$

48. $f(x)=\frac{3}{1-x}$

49. $f(x)=\frac{1}{x^2}$

50. $f(x)=\frac{2}{x+5}$

51. $f(x)=\frac{1}{4x-3}$

52. $f(x)=\frac{3x}{x+1}$

53. $f(x)=\frac{x}{x-9}$

54. $f(x)=\frac{x^2}{2x+1}$

In Exercises 55 – 78, let f be the function defined by

$$f=\{(-3,4),\ (-2,2),\ (-1,0),\ (0,1),\ (1,3),\ (2,4),\ (3,-1)\}$$

and let g be the function defined by

$$g=\{(-3,-2),\ (-2,0),\ (-1,-4),\ (0,0),\ (1,-3),\ (2,1),\ (3,2)\}$$

Compute the indicated value, if it exists.

55. $(f+g)(-3)$

56. $(f-g)(2)$

57. $(f\cdot g)(-1)$

58. $(g+f)(1)$

59. $(g-f)(3)$

60. $(g\cdot f)(-3)$

61. $\left(\frac{f}{g}\right)(-2)$

62. $\left(\frac{f}{g}\right)(-1)$

63. $\left(\frac{f}{g}\right)(2)$

64. $\left(\frac{g}{f}\right)(-1)$

65. $\left(\frac{g}{f}\right)(3)$

66. $\left(\frac{g}{f}\right)(-3)$

67. $(f\circ g)(3)$

68. $f(g(-1))$

69. $(f\circ f)(0)$

70. $(f\circ g)(-3)$

71. $(g\circ f)(3)$

72. $g(f(-3))$

73. $(g\circ g)(-2)$

74. $(g\circ f)(-2)$

75. $g(f(g(0)))$

76. $f(f(f(-1)))$

77. $f(f(f(f(f(1)))))$

78. $(\underbrace{g\circ g\circ\cdots\circ g}_{\text{n times}})(0)$

In Exercises 79 – 90, use the given pair of functions to find the following values, if they exist.

(a) $(g\circ f)(0)$

(b) $(f\circ g)(-1)$

(c) $(f\circ f)(2)$

(d) $(g \circ f)(-3)$ (e) $(f \circ g)\left(\frac{1}{2}\right)$ (f) $(f \circ f)(-2)$

79. $f(x)=x^2,\ g(x)=2x+1$

80. $f(x)=4-x,\ g(x)=1-x^2$

81. $f(x)=4-3x,\ g(x)=|x|$

82. $f(x)=|x-1|,\ g(x)=x^2-5$

83. $f(x)=4x+5,\ g(x)=\sqrt{x}$

84. $f(x)=\sqrt{3-x},\ g(x)=x^2+1$

85. $f(x)=6-x-x^2,\ g(x)=x\sqrt{x+10}$

86. $f(x)=\sqrt[3]{x+1},\ g(x)=4x^2-x$

87. $f(x)=\dfrac{3}{1-x},\ g(x)=\dfrac{4x}{x^2+1}$

88. $f(x)=\dfrac{x}{x+5},\ g(x)=\dfrac{2}{7-x^2}$

89. $f(x)=\dfrac{2x}{5-x^2},\ g(x)=\sqrt{4x+1}$

90. $f(x)=\sqrt{2x+5},\ g(x)=\dfrac{10x}{x^2+1}$

In Exercises 91 – 100, use the given pair of functions to find and simplify expressions for the following functions. State the domain of each using interval notation.

(a) $(g \circ f)(x)$ (b) $(f \circ g)(x)$ (c) $(f \circ f)(x)$

91. $f(x)=2x+3,\ g(x)=x^2-9$

92. $f(x)=x^2-x+1,\ g(x)=3x-5$

93. $f(x)=x^2-4,\ g(x)=|x|$

94. $f(x)=3x-5,\ g(x)=\sqrt{x}$

95. $f(x)=|x+1|,\ g(x)=\sqrt{x}$

96. $f(x)=|x|,\ g(x)=\sqrt{4-x}$

97. $f(x)=\dfrac{1}{x},\ g(x)=x-3$

98. $f(x)=3x-1,\ g(x)=\dfrac{1}{x+3}$

99. $f(x)=\dfrac{3x}{x-1},\ g(x)=\dfrac{x}{x-3}$

100. $f(x)=\dfrac{x}{2x+1},\ g(x)=\dfrac{2x+1}{x}$

In Exercises 101 – 104, use the given pair of functions to find and simplify expressions for the following composite functions. You are not required to find the domain.

(a) $(g \circ f)(x)$ (b) $(f \circ g)(x)$ (c) $(f \circ f)(x)$

101. $f(x)=3-x^2,\ g(x)=\sqrt{x+1}$

102. $f(x)=x^2-x-1,\ g(x)=\sqrt{x-5}$

103. $f(x)=\dfrac{2x}{x^2-4},\ g(x)=\sqrt{1-x}$

104. $f(x)=\sqrt{2-4x},\ g(x)=-\dfrac{3}{x}$

In Exercises 105 – 110, use $f(x) = -2x$, $g(x) = \sqrt{x}$ and $h(x) = |x|$ to find and simplify expressions for the following composite functions. State the domain of each using interval notation.

105. $(h \circ g \circ f)(x)$
106. $(h \circ f \circ g)(x)$
107. $(g \circ f \circ h)(x)$
108. $(g \circ h \circ f)(x)$
109. $(f \circ h \circ g)(x)$
110. $(f \circ g \circ h)(x)$

In Exercises 111 – 122, write the given function as a composition of two or more non-identity[25] functions. (There are several correct answers so check your answer using function composition.)

111. $p(x) = (2x+3)^3$
112. $P(x) = (x^2 - x + 1)^5$
113. $h(x) = \sqrt{2x-1}$
114. $H(x) = |7-3x|$
115. $r(x) = \dfrac{2}{5x+1}$
116. $R(x) = \dfrac{7}{x^2-1}$
117. $q(x) = \dfrac{|x|+1}{|x|-1}$
118. $Q(x) = \dfrac{2x^3+1}{x^3-1}$
119. $v(x) = \dfrac{2x+1}{3-4x}$
120. $V(x) = \dfrac{x^2}{x^4+1}$
121. $w(x) = \dfrac{1}{(x-2)^3}$
122. $W(x) = \left(\dfrac{1}{2x-3}\right)^2$

123. Write the function $F(x) = \sqrt{\dfrac{x^3+6}{x^3-9}}$ as a composition of three or more non-identity functions.

124. Let $g(x) = -x$, $h(x) = x+2$, $j(x) = 3x$ and $k(x) = x-4$. In what order must these functions be composed with $f(x) = \sqrt{x}$ to create $F(x) = 3\sqrt{-x+2} - 4$?

125. What linear functions could be used to transform $f(x) = x^3$ into $F(x) = -\frac{1}{2}(2x-7)^3 + 1$? What is the proper order of composition?

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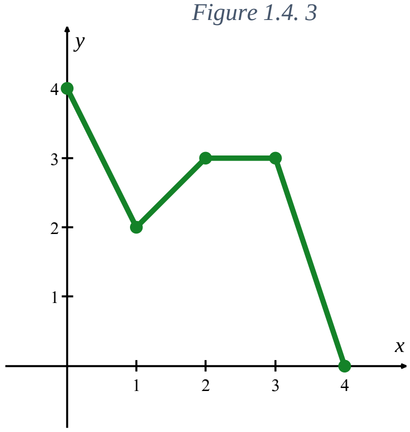
[25] The identity function is $I(x) = x$.

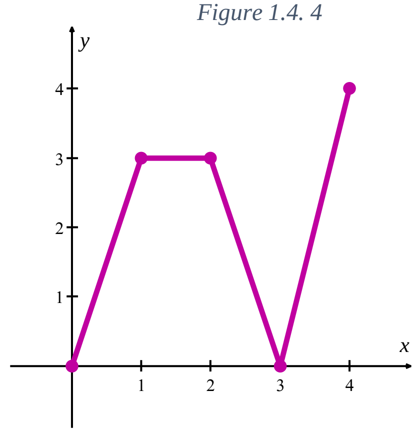
In Exercises 126 – 131, use the graphs of $y = f(x)$ and $y = g(x)$ below to find the function value.

Figure 1.4. 3



$y = f(x)$

Figure 1.4. 4



$y = g(x)$

126. $(g \circ f)(1)$
127. $(f \circ g)(3)$
128. $(g \circ f)(2)$
129. $(f \circ g)(0)$
130. $(f \circ f)(1)$
131. $(g \circ g)(1)$

132. The volume *V* of a cube is a function of its side length *x*. Let's assume that $x = t + 1$ is also a function of time *t*, where *x* is measured in inches and *t* is measured in minutes. Find a formula for *V* as a function of *t*.

133. A store offers a 30% discount on the price *x* of selected items. Then, the store takes off an additional 15% at the cash register. Use function composition to find a price function $P(x)$ that computes the final price of the item in terms of the original price *x*.

134. A rain drop hitting a lake makes a circular ripple. If the radius, in inches, grows as a function of time in minutes according to $r(t) = 25\sqrt{t+2}$, find the area of the ripple as a function of time. Find the area of the ripple at $t = 2$.

135. Use the function you found in the previous exercise to find the area of the ripple after 5 minutes.

136. The number of bacteria in a refrigerated food product is given by $N(T) = 23T^2 - 56T + 1$, $3 < T < 33$, where *T* is the temperature of the food. When the food is removed from the refrigerator, the temperature is given by $T(t) = 5t + 1.5$, where *t* is the time in hours. Find the composite function $N(T(t))$, and use it to find the time when the bacteria count reaches 6752. Round your final answer to two decimal places.

137. Discuss with your classmates how real-world processes such as filling out federal income tax forms or computing your final course grade could be viewed as a use of function composition. Find a process for which composition with itself (iteration) makes sense.

1.5 Inverses of Functions

Learning Objectives

- Verify that two functions are inverses of each other.
- Determine if a function is one-to-one.
- Use the graph of a one-to-one function to graph its inverse function.
- Find the inverse of a one-to-one function.

Thinking of a function as a process, we seek another function which might reverse that process. As in real life, we will find that some processes (like the process of putting socks on followed by putting shoes on) are reversible while some (like cooking a steak) are not. We start by discussing a very basic function which is reversible, $f(x)=3x+4$. Thinking of f as a process, we start with an input x and apply two steps:

1. Multiply by 3.
2. Add 4.

To reverse this process, we seek a function g that will undo each of these steps by taking the output $3x+4$ from f and returning the input x. If we think of the real-world reversible two-step process of first putting on socks then putting on shoes, to reverse the process we first take off the shoes and then we take off the socks. In much the same way, the function g should undo the second step of f first. That is, g should

1. Subtract 4.
2. Divide by 3.

Following this procedure, we get $g(x)=\frac{x-4}{3}$. Let's check to see if the function g does the job. For the specific value of $x=5$, we have $f(5)=3(5)+4=19$. We next find $g(19)=\frac{19-4}{3}=5$, which is the original input to f. So g works for an x value of 5. Now, to verify that g is the inverse of f, we must show that it works for any value of x.

For any real number *x*, we have $f(x)=3x+4$. Then $g(f(x))=\frac{(3x+4)-4}{3}=\frac{3x}{3}=x$, which is the original input to f. So g is indeed the inverse of f. By carefully examining the arithmetic, we see g first 'undoing' the addition of 4, and then 'undoing' the multiplication by 3.

Not only does g undo f, but f also undoes g. That is, if we take the output from g, $g(x)=\frac{x-4}{3}$, and put it into f, we get $f(g(x))=f\left(\frac{x-4}{3}\right)=3\left(\frac{x-4}{3}\right)+4=(x-4)+4=x$.

Defining Inverse Functions

Using the language of function composition developed in **Section 1.4**, the statements $g(f(x))=x$ and $f(g(x))=x$ can be written as $(g\circ f)(x)=x$ and $(f\circ g)(x)=x$, respectively. The main idea is that g takes the outputs from f and returns them to their respective inputs and, conversely, f takes outputs from g and returns them to their respective inputs. We now have enough background to state the central definition of this section.

Definition 1.10. Function g is the **inverse** of function f if

1. $(g\circ f)(x)=x$ for all *x* in the domain of f **and**
2. $(f\circ g)(x)=x$ for all *x* in the domain of g.

If such a function g exists, function f is said to be **invertible** and function f is also the inverse of function g. Additionally, f and g are **inverses** of each other.

Example 1.5.1. Verify that functions $f(x)=\frac{1}{x+2}$ and $g(x)=\frac{1}{x}-2$ are inverses of each other.

Solution. We first show that $(g\circ f)(x)=x$ for all *x* in the domain of f, which is $\{x\mid x\neq -2\}$. For $x\neq -2$,

$$\begin{aligned}(g\circ f)(x)&=g(f(x))\\&=\frac{1}{\left(\frac{1}{x+2}\right)}-2\\&=x+2-2\\&=x\end{aligned}$$

With $(g \circ f)(x) = x$, we've met the first condition of **Definition 1.10**. Now, we show that $(f \circ g)(x) = x$ for all x in the domain of g, which is $\{x \mid x \neq 0\}$. For $x \neq 0$,

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= \frac{1}{\left(\frac{1}{x} - 2\right) + 2} \\ &= \frac{1}{\left(\frac{1}{x}\right)} \\ &= x\end{aligned}$$

With $(f \circ g)(x) = x$, we have met the second condition of **Definition 1.10**. So, we have verified that f and g are inverses of each other.

□

At this point, we may wonder if every function is invertible or if there is a function equivalent to cooking a steak that is not reversible.

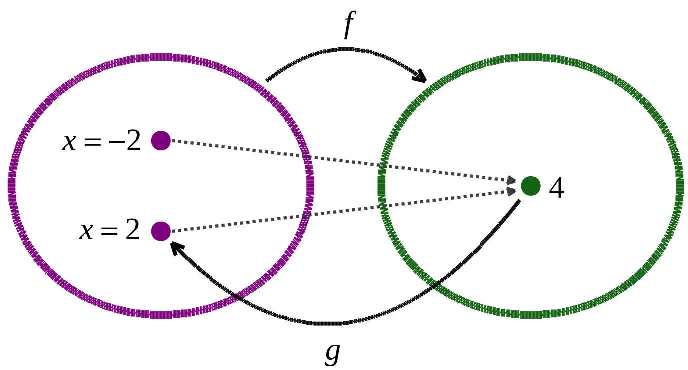
Functions that are Invertible

Let's consider the function $f(x) = x^2$ with domain $(-\infty, \infty)$. Is there a function that is the inverse of the function f? A likely candidate for reversing the process of squaring is $g(x) = \sqrt{x}$. By **Definition 1.10**, $(g \circ f)(x)$ must be equal to x for all x in the domain of f. We find that this is not always the case. For example, when $x = -2$,

$$\begin{aligned}g(f(-2)) &= g\left((-2)^2\right) \\ &= g(4) \\ &= \sqrt{4} \\ &= 2\end{aligned}$$

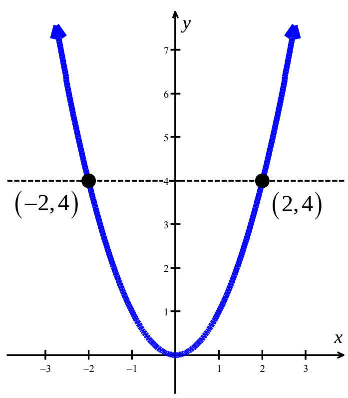
Since $g(f(-2)) \neq -2$, function g is not the inverse of f. For further insight into the situation, we note that $f(2) = 4$, from which it follows that $g(f(2)) = 2$. This is presented schematically in the following picture.

Figure 1.5. 1



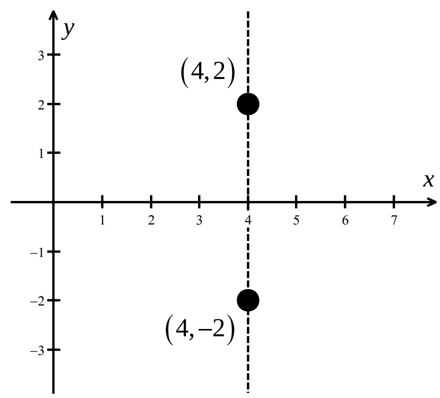
We see from the diagram that both $f(-2)$ and $f(2)$ are 4. By definition, a function g can take 4 back to only −2 or 2, not both. From a graphical standpoint, points $(-2,4)$ and $(2,4)$, which lie on a horizontal line, are on the graph of f, but by the vertical line test, no function can contain both points $(4,-2)$ and $(4,2)$.

Figure 1.5. 2



$y = f(x) = x^2$

Figure 1.5. 3



$y = g(x)$: a function?

Therefore, for a function to have an inverse, different inputs must go to different outputs or else we will run into the same problem we did with $f(x) = x^2$. We give this property a name.

Definition 1.11. A function f is said to be **one-to-one** if f matches different inputs to different outputs. Equivalently, f is one-to-one if and only if whenever $f(c) = f(d)$, then $c = d$.

Graphically, we detect one-to-one functions using the test below.

Theorem 1.2. The Horizontal Line Test: A function f is one-to-one if and only if no horizontal line intersects the graph of f more than once.

A function being one-to-one is enough to guarantee invertibility since every output value corresponds to a unique input value and, hence, the process is reversible. We summarize these results below.

Theorem 1.3. Equivalent Conditions for Invertibility: Suppose f is a function. The following statements are equivalent.

- f is invertible.
- f is one-to-one.
- The graph of f passes the horizontal line test.

We put this result to work in the next example.

Example 1.5.2. Determine if the following functions are one-to-one in two ways: (a) analytically using **Definition 1.11** and (b) graphically using the horizontal line test.

1. $f(x)=\dfrac{1}{x+2}$
2. $g(x)=x^2-2x+4$

Solution.

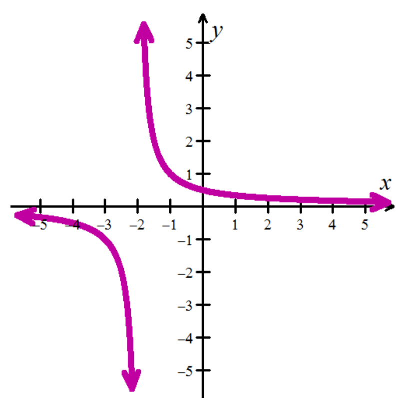
1. (a) We begin with the assumption that $f(c)=f(d)$ and try to show $c=d$.

$$\begin{aligned} f(c) &= f(d) \\ \frac{1}{c+2} &= \frac{1}{d+2} && \text{since } f(x)=\frac{1}{x+2} \\ (1)(d+2) &= (1)(c+2) \\ d+2 &= c+2 \\ d &= c \end{aligned}$$

We have shown that f is one-to-one.

(b) We can graph $f(x)=\dfrac{1}{x+2}$ through transformations of the toolkit function $y=\dfrac{1}{x}$. To graph $y=f(x)$, we shift $y=\dfrac{1}{x}$ to the left by 2 units. We see that the graph of f passes the horizontal line test, verifying f is one-to-one.

Figure 1.5. 4



$y=f(x)$

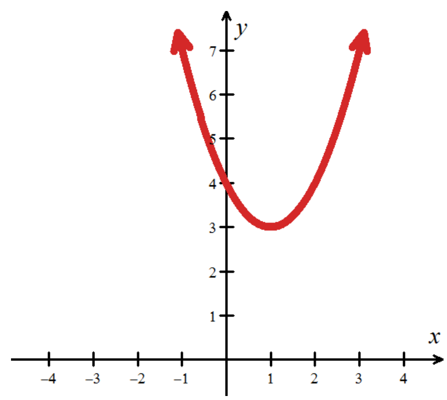
2. (a) We begin with $g(c)=g(d)$. As we work our way through the problem, we encounter a nonlinear equation. We move the non-zero terms to the left, leave a 0 on the right, and factor accordingly.

$$\begin{aligned} g(c)&=g(d) \\ c^2-2c+4&=d^2-2d+4 && \textbf{since } g(x)=x^2-2x+4 \\ c^2-2c&=d^2-2d && \textbf{subtract } 4 \\ c^2-d^2-2c+2d&=0 \\ (c+d)(c-d)-2(c-d)&=0 \\ ((c+d)-2)(c-d)&=0 && \textbf{factor by grouping} \end{aligned}$$

Setting each factor equal to zero results in $c=2-d$ or $c=d$. These two possibilities for c suggest that g may not be one-to-one. Taking $d=0$ results in $c=0$ or $c=2$. With $g(0)=4$ and $g(2)=4$, we have two different inputs with the same output meaning that g is not one-to-one.

(b) We note that $g(x)=x^2-2x+4$ can be written as $g(x)=(x-1)^2+3$ and as such may be graphed as a transformation of the toolkit function $y=x^2$. We see immediately from the graph[26] that g is not one-to-one since there are horizontal lines that cross the graph more than once.

Figure 1.5. 5



$y=g(x)$

□

We next observe some properties of inverse functions that follow from **Definition 1.10**.

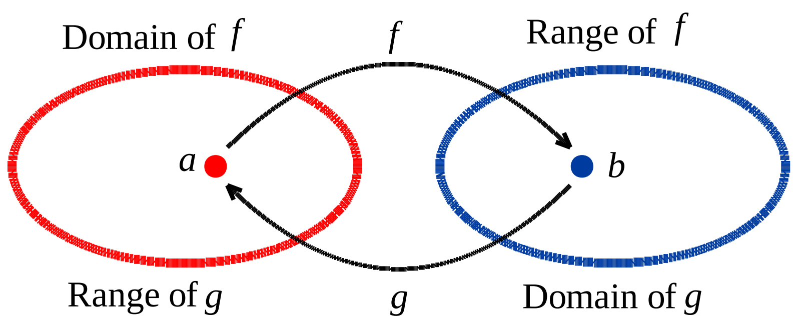
Properties of Inverse Functions

In writing $(g\circ f)(x)=g(f(x))=x$, for all x in the domain of f, it is implied that each output value of f, or $f(x)$, is an input value of g. Likewise, $(f\circ g)(x)=f(g(x))=x$, for all x in the domain of g, implies that each output value of g, or $g(x)$, is an input value of f and every input value of g, or x, is

[26] Practice with graphing quadratic functions is coming in **Chapter 2**.

an output value of f. Taken together, it means that the domain of f is the range of g and the range of f is the domain of g. We can visualize the situation in the following diagram.

Figure 1.5. 6



We next formalize this concept that inverse functions exchange inputs and outputs.

Theorem 1.4. Properties of Inverse Functions: Suppose functions f and g are inverses of each other.

- The range[27] of f is the domain of g and the domain of f is the range of g.
- $f(a)=b$ if and only if $g(b)=a$.
- Point (a,b) is on the graph of f if and only if point (b,a) is on the graph of g.

One implication of **Theorem 1.4** is that an inverse function g is uniquely defined by the invertible function f, since we know the domain of g and all its function values. Notice that points (a,b) and (b,a) are symmetric about the line $y=x$ since this line is a perpendicular bisector[28] of the line segment connecting them, as shown on the following graph, on the left.

[27] Recall that this is the set of all outputs of a function.

[28] This can be verified algebraically by showing that the slope of the line segment is -1 and the distance from (a,b) to a point on the line is the same as the distance from (b,a) to a point on the line.

Figure 1.5. 7

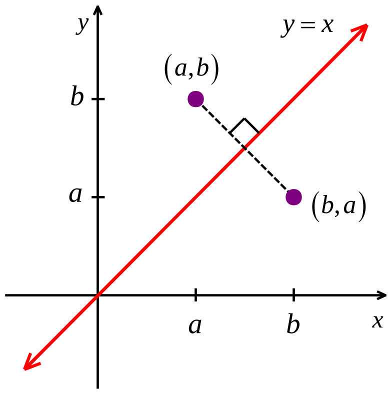
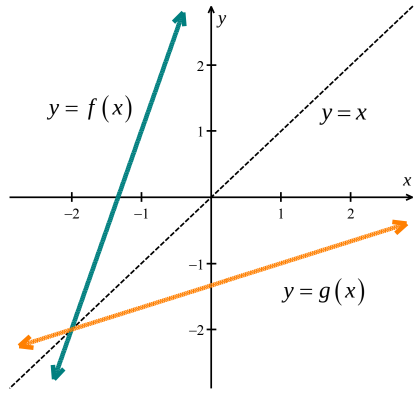


Figure 1.5. 8



Thus, another implication of **Theorem 1.4** is that graphs of an invertible function and its inverse are symmetric about the line $y = x$. We demonstrate this property above, to the right, with graphs of the inverse functions $f(x) = 3x + 4$ and $g(x) = \frac{x-4}{3}$.

We present the following theorem to summarize properties of uniqueness and graphing.

Theorem 1.5. Uniqueness of Inverse Functions and Their Graphs: Suppose f is an invertible function.

- There is exactly one inverse function for f, denoted f^{-1} (read '*f* inverse')
- The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across the line $y = x$.

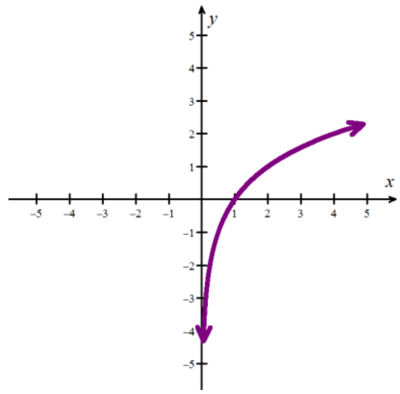
The notation f^{-1} is an unfortunate choice, having been used in prior math classes to represent $\frac{1}{f}$. This is most definitely not the case since, for instance, $f(x) = 3x + 4$ has as its inverse $f^{-1}(x) = \frac{x-4}{3}$, which is certainly different than $\frac{1}{f(x)} = \frac{1}{3x+4}$.

Finding the Inverse of a One-to-One Function

We begin by using the graph of a function to sketch the inverse function. From **Theorem 1.5**, the graph of the inverse function, f^{-1}, is the reflection of the graph of *f* about the line $y = x$.

Example 1.5.3. Given the following graph of f, sketch a graph of f^{-1}.

Figure 1.5. 9

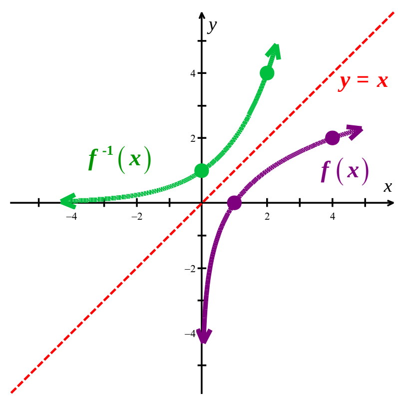


$y = f(x)$

Solution. We first observe that f is a one-to-one function, so will have an inverse. We also note that the graph has an apparent domain of $(0,\infty)$ and range of $(-\infty,\infty)$. From **Theorem 1.4**, the inverse will have a domain of $(-\infty,\infty)$ and range of $(0,\infty)$.

If we reflect the graph of $y = f(x)$ across the line $y = x$, the point $(1,0)$ reflects to $(0,1)$ and the point $(4,2)$ reflects to $(2,4)$. We sketch the inverse on the same axes as the original function, shown below.

Figure 1.5. 10



□

To find the inverse of the equation of a function, we follow a general methodology, as outlined below. **Theorem 1.4** tells us the equation $y = f^{-1}(x)$ is equivalent to $f(y) = x$ and this is the basis of our algorithm.

Steps for Finding the Inverse of a One-to-One Function
1. Write $y = f(x)$.
2. Interchange *x* and *y*.
3. Solve $x = f(y)$ for *y* to obtain $y = f^{-1}(x)$.

Note that we could have simply written 'Solve $x = f(y)$ for y' and be done with it. The act of interchanging the x and y is there to remind us that we are finding the inverse function by switching the inputs and the outputs.

Example 1.5.4. Find the inverse of the one-to-one function $f(x) = \dfrac{2-3x}{4}$ and graph $y = f(x)$ and $y = f^{-1}(x)$ on the same coordinate system.

Solution.

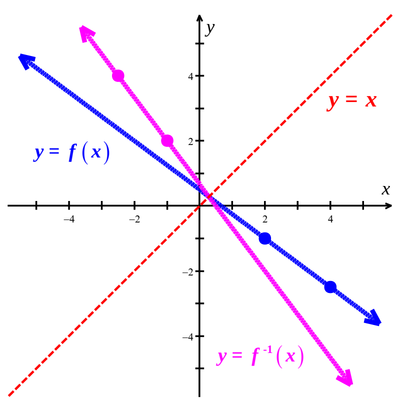
We will use the algorithm to find the inverse of $f(x) = \dfrac{2-3x}{4}$.

$$\begin{aligned} y &= \frac{2-3x}{4} && \text{from } y = f(x) \\ x &= \frac{2-3y}{4} && \text{switch } x \text{ and } y \\ 4x &= 2-3y \\ 4x-2 &= -3y \\ \frac{4x-2}{-3} &= y \end{aligned}$$

We can rewrite the resulting equation as $y = -\dfrac{4}{3}x + \dfrac{2}{3}$ and, last of all, will refer to this inverse function as $f^{-1}(x) = -\dfrac{4}{3}x + \dfrac{2}{3}$.

Using points $(2,-1)$ and $\left(4,-\dfrac{5}{2}\right)$ on the graph of f, along with corresponding points $(-1,2)$ and $\left(-\dfrac{5}{2},4\right)$ on the graph of f^{-1}, the two functions are graphed below. Notice that their graphs are symmetric about the line $y = x$.

Figure 1.5. 11



□

Example 1.5.5. Find the inverse of the one-to-one function $g(x)=\dfrac{2x}{1-x}$ and use it to determine the range of the function g. Verify your answer analytically.

Solution. We will use the algorithm to find the inverse of $g(x)=\dfrac{2x}{1-x}$.

$$
\begin{aligned}
y &= \frac{2x}{1-x} && \textbf{from } y=g(x)\\
x &= \frac{2y}{1-y} && \textbf{switch } x \textbf{ and } y\\
x(1-y) &= 2y && \textbf{multiply by } (1-y)\\
x-xy &= 2y && \textbf{expand}\\
x &= 2y+xy && \textbf{isolate terms containing } y \textbf{ on one side}\\
x &= (2+x)y && \textbf{factor}\\
\frac{x}{2+x} &= y && \textbf{solve for } y
\end{aligned}
$$

We have $g^{-1}(x)=\dfrac{x}{x+2}$. The range of g is the domain of g^{-1}, which is $(-\infty,-2)\cup(-2,\infty)$.

To check this result analytically, we first verify that $(g^{-1}\circ g)(x)=x$ for all x in the domain of g; that is, for all $x\neq 1$.

$$
\begin{aligned}
(g^{-1}\circ g)(x) &= g^{-1}(g(x))\\
&= g^{-1}\left(\frac{2x}{1-x}\right)\\
&= \frac{\left(\frac{2x}{1-x}\right)}{\left(\frac{2x}{1-x}\right)+2} && \textbf{since } g^{-1}(x)=\frac{x}{x+2}\\
&= \frac{\left(\frac{2x}{1-x}\right)}{\left(\frac{2x}{1-x}\right)+2}\cdot\frac{(1-x)}{(1-x)} && \textbf{clear denominators}\\
&= \frac{2x}{2x+2(1-x)}\\
&= \frac{2x}{2}\\
&= x
\end{aligned}
$$

Next, we verify that $(g\circ g^{-1})(x)=x$ for all $x\neq 2$.

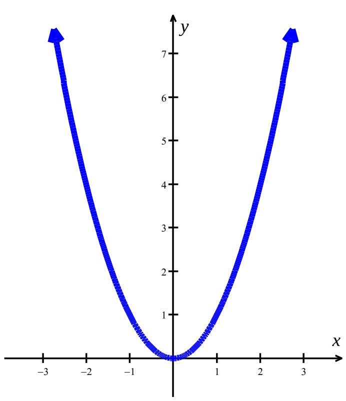
$$\begin{aligned}(g\circ g^{-1})(x) &= g\left(g^{-1}(x)\right)\\ &= g\left(\frac{x}{x+2}\right)\\ &= \frac{2\left(\frac{x}{x+2}\right)}{1-\left(\frac{x}{x+2}\right)} && \textbf{since } g(x)=\frac{2x}{1-x}\\ &= \frac{2\left(\frac{x}{x+2}\right)}{1-\left(\frac{x}{x+2}\right)}\cdot\frac{(x+2)}{(x+2)} && \textbf{clear denominators}\\ &= \frac{2x}{(x+2)-x}\\ &= \frac{2x}{2}\\ &= x\end{aligned}$$

We have verified that g and g^{-1} are inverses.

□

We return to $f(x)=x^2$. We know that f, with domain $(-\infty,\infty)$, is not one-to-one, and thus is not invertible. However, if we restrict the domain of f, we can produce a new function g which is one-to-one. If we define $g(x)=x^2, x\geq 0$, then we have the following.

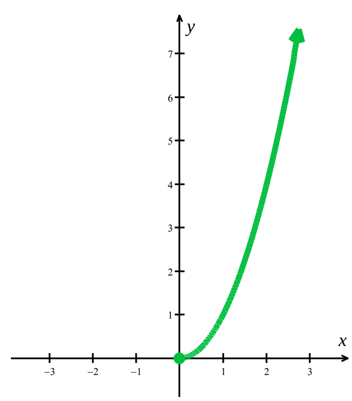
Figure 1.5. 12



$y=f(x)=x^2$

restrict domain to $x\geq 0$

Figure 1.5. 13



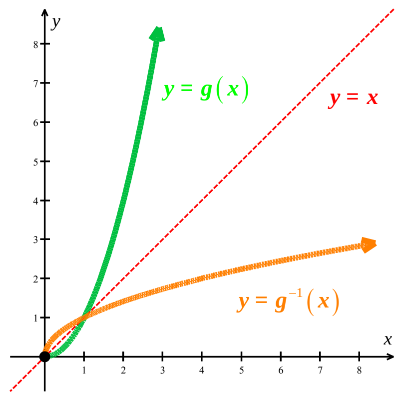
$y=g(x)=x^2,\ x\geq 0$

The graph of g passes the horizontal line test. To find an inverse of g, we proceed as usual.

$$\begin{aligned}y &= g(x)\\ y &= x^2, x\geq 0\\ x &= y^2, y\geq 0 && \textbf{switch } x \textbf{ and } y\\ y &= \pm\sqrt{x}\end{aligned}$$

Since $y \geq 0$, we find $y = \sqrt{x}$, and so $g^{-1}(x) = \sqrt{x}$. With the restriction on the domain of g, we find that $\left(g^{-1} \circ g\right)(x) = x$ and $\left(g \circ g^{-1}\right)(x) = x$. Graphing g and g^{-1} on the same set of axes shows they are reflections about the line $y = x$.

Figure 1.5. 14



Our next example continues the theme of domain restriction.

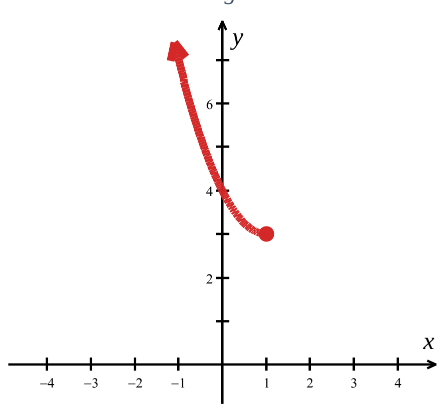
Example 1.5.6. Graph the following functions to show they are one-to-one and find their inverses.

1. $j(x) = x^2 - 2x + 4,\ x \leq 1$
2. $k(x) = \sqrt{x+2} - 1$

Solution.

1. The function j is a restriction of the function g from **Example 1.5.2**. Since the domain of j is restricted to $x \leq 1$, we are selecting only the left half of the parabola. We see that the graph of j passes the horizontal line test and thus j is invertible.

Figure 1.5. 15



$y = j(x)$

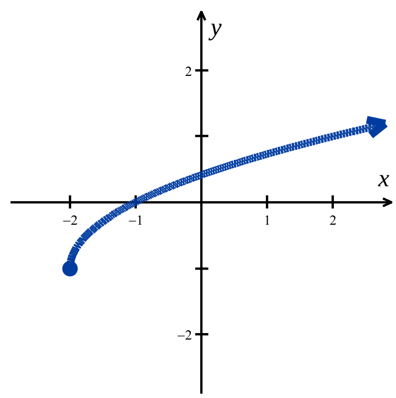
We now use our algorithm to find $j^{-1}(x)$.[29]

$$y = x^2 - 2x + 4,\ x \le 1 \qquad \text{from } y = j(x)$$
$$x = y^2 - 2y + 4,\ y \le 1 \qquad \text{switch } x \text{ and } y$$
$$0 = y^2 - 2y + 4 - x$$
$$y = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(4-x)}}{2(1)} \qquad \text{quadratic formula, } c = 4 - x$$
$$y = \frac{2 \pm \sqrt{4x - 12}}{2}$$
$$y = \frac{2 \pm 2\sqrt{x-3}}{2}$$
$$y = 1 \pm \sqrt{x-3}$$
$$y = 1 - \sqrt{x-3} \qquad \text{since } y \le 1$$

We have $j^{-1}(x) = 1 - \sqrt{x-3}$.

2. We graph $k(x) = \sqrt{x+2} - 1$ using transformations of the toolkit function $y = \sqrt{x}$.

Figure 1.5. 16



$y = k(x)$

We next find k^{-1}.

[29] Here, we use the Quadratic Formula to solve for *y*. We note that you can (and should!) also consider solving for *y* by completing the square.

$$
\begin{aligned}
y &= \sqrt{x+2}-1 \quad \textbf{from } y = k(x) \\
x &= \sqrt{y+2}-1 \quad \textbf{switch } \boldsymbol{x} \textbf{ and } \boldsymbol{y} \\
x+1 &= \sqrt{y+2} \\
(x+1)^2 &= \left(\sqrt{y+2}\right)^2 \\
x^2+2x+1 &= y+2 \\
y &= x^2+2x-1
\end{aligned}
$$

We have $k^{-1}(x) = x^2+2x-1$. Noting that k^{-1} is a quadratic function, we recall seeing several quadratic functions in this section that were not one-to-one unless their domains were suitably restricted. **Theorem 1.4** tells us that the domain of k^{-1} is the range of k. From the graph of k, we see that the range is $[-1,\infty)$, which means we restrict the domain of k^{-1} to $x \geq -1$. Our final answer for the inverse of k is $k^{-1}(x) = x^2+2x-1$, $x \geq -1$. This result can be verified through showing that $(k^{-1} \circ k)(x) = x$ and $(k \circ k^{-1})(x) = x$.

□

1.5 Exercises

1. Why do we restrict the domain of the function $f(x) = x^2$ to find the function's inverse?

2. Are one-to-one functions either always increasing or always decreasing? Why or why not?

In Exercises 3 – 4, use function composition to verify that $f(x)$ and $g(x)$ are inverse functions.

3. $f(x) = \sqrt[3]{x-1}$ and $g(x) = x^3 + 1$

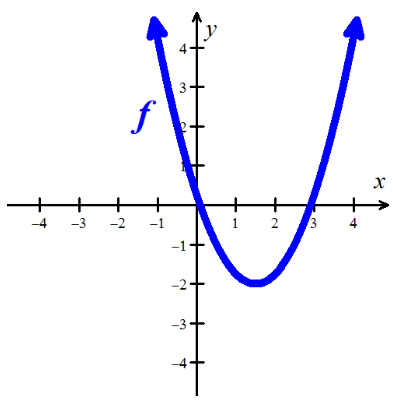
4. $f(x) = -3x + 5$ and $g(x) = \dfrac{x-5}{-3}$

5. Show that the function $f(x) = 3 - x$ is its own inverse.

In Exercises 6 – 7, determine whether the graph represents a one-to-one function.

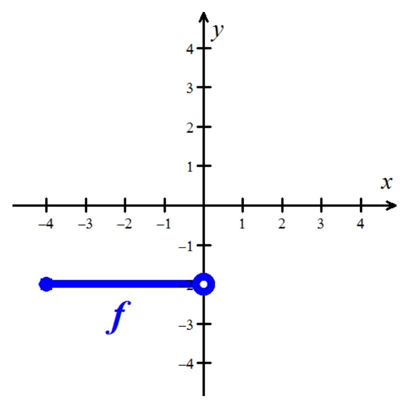
6.

Figure 1.5. 17



7.

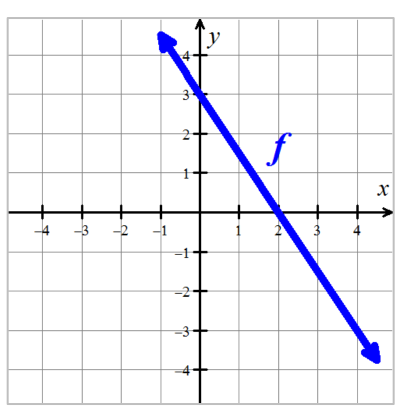
Figure 1.5. 18



In Exercises 8 – 9, sketch the graph of the inverse function.

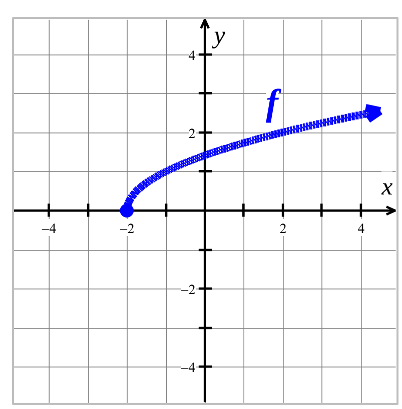
8.

Figure 1.5. 19



9.

Figure 1.5. 20



In Exercises 10 – 19, show that the given function is one-to-one and find its inverse. Check your answer algebraically and graphically. Verify that the range of f is the domain of f^{-1} and vice-versa.

10. $f(x)=6x-2$
11. $f(x)=42-x$
12. $f(x)=\dfrac{x-2}{3}+4$
13. $f(x)=1-\dfrac{4+3x}{5}$
14. $f(x)=\sqrt{3x-1}+5$
15. $f(x)=2-\sqrt{x-5}$
16. $f(x)=3\sqrt{x-1}-4$
17. $f(x)=1-2\sqrt{2x+5}$
18. $f(x)=3(x+4)^2-5,\ x\le -4$
19. $f(x)=\dfrac{3}{4-x}$

In Exercises 20 – 29, find the inverse of the given one-to-one function.

20. $f(x)=\sqrt[5]{3x-1}$
21. $f(x)=3-\sqrt[3]{x-2}$
22. $f(x)=x^2-10x,\ x\ge 5$
23. $f(x)=x^2-6x+5,\ x\le 3$
24. $f(x)=4x^2+4x+1,\ x<-1$
25. $f(x)=\dfrac{x}{1-3x}$
26. $f(x)=\dfrac{2x-1}{3x+4}$
27. $f(x)=\dfrac{4x+2}{3x-6}$
28. $f(x)=\dfrac{-3x-2}{x+3}$
29. $f(x)=\dfrac{x-2}{2x-1}$

With the help of your classmates, find the inverses of the functions in Exercises 30 – 33.

30. $f(x)=a\,x+b,\ a\ne 0$
31. $f(x)=a\sqrt{x-h}+k,\ a\ne 0,\ x\ge h$
32. $f(x)=a\,x^2+b\,x+c$ where $a\ne 0,\ x\ge -\dfrac{b}{2a}$
33. $f(x)=\dfrac{a\,x+b}{c\,x+d}$ (See **Exercise 39** below.)

34. Show that the Fahrenheit to Celsius conversion function, $C(F)=\dfrac{5}{9}(F-32)$, is invertible and that its inverse is the Celsius to Fahrenheit conversion formula, $F(C)=\dfrac{9}{5}C+32$.

35. A car travels at a constant speed of 50 miles per hour. The distance the car travels in miles is a function of time, t, in hours given by $d(t)=50t$. Find the inverse function by expressing the time of travel in terms of the distance traveled. Call this function $t(d)$. Find $t(180)$ and interpret its meaning.

36. The circumference C of a circle is a function of its radius given by $C(r)=2\pi r$. Express the radius of a circle as a function of its circumference. Call this function $r(C)$. Find $r(36\pi)$ and interpret its meaning.

37. With the help of your classmates, explain why a function which is either strictly increasing or strictly decreasing on its entire domain would have to be one-to-one, hence invertible.

38. What graphical feature must a function f possess for it to be its own inverse?

39. What conditions must you place on the values of a, b, c and d in **Exercise 33** in order to guarantee that the function is invertible?

Key Equations

Constant Function: $f(x)=c$, where c is a constant

Identity function: $f(x)=x$

Absolute Value Function: $f(x)=|x|$

Quadratic Function: $f(x)=x^2$

Cubic Function: $f(x)=x^3$

Reciprocal Function: $f(x)=\frac{1}{x}$

Reciprocal Squared Function: $f(x)=\frac{1}{x^2}$

Square Root Function: $f(x)=\sqrt{x}$

Cube Root Function: $f(x)=\sqrt[3]{x}$

Difference Quotient: $f(x)=\frac{f(x+h)-f(x)}{h}$

Key terms

Absolute Maximum: Largest function value over the entire graph

Absolute Minimum: Smallest function value over the entire graph

Applied Domain: Set of input values for which the function makes sense

Constant: Areas of the graph where the function values are constant

Decreasing: Areas of the graph where the function values are decreasing

Dependent Variable: Output values of a function

Domain: Set of all input values for which a function is defined

Even Function: Function whose graph is symmetric about the *y*-axis

Extrema: Maximum and minimum values of a function

Function Arithmetic: Adding, subtracting, multiplying, or dividing functions

Function Composition: Replacing the *x*-values of a function f with a function g ; $f(g(x))$

Function: A relation in which any two ordered pairs with the same first component have the same second component

Horizontal Line Test: A function is one-to-one if no horizontal line intersects the graph more than once

Horizontal Reflection: Transformation where the *x*-values of a function are multiplied by -1; the graph is reflected over the *y*-axis

Horizontal Scaling: Transformation where the *x*-values of a function are multiplied by the reciprocal of a non-zero constant; the graph is stretched or compressed horizontally

Horizontal Shift: Transformation where a constant is added to the *x*-value of the function; the graph is shifted right or left on the coordinate plane

Increasing: Areas of the graph where the function values are increasing

Independent Variable: Input values of a function

Inverse Functions: Two functions f and g where $f(g(x))=x$ and $g(f(x))=x$

Local Maximum: Largest function value on an open interval

Local Minimum: Smallest function value on an open interval

Non-rigid Transformations: Transformations that change the shape of the graph

Odd Function: Function whose graph is symmetric about the origin

One-to-One Function: A function f where if $f(c) = f(d)$, then $c = d$

Parent Functions: Basic functions (same as toolkit functions)

Piecewise-Defined Function: A function that uses more than one formula to define the output, where each formula has its own domain

Range: Set of all output values of a function

Relation: A set of ordered pairs

Rigid Transformations: Transformations that do not change the shape of the original graph

Toolkit Functions: Basic functions (same as parent functions)

Transformations: Changes to the graph of a function by modifying inputs and/or outputs

Vertical Line Test: A graph represents a function if no vertical line intersects it at more than one point

Vertical Reflection: Transformation where the *y*-values of a function are multiplied by -1; the graph is reflected over the *x*-axis

Vertical Scaling: Transformation where the *y*-values of a function are multiplied by a non-zero constant; the graph is stretched or compressed vertically

Vertical Shift: Transformation where a constant is added to the *y*-values of a function; the graph is shifted up or down on the coordinate plane

Zeros of a function: Solutions to the equation $f(x) = 0$; *x*-intercepts of the graph

CHAPTER 2
POLYNOMIAL FUNCTIONS

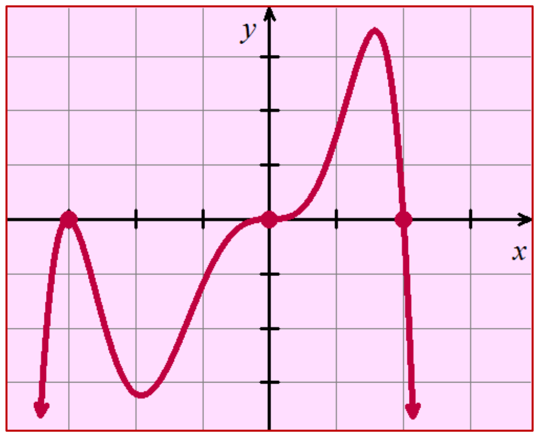


Figure 2.0. 1

Chapter Outline

2.1 Quadratic Functions

2.2 Graphs of Polynomials

2.3 Using Synthetic Division to Factor Polynomials

2.4 Real Zeros of Polynomials

2.5 Complex Zeros of Polynomials

2.6 Polynomial Inequalities

Introduction

Chapter 2 examines polynomial functions and their graphs using a variety of methods. Throughout the chapter, you will learn many strategies for working with polynomial functions. These strategies lay a foundation for future work, not only with polynomial functions, but with other functions as well. By the end of this chapter, you should be able to move fluidly among representations of polynomial functions and/or use information from one representation to gain information in another.

In Section 2.1, we start by reviewing what you know about quadratic functions in the form of $y = ax^2 + bx + c$ (where $a \neq 0$ and *a*, *b*, and *c* are rational numbers) including finding solutions by factoring, completing the square, or using the quadratic formula. You will find both real and complex solutions to quadratic equations and explore how the different methods for finding solutions are related to graphing quadratic functions. For example, you will realize the formula for finding the *x* value of the vertex of a quadratic, $x = -\frac{b}{2a}$, simply comes from the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, and the fact that parabolas are symmetric about a line.

Section 2.2 introduces polynomial functions of degree three and higher. It starts by familiarizing you with terms such as degree, leading term, leading coefficient, and constant term. You then explore ideas related to the Intermediate Value Theorem and build on those understandings to find solutions and their multiplicity. You also explore end behavior of polynomial functions numerically and by categorizing polynomials as either odd or even and positive or negative. Armed with understandings about zeros, multiplicity of zeros, *x*- and *y*-intercepts and end behavior, you learn to sketch graphs of polynomial functions, primarily in factored form. (Later you will learn to do this when the polynomial is not factored for you.)

Section 2.3 is devoted to helping you gain skills to factor polynomials of degree three or larger, and to use that information for finding solutions (real or complex), for finding *x*-intercepts, and for graphing polynomials. The section starts by developing skills with polynomial long division. You should relate division with polynomials to division with whole numbers; the process is parallel. Just like with division with whole numbers, if there is no remainder, the divisor is a factor; if there is a remainder, the divisor is not a factor. You then extend that understanding to the more efficient process of synthetic division. Vocabulary may become an issue in this section; you should take care to understand how the terms ‘factors’, ‘solutions’, ‘zeros’, and ‘*x*-intercepts’ are related to each other and to the graph of a polynomial function.

In sections 2.4 and 2.5, you put the ideas learned in the previous three sections together to work with, and understand, polynomials of degree 3 and larger in non-factored form. In 2.4, you will work with polynomials with all real solutions, where as in 2.5, some (or all) of the solutions may be complex. The primary goal of these sections (and the entire chapter) is not to turn you into a graphing calculator (though facility with graphing is very important), but rather to help you gain a deep understanding of polynomial behavior on which you will be able to rely in future math courses.

Section 2.6 focuses on solving inequalities involving polynomials. The section explores both analytical and graphical methods for solving inequalities. The section also gives you opportunities to revisit ideas of multiplicity of roots. By the end of this section, you should gain an appreciation of how fluidity with graphing is a useful tool in solving/understanding polynomial inequalities. The ideas developed in this section for solving inequalities will be extended later with rational functions.

2.1 Quadratic Functions

Learning Objectives

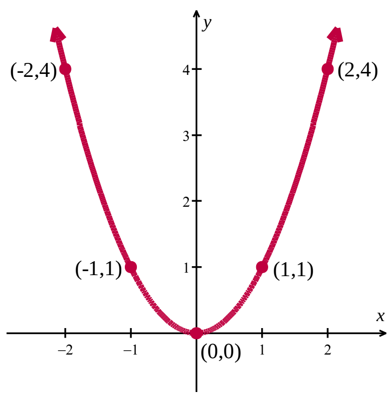
- Graph a quadratic function though transformations of $f(x)=x^2$.
- Change a quadratic function from general to standard form.
- Find the vertex and axis of symmetry of a quadratic function.
- Find the intercepts of a quadratic function.
- Graph a quadratic function using vertex, axis of symmetry and intercepts.
- Solve applications that require finding the maximum or minimum value of a quadratic function.
- Solve quadratic type equations.

You may recall studying quadratic equations in Intermediate Algebra. In this section, we review those equations in the context of quadratic functions.

Graphing Quadratic Functions through Transformations

The most basic quadratic function is $f(x)=x^2$, which you may recognize as one of the toolkit functions.

Figure 2.1. 1



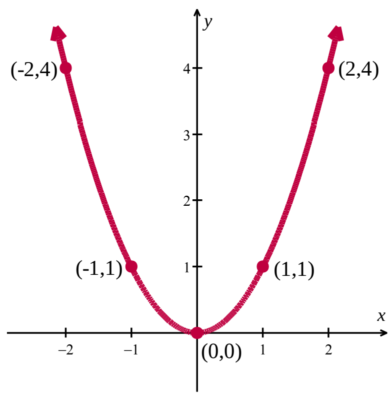
$f(x)=x^2$

Its shape should look familiar from Intermediate Algebra – it is called a **parabola**. The point $(0,0)$ is called the **vertex**. The minimum value of $f(x)=x^2$ is 0, and this value occurs when $x=0$. Knowing the graph of $f(x)=x^2$ allows us to graph other quadratic functions using transformations.

Example 2.1.1. Graph the function $g(x)=(x+2)^2-3$, starting with the graph of $f(x)=x^2$ and using transformations. Determine the vertex of $g(x)$.

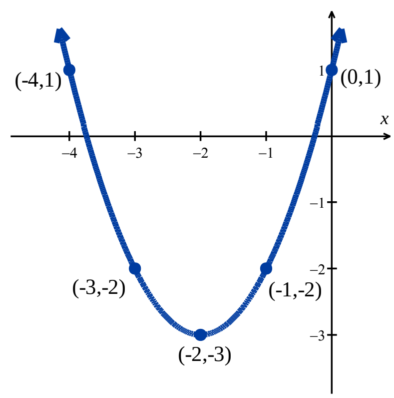
Solution. Since $g(x)=(x+2)^2-3=f(x+2)-3$, we can do this in two steps. First, we subtract 2 from each of the *x*-values of the points on $y=f(x)$. This shifts the graph of $y=f(x)$ to the left 2 units. Next, we subtract 3 from each of the *y*-values of these new points. This moves the graph down 3 units.

Figure 2.1. 2



$f(x)=x^2$

Figure 2.1. 3



$g(x)=(x+2)^2-3$

We see that the vertex has moved from $(0,0)$ on the graph of $y=f(x)$ to $(-2,-3)$ on the graph of $y=g(x)$.

□

A few remarks about **Example 2.1.1** are in order. We could convert $g(x)$ into a 'simpified' form by expanding and collecting like terms. Doing so, we find $g(x)=(x+2)^2-3=x^2+4x+1$. This 'simplified' form of $g(x)$ is referred to as **general form**. We note that a quadratic function in general form does not lend itelf easily to graphing via transformations. For that reason, the form of g presented in **Example 2.1.1** is often preferred and is referred to as **standard form**.

Definition 2.1. Standard and General Form of Quadratic Functions:

- A **quadratic function** f is a function of the form $f(x)=ax^2+bx+c$, where *a*, *b* and *c* are real numbers with $a\neq 0$. When written as $f(x)=ax^2+bx+c$, we say that f is in **general form**.
- If a quadratic function f is written in the form $f(x)=a(x-h)^2+k$, where *a*, *h* and *k* are real numbers with $a\neq 0$, we say that f is in **standard form**.

We continue with the Quadratic Formula, an important tool from Intermediate Algebra.

Equation 2.1. The Quadratic Formula: If *a*, *b* and *c* are real numbers with $a \neq 0$, then the solutions to $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Intermediate Algebra also introduced the discriminant, particularly useful in identifying the number and type of solutions to a quadratic equation.

Definition 2.2. If *a*, *b* and *c* are real numbers with $a \neq 0$, then the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$ is the quantity $b^2 - 4ac$.

By thinking about the consequences of taking the square root of the discriminant, along with the position of the discriminant in the Quadratic Formula, we have the following.

Determining the Number of Real Solutions to a Quadratic Equation

Let *a*, *b* and *c* be real numbers with $a \neq 0$.

- If $b^2 - 4ac < 0$, the equation $ax^2 + bx + c = 0$ has no real solutions.
- If $b^2 - 4ac = 0$, the equation $ax^2 + bx + c = 0$ has one real solution.[1]
- If $b^2 - 4ac > 0$, the equation $ax^2 + bx + c = 0$ has two real solutions.

Knowing the Quadratic Formula and paying attention to the discriminant will be useful throughout this chapter.

The Standard Form of a Quadratic Function

For a quadratic function that is written in standard form, the following theorem allows us to quickly identify the vertex.

Theorem 2.1. Vertex Formula for Quadratics in Standard Form: For the quadratic function $f(x) = a(x-h)^2 + k$, where *a*, *h* and *k* are real numbers with $a \neq 0$, the vertex of the graph of $y = f(x)$ is (h, k).

[1] In this case, technically, we have two real but equal solutions.

We can readily verify the formula given in **Theorem 2.1** with the function from **Example 2.1.1**. After a slight rewrite, $g(x)=(x+2)^2-3=(x-(-2))^2+(-3)$, and we identify $h=-2$ and $k=-3$. Sure enough, we found the vertex of the graph of $y=g(x)$ to be $(-2,-3)$.

To see why the formula in **Theorem 2.1** produces the vertex, we start with $y=x^2$ and its vertex of $(0,0)$. We can determine the vertex of $y=a(x-h)^2+k$ by determining the final destination of $(0,0)$ as it is moved through each transformation.

- To obtain the formula $f(x)=a(x-h)^2+k$, we start with $g(x)=x^2$ and define $g_1(x)=a\cdot g(x)=a\cdot x^2$. This results in a vertical scaling and/or reflection.[2] Since we multiply the output by *a*, we multiply the *y*-coordinates on the graph of g by *a*, so the point $(0,0)$ remains $(0,0)$ and is still the vertex.
- Next, we define $g_2(x)=g_1(x-h)=a(x-h)^2$. This induces a horizontal shift right or left *h* units[3] and moves the vertex, in either case, to $(h,0)$.
- Finally, $f(x)=g_2(x)+k=a(x-h)^2+k$ which effects a vertical shift up or down *k* units[4] resulting in the vertex moving from $(h,0)$ to (h,k).

In addition to verifying **Theorem 2.1**, we have also shown the role of the number *a* in the graphs of quadratic functions. The graph of $y=a(x-h)^2+k$ is a parabola opening upward if $a>0$ and opening downward if $a<0$. Moreover, the symmetry enjoyed by the graph of $y=x^2$ about the *y*-axis is translated to a symmetry about the vertical line $x=h$, which is the vertical line through the vertex.[5] This is called the **axis of symmetry** of the parabola and is dashed in the following figures.

[2] Just a scaling if $a>0$. If $a<0$, there is a reflection involved.
[3] Right if $h>0$, left if $h<0$.
[4] Up if $k>0$, down if $k<0$.
[5] You should use transformations to verify this!

Figure 2.1. 4

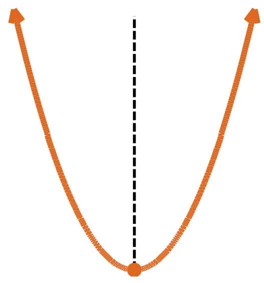
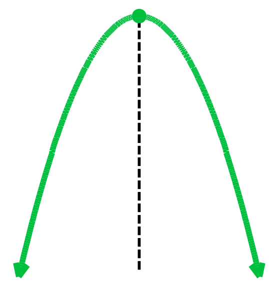


Figure 2.1. 5



Graphs of $y=a(x-h)^2+k$

Changing from General Form to Standard Form

Without a doubt, the standard form of a quadratic function allows us to list the attributes of the graphs of such functions quickly and elegantly. What remains to be shown, however, is the fact that every quadratic function can be written in standard form. To convert a quadratic function given in general form into standard form, we employ the ancient rite of 'completing the square'. We remind the reader how this is done in our next example.

Example 2.1.2. Convert the functions below from general form to standard form. Find the vertex, the axis of symmetry and any *x*- or *y*-intercepts. Graph each function and determine its range.

1. $f(x)=x^2-4x+3$
2. $g(x)=6-x-x^2$

Solution.

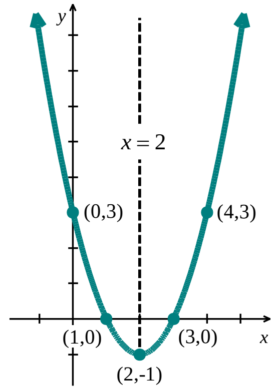
1. To convert $f(x)=x^2-4x+3$ from general form to standard form, we complete the square on x^2-4x by first taking half of -4 to get $\frac{1}{2}(-4)=-2$. This tells us that the target perfect square quantity is $(x-2)^2$. To get an expression equivalent to $(x-2)^2$, we need to add $(-2)^2=4$ to x^2-4x to create a perfect square trinomial, but to keep the balance we must also subtract it.

$$\begin{aligned} f(x) &= x^2-4x+3 \\ &= \left(x^2-4x+4-4\right)+3 \\ &= \left(x^2-4x+4\right)-4+3 && \textbf{group the perfect square trinomial} \\ &= (x-2)^2-1 && \textbf{factor the perfect square trinomial} \end{aligned}$$

In the form $f(x)=(x-2)^2-1$, we readily find the vertex to be $(2,-1)$ which makes the axis of symmetry $x=2$.

To find the x-intercepts, we set $y=f(x)=0$. We have the choice of two formulas for $f(x)$. Since we recognize $f(x)=x^2-4x+3$ to be easily factorable, we proceed to solve $x^2-4x+3=0$. Factoring gives $(x-3)(x-1)=0$ so that $x=3$ or $x=1$, giving us x-intercepts of $(1,0)$ and $(3,0)$. To find the y-intercept, we set $x=0$. Once again, the general form $f(x)=x^2-4x+3$ is easiest to work with, and we find $y=f(0)=3$. Hence, the y-intercept is $(0,3)$ and, finding its 'mirror image' about $x=2$, the axis of symmetry, we identify an additional point as being $(4,3)$. Putting all of this information together results in the following graph.

Figure 2.1. 6



$f(x)=x^2-4x+3$

We see that the range of f is $[-1,\infty)$ and we are done.

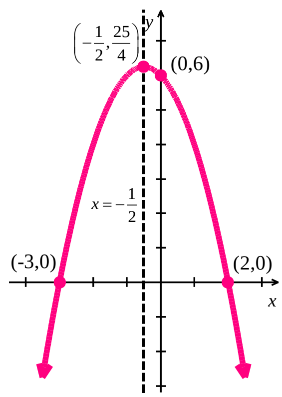
2. To get started, we rewrite $g(x)=6-x-x^2$ as $g(x)=-x^2-x+6$ and note that the coefficient of x^2 is -1, not 1. This means that our first step is to factor out the -1 from both the x^2 and x terms. We then follow the completing the square recipe as before.

$$\begin{aligned}
g(x) &= -x^2-x+6 \\
&= (-1)(x^2+x)+6 \\
&= (-1)\left(x^2+x+\frac{1}{4}-\frac{1}{4}\right)+6 && \textbf{add and subtract } \left(\frac{1}{2}\right)^2=\frac{1}{4} \textbf{ to } x^2+x \\
&= (-1)\left(x^2+x+\frac{1}{4}\right)+(-1)\left(-\frac{1}{4}\right)+6 && \textbf{group the perfect square trinomial} \\
&= -\left(x+\frac{1}{2}\right)^2+\frac{25}{4} && \textbf{factor the perfect square trinomial and simplify}
\end{aligned}$$

From $g(x) = -\left(x+\frac{1}{2}\right)^2 + \frac{25}{4}$, we get the vertex of $\left(-\frac{1}{2}, \frac{25}{4}\right)$ and the axis of symmetry $x = -\frac{1}{2}$.

To find the *x*-intercepts, we opt to set the given formula $g(x) = 6 - x - x^2 = 0$. Solving, we get $x = -3$ and $x = 2$, so the *x*-intercepts are $(-3,0)$ and $(2,0)$. Setting $x = 0$, we find $g(0) = 6$ for a *y*-intercept of $(0,6)$. Plotting these points gives us the following graph.

Figure 2.1. 7



$g(x) = 6 - x - x^2$

We see that the range of g is $\left(-\infty, \frac{25}{4}\right]$.

□

With **Example 2.1.2** fresh in our minds, we are now in a position to show that every quadratic function can be written in standard form. We begin with $f(x) = ax^2 + bx + c$, assume $a \neq 0$, and complete the square.

$$f(x) = ax^2 + bx + c$$

$$= a\left(x^2 + \frac{b}{a}x\right) + c \qquad \textbf{factor out the coefficient of } a \textbf{ from } x^2 \textbf{ and } x$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c \qquad \textbf{add and subtract } \left(\frac{1}{2}\cdot\frac{b}{a}\right)^2 \textbf{ to } x^2 + \frac{b}{a}x$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + a\left(-\frac{b^2}{4a^2}\right) + c \qquad \textbf{group the perfect square trinomial}$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \qquad \textbf{factor and simplify}$$

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \qquad \textbf{obtain common denominator}$$

Comparing this last expression with the standard form, we identify $(x-h)$ with $\left(x+\frac{b}{2a}\right)$ so that $h=-\frac{b}{2a}$. Instead of memorizing the value $k=\frac{4ac-b^2}{4a}$, we see that $f\left(-\frac{b}{2a}\right)=\frac{4ac-b^2}{4a}$. As such, we have derived a vertex formula for the general form. We next summarize both vertex formulas.

Equation 2.2. Vertex Formulas for Quadratic Functions: Suppose *a*, *b*, *c*, *h* and *k* are real numbers with $a \neq 0$.

- If $f(x)=a(x-h)^2+k$, the vertex of the graph of $y=f(x)$ is the point (h,k).
- If $f(x)=ax^2+bx+c$, the vertex of the graph of $y=f(x)$ is the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

We next incorporate the vertex formulas from **Equation 2.2** into a general strategy for graphing parabolas that are presented to us in either general form or standard form.

To Graph the Parabola $f(x)=ax^2+bx+c$ or $f(x)=a(x-h)^2+k$ with $a \neq 0$:

1. Determine if the graph opens upward (positive *a*) or downward (negative *a*).
2. Find the vertex, $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ or (h,k), and the axis of symmetry, $x=-\frac{b}{2a}$ or $x=h$.
3. Find all *x*- and *y*-intercepts.
4. Find additional points, if necessary. (For example, when there are no *x*-intercepts.)
5. Plot the points from the previous steps and draw the axis of symmetry.
6. Connect the points with a smooth curve and extend the curve to complete your graph.

In the next example, we apply our graphing strategy to a parabola of the form $f(x)=ax^2+bx+c$.

Example 2.1.3. Graph the function $f(x)=x^2+4x+7$ after finding the vertex, the axis of symmetry and any *x*- or *y*-intercepts.

Solution. We begin by noting that $a=1>0$ and so the parabola will open upwards. Next, we identify the *x*-coordinate of the vertex.

$$-\frac{b}{2a} = -\frac{4}{2(1)}$$
$$= -2$$

Substituting $x = -2$ into $f(x) = x^2 + 4x + 7$, we find the second coordinate of the vertex.

$$f(-2) = (-2)^2 + 4(-2) + 7$$
$$= 3$$

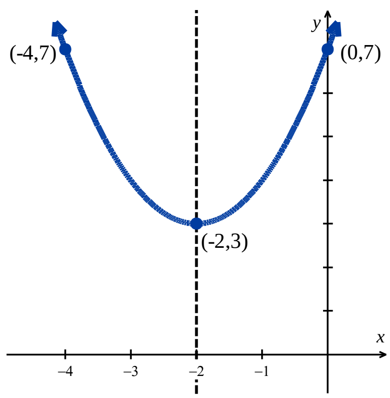
From the vertex of $(-2, 3)$, we see that the axis of symmetry is $x = -2$.

To find the *x*-intercpts, we set $f(x) = x^2 + 4x + 7 = 0$. Noting that $x^2 + 4x + 7$ is not easily factorable, we use the Quadratic Formula. With $a = 1$, $b = 4$ and $c = 7$, we have

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4 \cdot 1 \cdot 7}}{2 \cdot 1}$$
$$= \frac{-4 \pm \sqrt{-12}}{2}$$

Since the discriminant, -12, is less than zero, $x^2 + 4x + 7 = 0$ does not have any real solutions. Thus, there are no *x*-intercepts.[6] To find the *y*-intercept, we set $x = 0$ and find $f(0) = (0)^2 + 4(0) + 7 = 7$ for a *y*-intercept of $(0, 7)$. Reflecting the point $(0, 7)$ about the axis of symmetry, $x = -2$, gives us the additional point $(-4, 7)$. The graph appears below.

Figure 2.1. 8



$f(x) = x^2 + 4x + 7$

□

[6] Since the vertex is above the *x*-axis and the parabola opens upward, there are obviously no *x*-intercepts. Had we made this observation first, we could have skipped our attempt to calculate the *x*-intercepts.

For accuracy in sketching the curve it is helpful to plot additional points, such as the points $(-3,4)$ and $(-1,4)$ in the preceding example.

Solving Applications of Quadratic Functions

Example 2.1.4. The weekly profit, in dollars, made by selling *x* PortaBoy Game Systems is $P(x)=-1.5x^2+170x-150$ with the restriction that $0\le x\le 166$.

1. Graph $y=P(x)$. Include the *x*- and *y*-intercepts as well as the vertex and axis of symmetry.

2. Interpret the zeros of P.

3. Interpret the vertex of the graph of $y=P(x)$.

Solution.

1. To find the *x*-intercepts, we set $P(x)=0$ and solve $-1.5x^2+170x-150=0$ with the aid of the quadratic formula.

$$\begin{aligned} x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{-170\pm\sqrt{170^2-4(-1.5)(-150)}}{2(-1.5)} \\ &= \frac{-170\pm\sqrt{28000}}{-3} \\ &= \frac{(-1)\left(170\mp 20\sqrt{70}\right)}{(-1)(3)} \\ &= \frac{170\mp 20\sqrt{70}}{3} \end{aligned}$$

We get two *x*-intercepts: $\left(\frac{170-20\sqrt{70}}{3},0\right)$ and $\left(\frac{170+20\sqrt{70}}{3},0\right)$. To find the *y*-intercept, we set $x=0$ and find $y=P(0)=-150$ for a *y*-intercept of $(0,-150)$.

To find the vertex, we use the fact that $P(x)=-1.5x^2+170x-150$ is in the general form of a quadratic function and appeal to **Equation 2.2**. We first determine the value of the *x*-coordinate.

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{170}{2(-1.5)} \\ &= \frac{170}{3} \end{aligned}$$

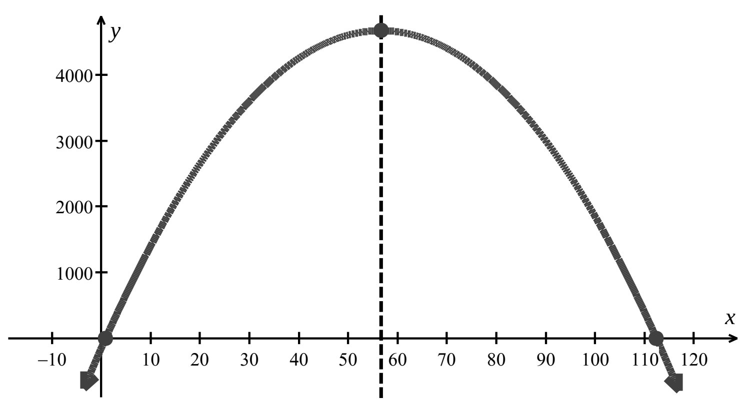
We next compute the *y*-coordinate of the vertex.

$$\begin{aligned} P\left(\frac{170}{3}\right) &= -1.5\left(\frac{170}{3}\right)^2 + 170\left(\frac{170}{3}\right) - 150 \\ &= -\frac{3}{2}\cdot\frac{28900}{9} + \frac{28900}{3} - 150 \\ &= -\frac{14450}{3} + \frac{28900}{3} - \frac{450}{3} \\ &= \frac{14000}{3} \end{aligned}$$

Our vertex is $\left(\frac{170}{3}, \frac{14000}{3}\right)$. The axis of symmetry is the vertical line passing through the vertex so it is the line $x = \frac{170}{3}$.

To sketch a reasonable graph, we approximate the *x*-intercepts, $(0.89, 0)$ and $(112.44, 0)$, and the vertex, $(56.67, 4666.67)$. We adjust the scales on the *x*-axis and the y-axis so that intercepts and vertex are visible.

Figure 2.1. 9



$y = P(x)$

2. The zeros of P are the solutions to $P(x) = 0$, which we have found to be approximately 0.89 and 112.44. We see from the graph that as long as *x* is between 0.89 and 112.44, the graph $y = P(x)$ is

above the *x*-axis, meaning $y = P(x) > 0$. This tells us that for these values of *x*, a profit is being made. Since *x* represents the weekly sales of PortaBoy Game Systems, we round the zeros to positive integers and have that as long as at least 1, but no more than 112, game systems are sold weekly, the retailer will make a profit.

3. From the graph, we see that the maximum value of P occurs at the vertex, which is approximately $(56.67, 4666.67)$. As above, *x* represents the weekly sales of PortaBoy systems, so we can't sell 56.67 game systems. Comparing $P(56) = 4666$ and $P(57) = 4666.5$, we conclude that we will make a maximum profit of \$4666.50 if we sell 57 game systems.

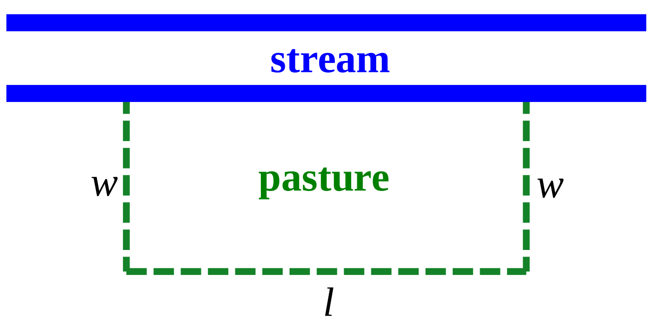
□

Our next example is a classic application of quadratic functions.

Example 2.1.5. Much to Kyle's surprise and delight, he inherits a large parcel of land in Wasatch County from one of his (e)strange(d) relatives. The time is finally right for him to pursue his dream of farming alpacas. He wishes to build a rectangular pasture, and estimates that he has enough money for 200 linear feet of fencing material. If he makes the pasture adjacent to a stream (so no fencing is required on that side), what are the dimensions of the pasture with the maximum area? If an average alpaca needs 25 square feet of grazing area, how many alpacas can Kyle keep in his pasture?

Solution. It is always helpful to sketch the problem situation, as we do below.

Figure 2.1. 10



We are tasked to find the dimensions of the pasture which would give a maximum area. We let *w* denote the width of the pasture and we let *l* denote the length of the pasture. The area of the pasture, which we'll call A, is related to *w* and *l* by the equation $A = w \cdot l$. Since our objective is to maximize A, we refer to $A = w \cdot l$ as the **objective function**.

We are given that the total amount of fencing available is 200 feet, which means $w+l+w=200$, or $l+2w=200$. This equation limits the possibilities for dimensions and area, and is thus referred to as a **constraint**.

In order to use the tools given to us in this section to maximize A, we need to write A as a function of just one variable, either *w* or *l*. This is where we use the constraint $l+2w=200$. Solving for *l*, we find $l=200-2w$, and we substitute this into our objective function, $A=l\cdot w$.

$$\begin{aligned} A &= (200-2w)(w) \\ &= 200w-2w^2 \\ &= -2w^2+200w \end{aligned}$$

Before we go any further, we need to find the applied domain of A so that we know what values of *w* make sense in this problem situation. Since *w* represents the width of the pasture, $w>0$. Likewise, *l* represents the length of the pasture so $l=200-2w>0$. Solving this latter inequality, we find $w<100$. Hence, the function we wish to maximize is $A(w)=-2w^2+200w$ for $0<w<100$.

Since A is a quadratic function (of *w*), we know that the graph of $y=A(w)$ is a parabola. With the coefficient of w^2 being -2, we know that this parabola opens downward. This means that there is a maximum value to be found, and we know it occurs at the vertex. We use the vertex formula to find *w*.

$$\begin{aligned} w &= -\frac{b}{2a} \\ &= -\frac{200}{2(-2)} \\ &= 50 \end{aligned}$$

Since $w=50$ lies in the applied domain, $0<w<100$, we have that the area of the pasture is maximized when the width is 50 feet. To find the length, we use $l=200-2w$, from which $l=200-2(50)=100$, so the length of the pasture is 100 feet. The maximum area occurs at the vertex, when $w=50$ feet:

$$\begin{aligned} A(50) &= -2(50)^2+200(50) \\ &= 5000 \end{aligned}$$

Thus, the maximum area is 5000 square feet. If an average alpaca requires 25 square feet of pasture, Kyle can raise $\frac{5000}{25}=200$ average alpacas.

□

Solving Quadratic Type Equations

Throughout this textbook, you will find equations that may be rewritten in the form $au^2 + bu + c = 0$ where u is an expression of the variable we are solving for. Such equations are referred to as **quadratic in form**. Strategies for finding solutions to quadratic equations can be extended to solving equations that are quadratic in form.

Example 2.1.6. Solve the equation $5x^6 = 4 - 8x^3$.

Solution. It is often helpful when solving a non-linear equation to reorganize the equation so that we have an expression equal to zero.

$$5x^6 = 4 - 8x^3$$
$$5x^6 + 8x^3 - 4 = 0$$

Noting that the highest degree term of the equation is the square of the next highest degree term, in other words $\left(x^3\right)^2 = x^6$, we may rewrite the equation as a quadratic. We let $u = x^3$ to get

$$5\left(x^3\right)^2 + 8\left(x^3\right) - 4 = 0$$
$$5u^2 + 8u - 4 = 0$$

The equation $5u^2 + 8u - 4 = 0$ is now in a form we know how to solve. Because this quadratic is easily factorable, we will solve in this manner. However, completing the square or applying the quadratic formula may also be used.

$$5u^2 + 8u - 4 = 0$$
$$(5u - 2)(u + 2) = 0$$

Thus, $u = \frac{2}{5}$ or $u = -2$. However, recall that $5u^2 + 8u - 4 = 0$ is not the original equation, it is the equation we got by substituting $u = x^3$. We still need to solve for x, so there is one more step. For $x^3 = \frac{2}{5}$, we solve for x to find $x = \sqrt[3]{\frac{2}{5}}$, and for $x^3 = -2$ we have $x = \sqrt[3]{-2}$.

□

Example 2.1.7. Solve the equation $x^{\frac{1}{2}} - x^{\frac{1}{4}} = 6$.

Solution. Again we notice that we can rearrange the equation to set it equal to zero, and we see that the highest power term is the square of the next highest power term.

$$x^{\frac{1}{2}} - x^{\frac{1}{4}} = 6$$
$$x^{\frac{1}{2}} - x^{\frac{1}{4}} - 6 = 0$$
$$\left(x^{\frac{1}{4}}\right)^2 - x^{\frac{1}{4}} - 6 = 0$$

Using a similar strategy, we let $u = x^{\frac{1}{4}}$ so that $u^2 - u - 6 = 0$. We solve again by factoring, although any method of solving for u may be used.

$$u^2 - u - 6 = 0$$
$$(u-3)(u+2) = 0$$

Thus, $u = 3$ or $u = -2$. Substituting x back in, we get $x^{\frac{1}{4}} = 3$ or $x^{\frac{1}{4}} = -2$. Let's look at each case individually.

- For $x^{\frac{1}{4}} = 3$, noting that the range of an even root is non-negative, it is possible to have $x^{\frac{1}{4}} = 3$ so we raise both sides to the fourth power to find $x = 81$.
- For $x^{\frac{1}{4}} = -2$, there is a problem; in the real numbers, even roots do not give negative outputs. Thus, there is no solution for $x^{\frac{1}{4}} = -2$.

The only solution is $x = 81$.

□

2.1 Exercises

1. How can the vertex of a parabola be used in solving real world problems?

2. Explain why the condition of $a \neq 0$ is imposed in the definition of the quadratic function.

In Exercises 3 – 6, convert the function from general form to standard form.

3. $f(x) = x^2 - 2x - 8$

4. $f(x) = x^2 - 6x - 1$

5. $f(x) = 2x^2 - 4x + 7$

6. $f(x) = -4x^2 - 12x + 3$

In Exercises 7 – 22, find the vertex, the axis of symmetry and any *x*- or *y*-intercepts. Graph each function and determine its range.

7. $f(x) = x^2 + 2$

8. $f(x) = -(x+2)^2$

9. $f(x) = -(x+2)^2 + 2$

10. $f(x) = x^2 - 2x$

11. $f(x) = x^2 - 2x - 8$

12. $f(x) = x^2 - 6x - 1$

13. $f(x) = x^2 - 5x - 6$

14. $f(x) = x^2 - 7x + 3$

15. $f(x) = -2(x+1)^2 + 4$

16. $f(x) = 2x^2 - 4x - 1$

17. $f(x) = 2x^2 - 4x + 7$

18. $f(x) = -2x^2 + 5x - 8$

19. $f(x) = 4x^2 - 12x - 3$

20. $f(x) = -3x^2 + 4x - 7$

21. $f(x) = x^2 + x + 1$

22. $f(x) = -3x^2 + 5x + 4$

In Exercises 23 – 28, find the real solution(s) of the given equation.

23. $(x+3)^2 - 4(x+3) - 5 = 0$

24. $\left(\frac{1}{x+1}\right)^2 - 2\left(\frac{1}{x+1}\right) - 3 = 0$

25. $x - 3x^{\frac{1}{2}} + 2 = 0$

26. $4x^4 + 9 = 13x^2$

27. $2x^{-2} = x^{-1} + 1$

28. $y^{\frac{1}{3}} + y^{\frac{1}{6}} - 2 = 0$

In Exercises 29 – 33, the profit function $P(x)$ is given.

- Find the number of items which need to be sold in order to maximize profit.
- Find the maximum profit.

- Find and interpret the zeros of $P(x)$.

29. The profit, in dollars, made by selling *x* "I'd rather be a Sasquatch" t-shirts is $P(x) = -2x^2 + 28x - 26$, $0 \le x \le 15$.

30. The profit, in dollars, made by selling *x* bottles of 100% all-natural certified free-trade organic Sasquatch Tonic is $P(x) = -x^2 + 25x - 100$, $0 \le x \le 35$.

31. The profit, in cents, made by selling *x* cups of Mountain Thunder Lemonade at Junior's lemonade stand is $P(x) = -3x^2 + 72x - 240$, $0 \le x \le 30$.

32. The daily profit, in dollars, made by selling *x* Sasquatch Berry Pies is $P(x) = -0.5x^2 + 9x - 36$, $0 \le x \le 24$.

33. The monthly profit, in hundreds of dollars, made by selling *x* custom built electric scooters is $P(x) = -2x^2 + 120x - 1000$, $0 \le x \le 70$.

34. Using data from the Bureau of Transportation statistics, the average fuel economy F in miles per gallon for passenger cars in the US can be modeled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \le t \le 28$, where *t* is the number of years since 1980. Find and interpret the coordinates of the vertex of the graph of $y = F(t)$.

35. The temperature T, in degrees Fahrenheit, *t* hours after 6 AM is given by

$$T(t) = -\frac{1}{2}t^2 + 8t + 32, \quad 0 \le t \le 12$$

What is the warmest temperature of the day? When does this happen?

36. Suppose $C(x) = x^2 - 10x + 27$ represents the cost, in hundreds, to produce *x* thousand pens. How many pens should be produced to minimize the cost? What is the minimum cost?

37. Dani wishes to plant a vegetable garden along one side of her house. In her garage, she found 32 linear feet of fencing. Since one side of the garden will border the house, Dani doesn't need fencing along that side. What are the dimensions of the garden which will maximize the area of the garden? What is the maxium area of the garden?

38. In the situation of **Example 2.1.5**, Kyle has a nightmare that one of his alpacas fell into the steam and was injured. To avoid this, he wants to move his rectangular pasture away from the stream. This means that all four sides of the pasture require fencing. If the total amount of fencing available is still 200 linear feet, what dimensions maximize the area of the pasture now? What is the maximum area? Assuming an average alpaca requires 25 square feet of pasture, how many alpacas can he raise now?

39. What is the largest rectangular area one can enclose with 14 inches of string?

40. The height of an object dropped from the roof of an eight story building is modeled by $h(t) = -16t^2 + 64$, $0 \le t \le 2$. Here, h is the height of the object off the ground, in feet, t seconds after the object is dropped. How long before the object hits the ground?

41. The height h, in feet, of a model rocket above the ground t seconds after lift-off is given by $h(t) = -5t^2 + 100t$, for $0 \le t \le 20$. When does the rocket reach its maximum height above the ground? What is its maximum height?

42. Jason participates in the Highland Games. In one event, the hammer throw, the height h, in feet, of the hammer above the ground t seconds after Jason lets it go is modeled by $h(t) = -16t^2 + 22.08t + 6$. What is the hammer's maximum height? What is the hammer's total time in the air? Round your answers to two decimal places.

43. Assuming no air resistance or forces other than the Earth's gravity, the height above the ground at time t of a falling object is given by $s(t) = -4.9t^2 + v_0 t + s_0$ where s is in meters, t is in seconds, v_0 is the object's initial velocity in meters per second and s_0 is its initial position in meters.

(a) What is the applied domain of this function?

(b) Discuss with your classmates what each of $v_0 > 0$, $v_0 = 0$ and $v_0 < 0$ would mean.

(c) Come up with a scenario in which $s_0 < 0$.

(d) Let's say a slingshot is used to shoot a marble straight up from the ground ($s_0 = 0$) with an initial velocity of 15 meters per second. What is the marble's maximum height above the ground? At what time will it hit the ground?

(e) Now shoot the marble from the top of a tower which is 25 meters tall. When does it hit the ground?

(f) What would the height function be if instead of shooting the marble up off of the tower, you were to shoot it straight DOWN from the top of the tower?

44. The two towers of a suspension bridge are 400 feet apart. The parabolic cable attached to the tops of the towers is 10 feet above the point on the bridge deck that is midway between the towers. If the towers are 100 feet tall, find the height of the cable directly above a point of the bridge deck that is 50 feet to the right of the left-hand tower.

45. Find all of the points on the line $y = 1 - x$ which are 2 units from $(1, -1)$.

46. Let L be the line $y = 2x + 1$. Find a function $D(x)$ which measures the distance squared from a point on L to $(0,0)$. Use this to find the point on L closest to $(0,0)$.

47. With the help of your classmates, show that if a quadratic function $f(x) = ax^2 + bx + c$ has two real zeros then the *x*-coordinate of the vertex is the midpoint of the zeros.

2.2 Graphs of Polynomials

Learning Objectives

- Determine whether or not a function is a polynomial.
- Identify the degree, leading term, leading coefficient and constant term of a polynomial.
- Determine the existence of zeros using the Intermediate Value Theorem.
- Find the zeros and multiplicities of a polynomial; use multiplicity to determine the behavior of the graph at each zero.
- Identify the end behavior of a polynomial function.
- Sketch the graph of a polynomial function using zeros, multiplicities and end behavior.
- Solve applications that require finding the maximum or minimum value of a polynomial function.

Quadratic functions belong to a much larger group of functions called **polynomials**. We begin our formal study of general polynomials with a definition and some examples.

Definition 2.3. A **polynomial function** is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0,$$

where a_0, a_1, …, a_n are real numbers with $a_n \neq 0$ and n is a nonnegative integer. The domain of a polynomial function is $(-\infty, \infty)$.

In an effort to understand **Definition 2.3**, we look at an example of a polynomial function, $f(x) = 4x^5 - 3x^2 + 2x - 5$. We can rewrite f as $f(x) = 4x^5 + 0x^4 + 0x^3 + (-3)x^2 + 2x + (-5)$. Comparing this with **Definition 2.3**, we identify $n = 5$, $a_5 = 4$, $a_4 = 0$, $a_3 = 0$, $a_2 = -3$, $a_1 = 2$ and $a_0 = -5$. The subscript on a merely indicates to which power of x the coefficient belongs.

Functions that are Polynomials

The following example provides some insight into determining whether or not a function represents a polynomial.

Example 2.2.1. Determine if the following functions are polynomials. Explain your reasoning.

1. $g(x)=\sqrt{2}\,x-\pi x+3$
2. $p(x)=\sqrt{2x}-\pi x+3$
3. $q(x)=\dfrac{x^2}{2}+5x$
4. $f(x)=3^x+5x^2$
5. $h(x)=\dfrac{2}{x^2}+7$
6. $z(x)=3$

Solution.

1. We note $g(x)=\sqrt{2}\,x-\pi x+3$ can be written as $g(x)=\left(\sqrt{2}-\pi\right)x+3$. Since $\sqrt{2}-\pi$ is a real number, as is 3, we find that g is of the form $g(x)=a_1x+a_0$ and is therefore a polynomial by **Definition 2.3**.

2. We rewrite $p(x)=\sqrt{2x}-\pi x+3$ as $p(x)=\sqrt{2}\,x^{\frac{1}{2}}-\pi x+3$ and note that in the term $\sqrt{2}\,x^{\frac{1}{2}}$ the power $\dfrac{1}{2}$ is not an integer. Thus, p is not a polynomial.

3. Once again, we start by rewriting $q(x)=\dfrac{x^2}{2}+5x$ to match the format given to us in **Definition 2.3**. We get $q(x)=\dfrac{1}{2}x^2+5x+0$, a polynomial of the form $q(x)=a_2x^2+a_1x+a_0$.

4. We note that the function $f(x)=3^x+5x^2$ has a first term of 3^x. While it may be tempting to think of 3^x as being somehow related to x^3, these two terms are very different. Later on, we will study functions that include terms like 3^x, but for now we simply note that this term does not belong in a polynomial function, and so f is not a polynomial.

5. The function $h(x)=\dfrac{2}{x^2}+7$ can be rewritten as $h(x)=2x^{-2}+7$. This function is not a polynomial since the term $2x^{-2}$ contains a negative power of *x*. **Definition 2.3** requires powers to be nonnegative integers.

6. There's nothing in **Definition 2.3** that prevents a constant function from being a polynomial. For $z(x)=3$, we can think of the degree as being zero and the function *z* can be written as $z(x)=3x^0$ (assuming $x\neq 0$). Hence, $z(x)=3$ is a polynomial.

□

We continue with the introduction of some terminology involving characteristics of polynomials.

Characteristics of Polynomials

Definition 2.4. Suppose f is a polynomial function.

- Given $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0$ with $a_n\neq 0$, we say
  - The **degree** of f is *n*, the highest power of the variable that appears in the polynomial.
  - The **leading term** of f is a_nx^n, the term containing the highest power of the variable.
  - The **leading coefficient** of f is a_n, the coefficient of the leading term.
  - The **constant term** of f is a_0, the term with no variable.
- If $f(x)$ is just a nonzero constant, we say the degree of f is 0.

One good thing that comes from **Definition 2.4** is that we can now think of linear functions as degree 1 (or 'first degree') polynomial functions and quadratic functions as degree 2 (or 'second degree') polynomial functions. We continue with an example that puts **Definition 2.4** to good use.

Example 2.2.2. Find the degree, leading term, leading coefficient and constant term of the following polynomial functions.

1. $f(x)=4x^5-3x^2+2x-5$
2. $g(x)=12x+x^3$
3. $h(x)=\dfrac{4-x}{5}$
4. $p(x)=(2x-1)^3(x-2)(3x+2)$

Solution.

1. There are no surprises with $f(x)=4x^5-3x^2+2x-5$. It is written in the form of **Definition 2.4** and we see that the degree is 5, the leading term is $4x^5$, the leading coefficient is 4 and the constant term is -5.

2. For $g(x)=12x+x^3$, we find the highest power of the variable x to be 3, and so the polynomial has degree 3. The leading term is x^3 since it has degree 3. The leading coefficient, or the coefficient of the leading term, is 1. Finally, there is no term without a variable and so the constant term is 0.

3. We need to rewrite the formula for h so that it resembles the form given in **Definition 2.4**.

$$\begin{aligned} h(x) &= \frac{4-x}{5} \\ &= \frac{4}{5}-\frac{x}{5} \\ &= -\frac{1}{5}x+\frac{4}{5} \end{aligned}$$

   We can now identify the degree of h as being 1, from which the leading term is $-\frac{1}{5}x$ and the leading coefficient is $-\frac{1}{5}$. The constant term is $\frac{4}{5}$.

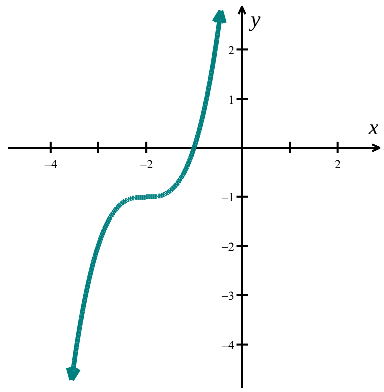
4. It is not necessary to get $p(x)=(2x-1)^3(x-2)(3x+2)$ in the form of **Definition 2.4** since we can glean the information requested about p without multiplying out the entire expression. The leading term of p will be the term which has the highest power of x. The way to get this term is to multiply together the terms with the highest power of x from each factor. In other words, the leading term of $p(x)$ is the product of the leading terms of the factors of $p(x)$. Hence, the leading term of p is $(2x)^3(x)(3x)=24x^5$. This means that the degree of p is 5 and the leading coefficient is 24.

   As for the constant term, we can perform a similar trick. The constant term is obtained by multiplying the constant terms from each of the factors: $(-1)^3(-2)(2)=4$.

□

We turn our attention to graphs of polynomials in general. The toolkit functions that are polynomials include the constant function, the identity function, the quadratic function and the cubic function. Transformations of these functions are also polynomials. An example is $f(x)=(x+2)^3-1$.

Figure 2.2. 1



$$f(x) = (x+2)^3 - 1$$

The graph of $f(x) = (x+2)^3 - 1$ demonstrates the properties of being **continuous** and **smooth**.[7] The following graphs are not continuous and smooth. The first three graphs are not continuous at $x = 2$ while the fourth graph is not smooth at $x = 2$.

Figure 2.2. 2

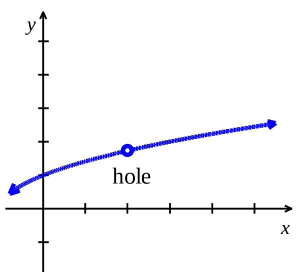


Figure 2.2. 3

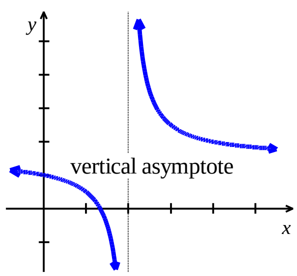


Figure 2.2. 4

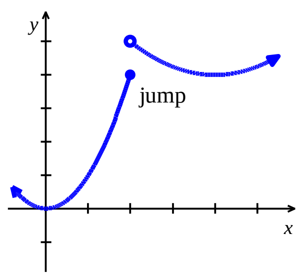
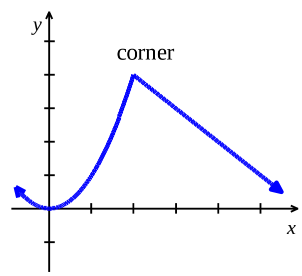


Figure 2.2. 5



None of these four graphs is that of a polynomial function. Before moving on, we note that the graphs of polynomial functions are continuous and smooth everywhere.

The Intermediate Value Theorem

Due to the continuity of polynomials, we may use the following theorem.

Theorem 2.2. The Intermediate Value Theorem: Suppose f is a continuous function on an interval containing $x = a$ and $x = b$ with $a < b$. If $f(a)$ and $f(b)$ have different signs, then f has at least one zero between $x = a$ and $x = b$; that is, for at least one real number c such that $a < c < b$, we have $f(c) = 0$.

Most students see the Intermediate Value Theorem as common sense since it says, geometrically, that the graph of a continuous function cannot be above the *x*-axis at one point and below the *x*-axis at another

[7] The terms 'continuous' and 'smooth' will be discussed more precisely in future math classes.

point without crossing the *x*-axis somewhere in between. The following example uses the Intermediate Value Theorem to establish a fact that most students take for granted.

Example 2.2.3. Use the Intermediate Value Theorem to establish that there is a positive number whose square is 2.

Solution. Consider the polynomial function $f(x)=x^2-2$. Then $f(1)=-1$ and $f(3)=7$. Since $f(1)$ and $f(3)$ have different signs, and f is a polynomial so is continuous, the Intermediate Value Theorem guarantees us a real number *c* between 1 and 3 with $f(c)=0$. If $f(c)=c^2-2=0$ then $c^2=2$. It follows that, since *c* is between 1 and 3, *c* must be positive and so *c* is the number that we seek.

□

Our primary use of the Intermediate Value Theorem is in graphing polynomial functions since it guarantees that a polynomial function is always positive or always negative on intervals which do not contain any of its zeros.

Graphing Polynomials

Example 2.2.4. Sketch a rough graph of the polynomial function $f(x)=(x-2)(x+3)^2$.

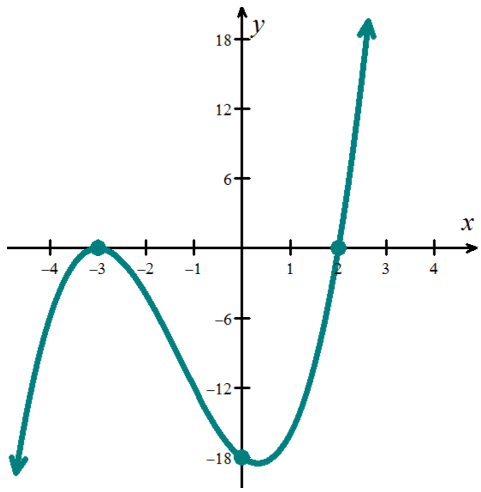
Solution. Our goal in sketching a 'rough' graph is to draw a smooth, continuous curve that includes *x*-intercepts and that shows intervals where the graph is above the *x*-axis and intervals where the graph is below the *x*-axis. First, we find the zeros, or *x*-intercepts, of f by solving $(x-2)(x+3)^2=0$. We get $x=2$ and $x=-3$. These two points divide the *x*-axis into three intervals: $(-\infty,-3)$, $(-3,2)$ and $(2,\infty)$.

We next test a point in each of these intervals to determine if the function f is positive or negative. Where f is positive, its graph is above the *x*-axis; where f is negative, its graph is below the *x*-axis.

Interval	Test Value	Function Value	Location of Graph
$(-\infty,-3)$	$x=-4$	$f(-4)=-6$	Below *x*-axis
$(-3,2)$	$x=0$	$f(0)=-18$	Below *x*-axis
$(2,\infty)$	$x=3$	$f(3)=+36$	Above *x*-axis

Knowing that the graph is smooth and continuous, we use the *x*-intercepts and intervals where f is above/below the *x*-axis to sketch a graph of the function. To include the *y*-intercept in our graph, we set $x=0$ to find $y=-18$. We don't know how low the graph of our function goes, but we do know it goes at least to −18.

Figure 2.2. 6



A sketch of $y=f(x)$

□

In **Example 2.2.4**, if we took the time to find the leading term of f, we would find it to be x^3. We have yet to look at the end behavior of polynomial functions, but we will find that the leading term gives us important information about the direction of the graph's tails.

The Multiplicity of a Zero

Before discussing end behavior, we look at the multiplicity of the zero of a polynomial and its effect on the graph. If a polynomial is written as a product of linear factors, then the multiplicity of a zero would be the number of occurences of the factor corresponding to that zero. From **Example 2.2.4**, we have $f(x)=(x-2)(x+3)(x+3)$. Since $(x-2)$ occurs one time, $x=2$ is a zero of multiplicity one. The factor $(x+3)$ occurs twice, and so $x=-3$ is a zero of multiplicity two.

We note that $(x-2)$ changes from negative to positive at $x=2$, resulting in a change of sign for the function f. On the other hand, $(x+3)^2$ remains positive on both sides of $x=-3$ so there is not a change in sign for the function f at that point. The effects of multiplicity are summarized below.

The Role of Multiplicity

Suppose f is a polynomial function and $x = c$ is a zero of multiplicity *m*.

- If *m* is even, the graph of $y = f(x)$ touches and rebounds from the *x*-axis at $(c, 0)$.
- If *m* is odd, the graph of $y = f(x)$ crosses through the *x*-axis at $(c, 0)$.

The End Behavior of Graphs of Polynomials

We start by observing the behavior of the graphs of $f(x) = x^2$, $h(x) = x^6$, $g(x) = -x^2$ and $j(x) = -x^6$. We note that these graphs have similar shapes. However, as the degree increases, the graphs flatten somewhat near the origin and become steeper away from the origin. Additionally, the graphs of g and j are reflections of the graphs of f and h, respectively, about the *x*-axis and thus have tails that head off in directions opposite to those of f and h.

Figure 2.2. 7

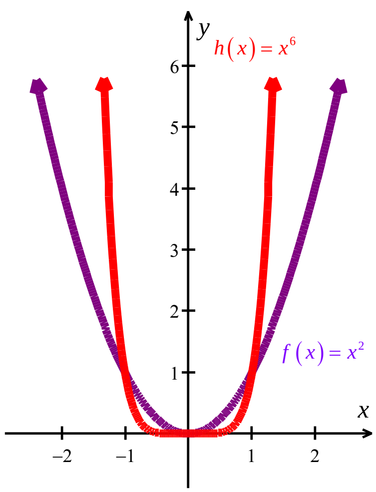
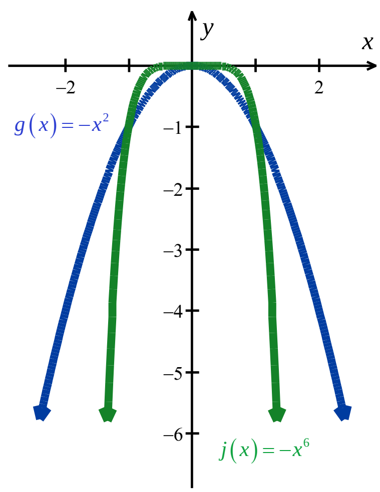


Figure 2.2. 8



The **end behavior** of a function is a way to describe what is happening to the function values (the *y*-values) as the *x*-values approach the 'ends' of the *x*-axis.[8] That is, what happens to *y* as *x* becomes small without bound[9] (written $x \to -\infty$) and, on the flip side, as *x* becomes large without bound[10] (written $x \to \infty$).

[8] Of course, there are no ends to the *x*-axis.
[9] We think of *x* as becoming a very large (in the sense of its absolute value) negative number, far to the left of zero.
[10] We think of *x* as moving far to the right of zero and becoming a very large positive number.

For example, given $f(x)=x^2$, as $x\to-\infty$, we imagine substituting $x=-100$, $x=-1000$, etc., into f to get $f(-100)=10000$, $f(-1000)=1000000$, and so on. Thus, the function values are becoming larger and larger positive numbers, without bound. To describe this behavior, we write

$$\text{as } x\to-\infty, f(x)\to\infty$$

If we study the behavior of f as $x\to\infty$, we see that, in this case too, $f(x)\to\infty$. The same can be said for any function of the form $f(x)=x^n$ where *n* is an even natural number. If we generalize just a bit to include vertical scalings and reflections across the *x*-axis, we have the following.

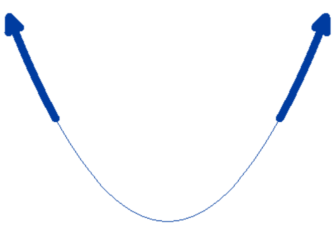
End Behavior of functions $f(x)=ax^n$, n even

Suppose $f(x)=ax^n$ where $a\neq 0$ is a real number and *n* is an even natural number. The end behavior of the graph of $y=f(x)$ matches one of the following:

1. For $a>0$, as $x\to-\infty$, $f(x)\to\infty$ and as $x\to\infty$, $f(x)\to\infty$
2. For $a<0$, as $x\to-\infty$, $f(x)\to-\infty$ and as $x\to\infty$, $f(x)\to-\infty$

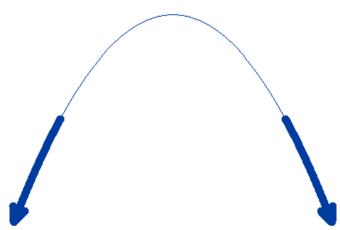
Graphically:

Figure 2.2. 9



$a>0$

Figure 2.2. 10



$a<0$

We now turn our attention to functions of the form $f(x)=x^n$ where $n\geq 3$ is an odd natural number. (We ignore the case when $n=1$ since the graph of $f(x)=x$ is a line and doesn't fit the general pattern of higher-degree odd polynomials.) Below are the graphs of $f(x)=x^3$, $h(x)=x^7$, $g(x)=-x^3$ and $j(x)=-x^7$. The flattening and steepening that we saw with the even powers presents itself here as well.

Figure 2.2. 11

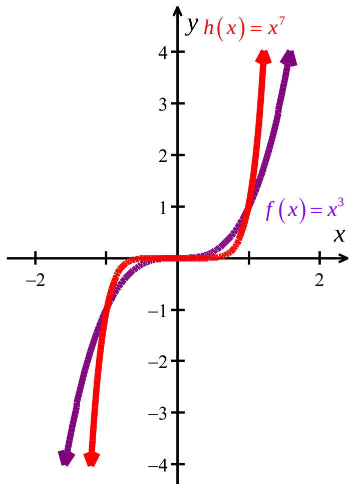
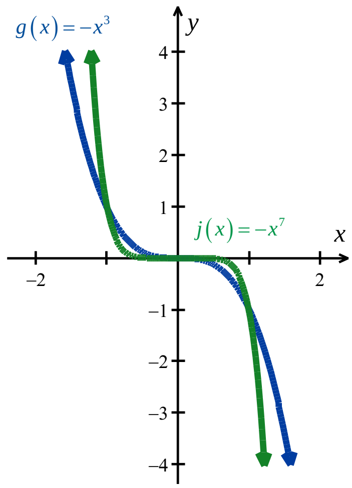


Figure 2.2. 12



The end behavior of the functions f and h is the same: $y \to -\infty$ as $x \to -\infty$ and $y \to \infty$ as $x \to \infty$. We note that the graphs of the functions g and j result from reflections of the graphs of f and h, respectively, about the *x*-axis, resulting in 'opposite' end behavior. As with the even degreed functions, we can generalize their end behavior.

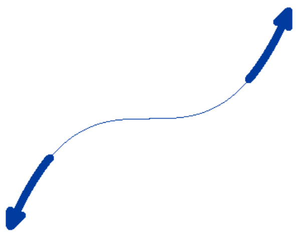
End Behavior of functions $f(x) = ax^n$, n odd

Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and $n \geq 3$ is an odd natural number. The end behavior of the graph of $y = f(x)$ matches one of the following:

- For $a > 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$
- For $a < 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) = \to -\infty$

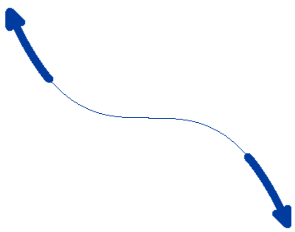
Graphically:

Figure 2.2. 13



$a > 0$

Figure 2.2. 14



$a < 0$

The following allows us to move beyond the end behavior of simple monomials and apply our results to all polynomial functions.

End Behavior for Polynomial Functions

The end behavior of a polynomial $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0$, with $a_n\neq 0$, matches the end behavior of $y=a_nx^n$.

To see why the end behavior of a polynomial matches the end behavior of its leading term, let's look at the specific example $f(x)=4x^3-x+5$. To examine the end behavior, we look at the behavior of f and the behavior of its leading term, $4x^3$, as $x\to\pm\infty$. The following table shows values of $4x^3$ and values of $f(x)$ for various input values of *x*.

x	$4x^3$	$f(x)=4x^3-x+5$
–1000	–4,000,000,000	–3,999,998,995
–100	–4,000,000	–3,999.895
–10	–4,000	–3,985
10	4,000	3,995
100	4,000,000	3,999,905
1000	4,000,000,000	3,999,999,005

As $x\to\pm\infty$, the table shows us that $f(x)\approx 4x^3$. Our next example shows how end behavior and multiplicity allow us to sketch a decent graph without calculating function values.

Example 2.2.5. Sketch a graph of $f(x)=x^3(2-x)(x+3)^2$ using end behavior and multiplicities of zeros.

Solution. The end behavior of the graph of f will match that of its leading term. To find the leading term, we multiply the leading terms of each factor to get $x^3(-x)(x)^2=-x^6$. Thus, the end behavior will be the same as that of $-x^6$. This tells us that the graph of $y=f(x)$ starts and ends below the *x*-axis. In other words, as $x\to-\infty$, $f(x)\to-\infty$ and as $x\to\infty$, $f(x)\to-\infty$.

Next, we find the zeros of $f(x)=x^3(2-x)(x+3)^2$. Fortunately, f is factored.[11] Setting each factor equal to 0 gives us $x=0$, $x=2$ and $x=-3$ as zeros. As far as multiplicities, we have the following:

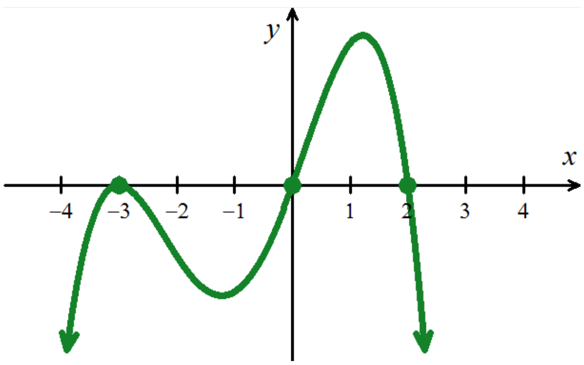
- $x=0$ is a zero of odd multiplicity 3, so the graph will cross through the *x*-axis at $(0,0)$.
- $x=2$ is a zero of odd multiplicity 1, so the graph will cross through the *x*-axis at $(2,0)$.

[11] Obtaining the factored form of a polynomial is the main focus of the next few sections.

- $x = -3$ is a zero of even multiplicity 2, so the graph will only touch the *x*-axis at $(-3,0)$.

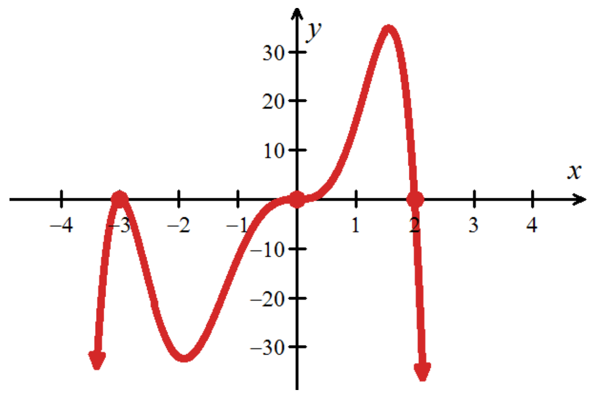
Putting all of this information together, we sketch a smooth, continuous, curve that includes the *x*-intercepts and intervals above/below the *x*-axis. Following is a rough sketch along with the actual graph of $f(x) = x^3(2-x)(x+3)^2$.

Figure 2.2. 15



A rough sketch of $y = f(x)$

Figure 2.2. 16



Actual Scaled Graph of $y = f(x)$

□

A general strategy for graphing polynomial functions appears below.[12]

To Graph a Polynomial Function:

1. Find the *x*-intercepts.
2. Find the multiplicity at each *x*-intercept.
3. Find the *y*-intercept.
4. Determine the end behavior.
5. Sketch the smooth curve that results from all of the above.

Applications

We end this section with an example that shows how polynomials of higher degree arise 'naturally' in even the most basic geometric applications.

[12] As in **Section 2.1**, we are interested in creating a rough sketch. For a more accurate graph, it is useful to plot additional points.

Example 2.2.6. A box with no top is to be fashioned from a 10 inch by 12 inch piece of cardboard by cutting out congruent squares from each corner of the cardboard and then folding the resulting tabs. Let x denote the length of the side of the square which is removed from each corner.

Figure 2.2. 17

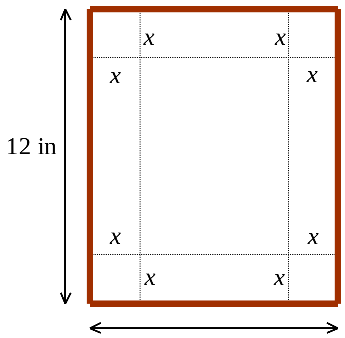
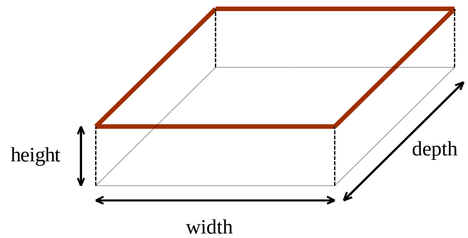


Figure 2.2. 18



1. Find the volume V of the box as a function of x. Include an appropriate applied domain.
2. Use graphing technology to graph $y = V(x)$ on the domain you found in part 1 and approximate the dimensions of the box with maximum volume to two decimal places. What is the maximum volume?

Solution.

1. From Geometry, we know that $\text{volume} = \text{width} \times \text{height} \times \text{depth}$. The key is to find each of these quantities in terms of x. From the figure, we see that the height of the box is x itself. The cardboard piece is initially 10 inches wide. Removing squares with a side length of x inches from each corner leaves $10-2x$ inches for the width.[13] As for the depth, the cardboard is initially 12 inches long, so after cutting out x inches from each side, we would have $12-2x$ inches remaining. As a function[14] of x, the volume is

$$\begin{aligned} V(x) &= x(10-2x)(12-2x) \\ &= x(120-44x+4x^2) \\ &= 120x-44x^2+4x^3 \end{aligned}$$

To find a suitable domain, we note that to make a box at all we need $x > 0$. Also, the shorter of the two dimensions of the cardboard is 10 inches, and since we are removing $2x$ inches from this dimension, we also require $10-2x > 0$ or $x < 5$.

---

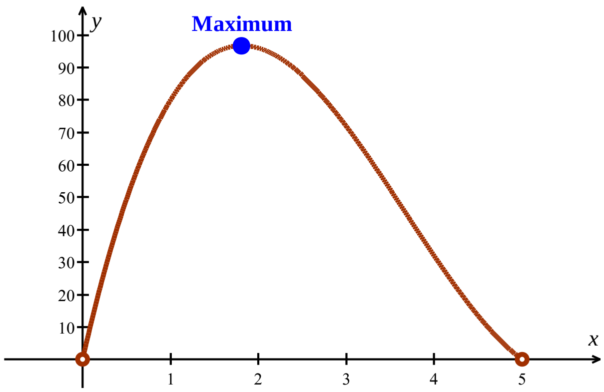
[13] There's no harm in taking an extra step here and making sure this makes sense. If we removed a 1×1 inch square from each corner, then the width would be 8 inches, so removing a square with sides of length x inches would leave 10-2x inches.

[14] When we write $V(x)$, it is in the context of function notation, not the volume V times the quantity x.

Hence, our volume is $V(x)=4x^3-44x^2+120x$ and the applied domain is $0<x<5$.

2. Using graphing technology, such as a graphing calculator or online graphing tool[15], we see that the graph of $y=V(x)$ has a relative maximum. For $0<x<5$, this is also the absolute maximum.

Figure 2.2. 19



Using graphing technology, we find the maximum occurs when $x\approx 1.81$ and $y\approx 96.77$. This yields a height of $x\approx 1.81$ inches, a width of $10-2x\approx 6.38$ inches and a depth of $12-2x\approx 8.38$ inches. The *y*-coordinate gives us the maximum volume, which is approximately 96.77 cubic inches.

□

[15] Wolfram Alpha or Desmos are possiblilities. Many free online graphing calculators are available as well.

2.2 Exercises

1. If a polynomial function is in factored form, what would be a good first step in order to determine the degree of the function?
2. In general, explain the end behavior of the graph of a polynomial function with odd degree if the leading coefficient is positive.

In Exercises 3 – 12, find the degree, the leading term, the leading coefficient, the constant term and the end behavior of the given polynomial.

3. $f(x) = 4 - x - 3x^2$
4. $g(x) = 3x^5 - 2x^2 + x + 1$
5. $q(r) = 1 - 16r^4$
6. $Z(b) = 42b - b^3$
7. $f(x) = \sqrt{3}x^{17} + 22.5x^{10} - \pi x^7 + \frac{1}{3}$
8. $s(t) = -4.9t^2 + v_0 t + s_0$
9. $P(x) = (x-1)(x-2)(x-3)(x-4)$
10. $p(t) = -t^2(3-5t)(t^2+t+4)$
11. $f(x) = -2x^3(x+1)(x+2)^2$
12. $G(t) = 4(t-2)^2\left(t+\frac{1}{2}\right)$

In Exercises 13 – 22, find the *x*-intercept(s) of the given polynomial, their corresponding multiplicities, and the *y*-intercept. Use this information to sketch a rough graph of the polynomial.

13. $a(x) = x(x+2)^2$
14. $g(x) = x(x+2)^3$
15. $f(x) = -2(x-2)^2(x+1)$
16. $g(x) = (2x+1)^2(x-3)$
17. $F(x) = x^3(x+2)^2$
18. $P(x) = (x-1)(x-2)(x-3)(x-4)$
19. $Q(x) = (x+5)^2(x-3)^4$
20. $h(x) = x^2(x-2)^2(x+2)^2$
21. $H(t) = (3-t)(t+1)^2$
22. $Z(b) = b(49-b^2)$

In Exercises 23 – 28, given the pair of functions f and g, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and using transformations. Track at least three points of your choice through the transformations. State the domain and range of g.

23. $f(x)=x^3$, $g(x)=(x+2)^3+1$

24. $f(x)=x^4$, $g(x)=(x+2)^4+1$

25. $f(x)=x^4$, $g(x)=2-3(x-1)^4$

26. $f(x)=x^5$, $g(x)=-x^5-3$

27. $f(x)=x^5$, $g(x)=(x+1)^5+10$

28. $f(x)=x^6$, $g(x)=8-x^6$

29. Use the Intermediate Value Theorem to prove that $f(x)=x^3-9x+5$ has a real zero in each of the following intervals: $(-4,-3)$, $(0,1)$ and $(2,3)$.

30. Use the Intermediate Value Theorem to confirm that $f(x)=x^3-100x+2$ has at least one real zero between $x=0.01$ and $x=0.1$.

31. Show that the function $f(x)=x^3-5x^2+3x+6$ has at least two real zeros between $x=1$ and $x=4$.

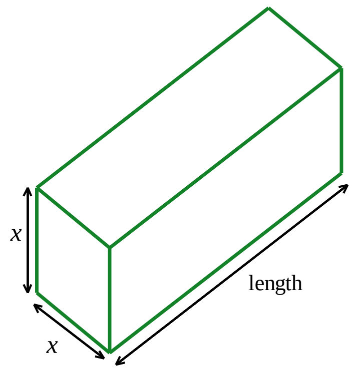
32. Rework **Example 2.2.6** assuming the box is to be made from an 8.5 inch by 11 inch sheet of paper. Using scissors and tape, construct the box. Are you surprised?

33. Suppose the profit P, in thousands of dollars, from producing and selling x hundred LCD TV's is given by $P(x)=-5x^3+35x^2-45x-25$ for $0\le x\le 10.07$. Use graphing technology to graph $y=P(x)$ and determine the number of TV's which should be sold to maximize profit. What is the maximum profit?

34. While developing their newest game, Sasquatch Attack!, the makers of the PortaBoy revised their profit function and now use $P(x)=-0.03x^3+3x^2+25x-250$, for $x\ge 0$. Use graphing technology to find the production level x that maximizes the profit made by producing and selling x PortaBoy game systems.

35. According to US Postal regulations, a rectangular shipping box must satisfy the inequality $\text{Length}+\text{Girth}\le 130\text{ inches}$ for Parcel Post and $\text{Length}+\text{Girth}\le 108\text{ inches}$ for other services. Let's assume we have a closed rectangular box with a square face of side length x as drawn below. The length is the longest side and is clearly labeled. The girth is the distance around the box in the other two dimensions so in our case it is the sum of the four sides of the square, $4x$.

Figure 2.2. 20



(a) Assuming that we'll be mailing a box via Parcel Post where $\text{Length} + \text{Girth} = 130 \text{ inches}$, express the length of the box in terms of *x* and then express the volume *V* of the box in terms of *x*.

(b) Find the dimensions of the box of maximum volume that can be shipped via Parcel Post.

(c) Repeat parts (a) and (b) if the box is shipped using 'other services'.

36. Show that the end behavior of a linear function $f(x) = mx + b$ is as it should be according to the results we've established in this section for polynomials of odd degree.[16] (That is, show that the graph of a linear function is 'up on one side' and 'down on the other' just like the graph of $y = a_n x^n$ for odd numbers *n*.)

37. Here are a few questions for you to discuss with your classmates.

(a) How many local extrema could a polynomial of degree *n* have? How few local extrema?

(b) Could a polynomial have two local maxima but no local minima?

(c) If a polynomial has two local maxima and two local minima, can it be of odd degree? Can it be of even degree?

(d) Can a polynomial have local extrema without having any real zeros?

(e) Why must every polynomial of odd degree have at least one real zero?

(f) Can a polynomial have two distinct real zeros and no local extrema?

(g) Can an *x*-intercept yield a local extrema? Can it yield an absolute extrema?

(h) If the *y*-intercept yields an absolute minimum, what can we say about the degree of the polynomial and the sign of the leading coefficient?

---

[16] Remember, to be a linear function, $m \neq 0$.

2.3 Using Synthetic Division to Factor Polynomials

Learning Objectives

- Use division to factor polynomials and determine zeros.
- Use synthetic division to simplify the division process.
- Use the Remainder Theorem to find function values of polynomials.
- Use the Factor Theorem to relate zeros to factors of polynomials.

Using Division to Find Zeros of Polynomials

Suppose we wish to find the zeros of $f(x)=x^3+4x^2-5x-14$. Setting $f(x)=0$ results in the polynomial equation $x^3+4x^2-5x-14=0$. Despite all of the factoring techniques we learned in Intermediate Algebra, this equation foils[17] us at every turn. Should we happen to guess (correctly) that $x=2$ is a zero, there must be a factor of $(x-2)$ lurking around in the factorization of $f(x)$. How could we use the factor $(x-2)$ to find this factorization? The answer comes from our old friend, polynomial division. Dividing $x^3+4x^2-5x-14$ by $x-2$ gives

$$
\begin{array}{r}
x^2+6x+7 \\
x-2\,\overline{)\,x^3+4x^2-5x-14} \\
-\underline{\left(x^3-2x^2\right)} \qquad\qquad \\
6x^2-5x \qquad \\
-\underline{\left(6x^2-12x\right)} \qquad \\
7x-14 \\
-\underline{\left(7x-14\right)} \\
0
\end{array}
$$

This means $\left(x^3+4x^2-5x-14\right)\div(x-2)=x^2+6x+7$ or, after multiplying through by $(x-2)$,

$x^3+4x^2-5x-14=\left(x^2+6x+7\right)(x-2)$.

[17] Pun intended.

To find the zeros of $f(x) = x^3 + 4x^2 - 5x - 14$, we now solve $(x^2 + 6x + 7)(x - 2) = 0$. We get $x^2 + 6x + 7 = 0$ and $x - 2 = 0$. Since $x^2 + 6x + 7$ doesn't factor nicely, we apply the Quadratic Formula to get $x = -3 \pm \sqrt{2}$. From $x - 2 = 0$, we get our 'known' zero of $x = 2$.

The point of this section is to generalize the technique applied here. First up is a friendly reminder of what we can expect when we divide polynomials.

Theorem 2.3. Polynomial Division: Suppose $d(x)$ and $p(x)$ are nonzero polynomials where the degree of *p* is greater than or equal to the degree of *d*. There exist two unique polynomials, $q(x)$ and $r(x)$, such that $p(x) = d(x)q(x) + r(x)$, where either $r(x) = 0$ or the degree of *r* is strictly less than the degree of *d*.

As you may recall, all of the polynomials in **Theorem 2.3** have special names. The polynomial *p* is called the **dividend**; *d* is the **divisor**; *q* is the **quotient**; *r* is the **remainder**. If $r(x) = 0$ then *d* is called a **factor** of *p*. The proof of **Theorem 2.3** is usually relegated to a course in Abstract Algebra, but we can still use the result to establish two important facts which are the basis for the rest of the chapter.

The Remainder and Factor Theorems

Theorem 2.4. The Remainder Theorem: Suppose *p* is a polynomial of degree at least 1 and *c* is a real number. When $p(x)$ is divided by $x - c$ the remainder is $p(c)$.

The proof of **Theorem 2.4** is a direct consequence of **Theorem 2.3**. When a polynomial is divided by $x - c$, the remainder is either 0 or has degree less than the degree of $x - c$. Since $x - c$ is degree 1, the degree of the remainder must be 0, which means the remainder is a constant. Hence, in either case, $p(x) = (x - c)q(x) + r$ where *r*, the remainder, is a real number, possibly 0. We have the following.

$$\begin{aligned} p(c) &= (c - c)q(c) + r \\ &= 0 \cdot q(c) + r \\ &= r \end{aligned}$$

So we get $r = p(c)$ as required. We have one more result to present.

Theorem 2.5. The Factor Theorem: Suppose *p* is a polynomial. The real number *c* is a zero of *p* if and only if $(x - c)$ is a factor of $p(x)$.

The proof of the Factor Theorem is a consequence of what we already know. If $(x-c)$ is a factor of $p(x)$, this means $p(x)=(x-c)q(x)$ for some polynomial *q*. Hence, $p(c)=(c-c)q(x)=0$, so *c* is a zero of *p*.

Conversely, if *c* is a zero of *p*, then $p(c)=0$. In this case, the Remainder Theorem tells us the remainder when $p(x)$ is divided by $(x-c)$, namely $p(c)$, is 0, which means that $(x-c)$ is a factor of *p*. What we have established is the fundamental connection between zeros of polynomials and factors of polynomials.

Synthetic Division – Why it Works

We can find a more efficient way to divide polynomials by quantities of the form $x-c$. Let's take a closer look at the long division we performed at the beginning of this section, with the single change of distributing through the negative ones to eliminate parentheses.

$$
\begin{array}{rrrrr}
 & & x^2 & +6x & +7 \\ \cline{2-5}
x-2 \big) & x^3 & +4x^2 & -5x & -14 \\
 & \underline{-x^3} & \underline{+2x^2} & & \\
 & & 6x^2 & -5x & \\
 & & \underline{-6x^2} & \underline{+12x} & \\
 & & & 7x & -14 \\
 & & & \underline{-7x} & \underline{+14} \\
 & & & & 0
\end{array}
$$

Observe that the terms $-x^3$, $-6x^2$ and $-7x$ are the exact opposites of the terms above them. The algorithm we use ensures this is always the case, so we can omit them without losing any information. Also note that the terms we 'bring down' (namely the $-5x$ and -14) aren't really necessary to recopy, so we omit them, too.

$$
\begin{array}{rrrrr}
 & & x^2 & +6x & +7 \\ \cline{2-5}
x-2 \big) & x^3 & +4x^2 & -5x & -14 \\
 & \underline{} & \underline{2x^2} & & \\
 & & 6x^2 & & \\
 & & \underline{} & \underline{12x} & \\
 & & & 7x & \\
 & & & \underline{} & \underline{14} \\
 & & & & 0
\end{array}
$$

Now let's 'move things up' a bit and, for reasons which will become clear in a moment, copy the x^3 into the last row.

$$\begin{array}{r|rrrr} & & x^2 & +6x & +7 \\ \hline x-2 & x^3 & +4x^2 & -5x & -14 \\ & & 2x^2 & 12x & 14 \\ \hline & x^3 & 6x^2 & 7x & 0 \end{array}$$

Note that each term in the last row is now obtained by adding the two terms above it. Notice also that the quotient polynomial can be obtained by dividing each of the first three terms in the last row by *x* and adding the results. If you take the time to work back through the original division problem, you will find that this is exactly the way we determined the quotient polynomial. This means that we no longer need to write the quotient polynomial down, nor the *x* in the divisor, to determine our answer.

$$\begin{array}{r|rrrr} -2 & x^3 & +4x^2 & -5x & -14 \\ & & 2x^2 & 12x & 14 \\ \hline & x^3 & 6x^2 & 7x & 0 \end{array}$$

To streamline things further, recall that the $2x^2$, $12x$ and 14 in the second row came from multiplying the terms in the quotient, x^2, $6x$ and 7, respectively, first by -2 and then by -1. The result is multiplication by 2, so we replace the -2 in the divisor with 2. Furthermore, the coefficients of the quotient polynomial match the coefficients of the first three terms in the last row, so we now take the plunge and write only the coefficients of the terms.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & 0 \end{array}$$

Synthetic Division – How it Works

We have constructed a **synthetic division tableau**. Let's rework our division problem using this tableau. To divide $x^3+4x^2-5x-14$ by $x-2$, we write 2 in the place of the divisor and the coefficients of $x^3+4x^2-5x-14$ in the place of the dividend. We then 'bring down' the first coefficient of the dividend.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & & & \\ \hline & & & & \end{array} \qquad\qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & & & \\ \hline & 1 & & & \end{array}$$

Next, we take the 2 from the divisor and multiply by the 1 that was brought down, resulting in 2. We write the 2 underneath the 4, then add to get 6.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & 6 & & \end{array}$$

Now we take the 2 from the divisor times the 6 to get 12, and add it to the -5 to get 7.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & 7 & \end{array}$$

Finally, we take the 2 in the divisor times the 7 to get 14, and add it to the -14 to get 0.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \mathbf{0} \end{array}$$

The first three numbers in the last row are the coefficients of the quotient polynomial. Having divided a third degree polynomial by a first degree polynomial, the quotient is the second degree polynomial x^2+6x+7. The 0 is the remainder.

Synthetic division is a time saver, but only works for divisors of the form $x-c$. Note that when a polynomial (of degree at least 1) is divided by $x-c$, the result will be a polynomial of exactly one less degree.

Example 2.3.1. Use synthetic division to perform the following polynomial divisions. Find the quotient and the remainder polynomials, then write the dividend, quotient and remainder in the form given in **Theorem 2.3**.

1. $\left(5x^3-2x^2+1\right)\div(x-3)$
2. $\left(x^3+8\right)\div(x+2)$
3. $\dfrac{4-8x-12x^2}{2x-3}$

Solution.

1. For $\left(5x^3-2x^2+1\right)\div(x-3)$, when setting up the synthetic division tableau, we need to enter 0 for the coefficient of *x* in the dividend. Doing so, we have the following.

$$\begin{array}{r|rrrr} 3 & 5 & -2 & 0 & 1 \\ & \downarrow & 15 & 39 & 117 \\ \hline & 5 & 13 & 39 & 118 \end{array}$$

Since the dividend was a third degree polynomial, the quotient is a quadratic polynomial with coefficients 5, 13 and 39. Our quotient is $q(x)=5x^2+13x+39$ and the remainder is $r(x)=118$. According to **Theorem 2.3**, we have $5x^3-2x^2+1=(x-3)\left(5x^2+13x+39\right)+118$.

2. In dividing x^3+8 by $x+2$, we rewrite $x+2$ as $x-(-2)$ and proceed as before.

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 0 & 8 \\ & \downarrow & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & 0 \end{array}$$

We get the quotient $q(x)=x^2-2x+4$ and the remainder $r(x)=0$. This gives us $x^3+8=(x+2)(x^2-2x+4)$.

3. To divide $4-8x-12x^2$ by $2x-3$, two things must be done. First, we write the dividend in descending powers of x as $-12x^2-8x+4$. Second, since synthetic division is designed only for factors of the form $x-c$, we factor $2x-3$ as $2\left(x-\frac{3}{2}\right)$. Our strategy is to first divide $-12x^2-8x+4$ by 2 to get $-6x^2-4x+2$. Next, we divide by $\left(x-\frac{3}{2}\right)$. The tableau becomes

$$\begin{array}{r|rrr} \frac{3}{2} & -6 & -4 & 2 \\ & \downarrow & -9 & -\frac{39}{2} \\ \hline & -6 & -13 & -\frac{35}{2} \end{array}$$

We get $-6x^2-4x+2=\left(x-\frac{3}{2}\right)(-6x-13)-\frac{35}{2}$. Multiplying both sides by 2 and distributing gives $12x^2-8x+4=(2x-3)(-6x-13)-35$.

□

The next example pulls together all of the concepts discussed in this section.

Example 2.3.2. Let $p(x)=2x^3-5x+3$.

1. Find $p(-2)$ using the Remainder Theorem. Check your answer by substitution.

2. Use the fact that $x=1$ is a zero of p to factor $p(x)$ and then find all of the real zeros of p.

Solution.

1. The Remainder Theorem states that $p(-2)$ is the remainder when $p(x)$ is divided by $x-(-2)$. We set up our synthetic division tableau below. We are careful to record the coefficient of x^2 as 0 and proceed as above.

$$\begin{array}{r|rrrr} -2 & 2 & 0 & -5 & 3 \\ & \downarrow & -4 & 8 & -6 \\ \hline & 2 & -4 & 3 & -3 \end{array}$$

According to the Remainder Theorem, $p(-2) = -3$. We can check this by direct substitution into the formula $p(x)$.

$$\begin{aligned} p(-2) &= 2(-2)^3 - 5(-2) + 3 \\ &= -16 + 10 + 3 \\ &= -3 \end{aligned}$$

2. The Factor Theorem tells us that since $x = 1$ is a zero of p, $x - 1$ is a factor of $p(x)$. To factor $p(x)$, we divide as follows.

$$\begin{array}{r|rrrr} 1 & 2 & 0 & -5 & 3 \\ & \downarrow & 2 & 2 & -3 \\ \hline & 2 & 2 & -3 & 0 \end{array}$$

We get a remainder of 0 which verifies that $p(1) = 0$. Our quotient polynomial is $q(x) = 2x^2 + 2x - 3$, from which we find $p(x) = (x-1)(2x^2 + 2x - 3)$. To find the remaining zeros of *p*, we need to solve $2x^2 + 2x - 3 = 0$ for *x*. Since this doesn't factor nicely, we use the Quadratic Formula to determine that the remaining zeros are $x = \frac{-1 \pm \sqrt{7}}{2}$.

□

In **Section 2.2**, we introduced the multiplicity of a zero. The role of multiplicity in finding zeros of a polynomial is illustrated in the next example.

Example 2.3.3. Let $p(x) = 4x^4 - 4x^3 - 11x^2 + 12x - 3$. Given that $x = \frac{1}{2}$ is a zero of multiplicity 2, find all of the real zeros of *p*.

Solution. We set up for synthetic division. Since we are told the multiplicity of $\frac{1}{2}$ is two, we continue our tableau and divide $\left(x - \frac{1}{2}\right)$ into the quotient polynomial.

$$
\begin{array}{c|ccccc}
\frac{1}{2} & 4 & -4 & -11 & 12 & -3 \\
 & \downarrow & 2 & -1 & -6 & 3 \\
\hline
\frac{1}{2} & 4 & -2 & -12 & 6 & 0 \\
 & \downarrow & 2 & 0 & -6 & \\
\hline
 & 4 & 0 & -12 & 0 &
\end{array}
$$

From the first division, we have

$$4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)\left(4x^3 - 2x^2 - 12x + 6\right)$$

The second division gives us

$$4x^3 - 2x^2 - 12x + 6 = \left(x - \frac{1}{2}\right)\left(4x^2 - 12\right)$$

We combine the results of the first and second division to get

$$4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)^2 \left(4x^2 - 12\right)$$

To find the remaining zeros of *p*, we set $4x^2 - 12 = 0$ and get $x = \pm\sqrt{3}$.

□

In the previous example, we found $x = \pm\sqrt{3}$ are zeros of *p*, from which the Factor Theorem guarantees that both $\left(x - \sqrt{3}\right)$ and $\left(x - \left(-\sqrt{3}\right)\right) = \left(x + \sqrt{3}\right)$ are factors of *p*, as demonstrated below.

$$
\begin{aligned}
4x^4 - 4x^3 - 11x^2 + 12x - 3 &= \left(x - \frac{1}{2}\right)^2 \left(4x^2 - 12\right) \\
&= \left(x - \frac{1}{2}\right)^2 (4)\left(x^2 - 3\right) \\
&= \left(x - \frac{1}{2}\right)^2 (4)\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)
\end{aligned}
$$

The next section provides some tools which help us identify real numbers that may be zeros. We close this section with a summary of several concepts previously presented.

Connections Between Zeros, Factors and Graphs of Polynomial Functions

Suppose *p* is a polynomial function of degree $n \geq 1$. The following statements are equivalent.

- The real number *c* is a zero of *p*
- $p(c) = 0$
- $x = c$ is a solution to the polynomial equation $p(x) = 0$
- $(x - c)$ is a factor of $p(x)$
- The point $(c, 0)$ is an *x*-intercept of the graph of $y = p(x)$

2.3 Exercises

1. If division of a polynomial, of degree at least 1, by $x+4$ results in a remainder of zero, what can we conclude?

2. If a polynomial of degree n is divided by a binomial of degree 1, what is the degree of the quotient?

In Exercises 3 – 8, use polynomial long division to perform the indicated division. Write the polynomial in the form $p(x)=d(x)q(x)+r(x)$.

3. $\left(4x^2+3x-1\right)\div(x-3)$
4. $\left(2x^3-x+1\right)\div\left(x^2+x+1\right)$
5. $\left(5x^4-3x^3+2x^2-1\right)\div\left(x^2+4\right)$
6. $\left(-x^5+7x^3-x\right)\div\left(x^3-x^2+1\right)$
7. $\left(9x^3+5\right)\div(2x-3)$
8. $\left(4x^2-x-23\right)\div\left(x^2-1\right)$

In Exercises 9 – 22, use synthetic division to perform the indicated division. Write the polynomial in the form $p(x)=d(x)q(x)+r(x)$.

9. $\left(3x^2-2x+1\right)\div(x-1)$
10. $\left(x^2-5\right)\div(x-5)$
11. $\left(3-4x-2x^2\right)\div(x+1)$
12. $\left(4x^2-5x+3\right)\div(x+3)$
13. $\left(x^3+8\right)\div(x+2)$
14. $\left(4x^3+2x-3\right)\div(x-3)$
15. $\left(18x^2-15x-25\right)\div\left(x-\frac{5}{3}\right)$
16. $\left(4x^2-1\right)\div\left(x-\frac{1}{2}\right)$
17. $\left(2x^3+x^2+2x+1\right)\div\left(x+\frac{1}{2}\right)$
18. $\left(3x^3-x+4\right)\div\left(x-\frac{2}{3}\right)$
19. $\left(2x^3-3x+1\right)\div\left(x-\frac{1}{2}\right)$
20. $\left(4x^4-12x^3+13x^2-12x+9\right)\div\left(x-\frac{3}{2}\right)$
21. $\left(x^4-6x^2+9\right)\div\left(x-\sqrt{3}\right)$
22. $\left(x^6-6x^4+12x^2-8\right)\div\left(x+\sqrt{2}\right)$

In Exercises 23 – 32, determine $p(c)$ using the Remainder Theorem for the given polynomial function and value of c. If $p(c)=0$, factor $p(x)=(x-c)q(x)$.

23. $p(x)=2x^2-x+1$, $c=4$

24. $p(x)=4x^2-33x-180$, $c=12$

25. $p(x)=2x^3-x+6$, $c=-3$

26. $p(x)=x^3+2x^2+3x+4$, $c=-1$

27. $p(x)=3x^3-6x^2+4x-8$, $c=2$

28. $p(x)=8x^3+12x^2+6x+1$, $c=-\frac{1}{2}$

29. $p(x)=x^4-2x^2+4$, $c=\frac{3}{2}$

30. $p(x)=6x^4-x^2+2$, $c=-\frac{2}{3}$

31. $p(x)=x^4+x^3-6x^2-7x-7$, $c=-\sqrt{7}$

32. $p(x)=x^2-4x+1$, $c=2-\sqrt{3}$

In Exercises 33 – 42, you are given a polynomial and one of its zeros. Use the techniques in this section fo find the rest of the real zeros and factor the polynomial.

33. $x^3-6x^2+11x-6$, $c=1$

34. $x^3-24x^2+192x-512$, $c=8$

35. $3x^3+4x^2-x-2$, $c=\frac{2}{3}$

36. $2x^3-3x^2-11x+6$, $c=\frac{1}{2}$

37. x^3+2x^2-3x-6, $c=-2$

38. $2x^3-x^2-10x+5$, $c=\frac{1}{2}$

39. $4x^4-28x^3+61x^2-42x+9$, $c=\frac{1}{2}$ is a zero of multiplicity 2

40. $x^5+2x^4-12x^3-38x^2-37x-12$, $c=-1$ is a zero of multiplicity 3

41. $125x^5-275x^4-2265x^3-3213x^2-1728x-324$, $c=-\frac{3}{5}$ is a zero of multiplicity 3

42. x^2-2x-2 , $c=1-\sqrt{3}$

In Exercises 43 – 47, create a polynomial *p* which has the desired characteristics. You may leave the polynomial in factored form.

43. • The zeros of *p* are $c=\pm 2$ and $c=\pm 1$.

    • The leading term of $p(x)$ is $117x^4$.

44. • The zeros of *p* are $c=1$ and $c=3$.

    • $c=3$ is a zero of multiplicity 2.

- The leading term of $p(x)$ is $-5x^3$.

45.
- The solutions to $p(x)=0$ are $x=\pm 3$ and $x=6$.
- The leading tem of $p(x)$ is $7x^4$.
- The point $(-3,0)$ is a local minimum on the graph of $y=p(x)$.

46.
- The solutions to $p(x)=0$ are $x=\pm 3$, $x=-2$ and $x=4$.
- The leading term of $p(x)$ is $-x^5$.
- The point $(-2,0)$ is a local maximum on the graph of $y=p(x)$.

47.
- *p* is degree 4.
- as $x\to\infty$, $p(x)\to-\infty$.
- *p* has exactly three *x*-intercepts: $(-6,0)$, $(1,0)$ and $(117,0)$.
- The graph of $y=p(x)$ crosses through the *x*-axis at $(1,0)$.

48. Find a quadratic polynomial with integer coefficients which has $x=\frac{3}{5}\pm\frac{\sqrt{29}}{5}$ as its real zeros.

2.4 Real Zeros of Polynomials

Learning Objectives

- Find possible (potential) rational zeros using the Rational Zeros Theorem.
- Find real zeros of a polynomial and their multiplicities.

In **Section 2.3**, we found that we can use synthetic division to determine if a given real number is a zero of a polynomial function. This section presents results which will help us determine good candidates to test using synthetic division.

Finding Potential Rational Zeros

The following theorem gives us a list of possible real zeros.

Theorem 2.6. Rational Zeros Theorem: Suppose $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ is a polynomial of degree *n* with $n\geq 1$, and a_0, a_1, …, a_n are integers. If *r* is a rational zero of f, then *r* is of the form $\frac{p}{q}$, where *p* is a factor of the constant term a_0 and *q* is a factor of the leading coefficient a_n.

The Rational Zeros Theorem gives us a list of numbers to try in our synthetic division and that is a lot nicer than simply guessing. If none of the numbers in our list are zeros, then either the polynomial has no real zeros at all, or all of the real zeros are irrational numbers. We will not offer a proof here but will note that you've been using this idea when you factor a quadratic equation.

Example 2.4.1. Let $f(x)=2x^3-3x^2-8x-3$. Use the Rational Zeros Theorem to list all of the possible rational zeros of f.

Solution. To generate a complete list of possible rational zeros, we need to take each of the factors of the constant term, $a_0=-3$, and divide them by each of the factors of the leading coefficient, $a_3=2$. The factors of -3 are ± 1 and ± 3. The factors of 2 are ± 1 and ± 2, so the Rational Zeros Theorem gives the list $\pm\frac{1}{1}$, $\pm\frac{1}{2}$, $\pm\frac{3}{1}$ and $\pm\frac{3}{2}$, which is the same as ± 1, $\pm\frac{1}{2}$, ± 3 and $\pm\frac{3}{2}$.

□

The next example pulls together the Rational Zeros Theorem and the use of synthetic division, from **Section 2.3**, to determine which potential rational zeros are actually rational zeros of the polynomial.

Example 2.4.2. Find all of the rational zeros of $f(x)=2x^3-3x^2-8x-3$.

Solution. From **Example 2.4.1**, we have potential rational zeros of ±1, $\pm\frac{1}{2}$, ±3 and $\pm\frac{3}{2}$. We first try our potential zero of 1.

$$\begin{array}{r|rrrr} 1 & 2 & -3 & -8 & -3 \\ & \downarrow & 2 & -1 & -9 \\ \hline & 2 & -1 & -9 & -12 \end{array}$$

Since the remainder is not 0, we know $x=1$ is not a zero. We continue to the next possible zero of -1.

$$\begin{array}{r|rrrr} -1 & 2 & -3 & -8 & -3 \\ & \downarrow & -2 & 5 & 3 \\ \hline & 2 & -5 & -3 & 0 \end{array}$$

The remainder of 0 tells us that $x=-1$ is a zero. We have the additional information that $f(x)=(x+1)(2x^2-5x-3)$. While we could continue with synthetic division, once we have a second degree factor it is easy enough to solve $2x^2-5x-3=0$. In this case, factoring gives us $(2x+1)(x-3)=0$, from which we get the additional zeros of $x=-\frac{1}{2}$ and $x=3$.

□

With the next example, we revisit multiplicities, while relying on synthetic division and the Rational Zeros Theorem to factor polynomials to the point where we can identify non-rational zeros.

Example 2.4.3. Find all of the real zeros of $f(x)=x^4-2x^3-7x^2+16x-8$ and their multiplicities.

Solution. We begin by determining the potential rational zeros of f. The factors of the constant term, $a_0=-8$, are ±1, ±2, ±4 and ±8. Dividing the factors of a_0 by the factors of $a_4=1$, which include only ±1, we have potential rational zeros of ±1, ±2, ±4 and ±8.

We begin by testing the potential positive zero of 1.

$$\begin{array}{r|rrrrr} 1 & 1 & -2 & -7 & 16 & -8 \\ & \downarrow & 1 & -1 & -8 & 8 \\ \hline 1 & 1 & -1 & -8 & 8 & 0 \\ & \downarrow & 1 & 0 & -8 & \\ \hline & 1 & 0 & -8 & 0 & \end{array}$$

We see that $x=1$ is a zero, and continue the tableau to discover that it is a zero of multiplicity two. The synthetic division additionally provides us with a factored version of f: $f(x)=(x-1)^2(x^2-8)$.

Setting $x^2-8=0$, we find $x=\pm\sqrt{8}$, or $x=\pm 2\sqrt{2}$. Thus, in addition to the zero $x=1$ of multiplicity two, we have zeros $x=2\sqrt{2}$ and $x=-2\sqrt{2}$, each of multiplicity one.

□

Our next example reminds us of the role finding zeros plays in solving equations.

Example 2.4.4. Find all of the real solutions to the equation $2x^5+6x^3+3=3x^4+8x^2$.

Solution. Finding the real solutions to $2x^5+6x^3+3=3x^4+8x^2$ is the same as finding the real solutions to $2x^5-3x^4+6x^3-8x^2+3=0$. In other words, we are looking for the real zeros of $p(x)=2x^5-3x^4+6x^3-8x^2+3$. Using the techniques developed in this section, we get

$$\begin{array}{r|rrrrrr} 1 & 2 & -3 & 6 & -8 & 0 & 3 \\ & \downarrow & 2 & -1 & 5 & -3 & -3 \\ \hline 1 & 2 & -1 & 5 & -3 & -3 & 0 \\ & \downarrow & 2 & 1 & 6 & 3 & \\ \hline & 2 & 1 & 6 & 3 & 0 & \end{array}$$

Thus, $x=1$ is a zero of multiplicity 2 and we have $p(x)=(x-1)^2(2x^3+x^2+6x+3)$. To find remaining zeros, we set $2x^3+x^2+6x+3=0$ and return to synthetic division.[18]

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & 1 & 6 & 3 \\ & \downarrow & -1 & 0 & -3 \\ \hline & 2 & 0 & 6 & 0 \end{array}$$

We have an additional zero of $x=-\frac{1}{2}$. The quotient polynomial is $2x^2+6$ which produces no real zeros, so our solutions to the original equation are $x=1$ and $x=-\frac{1}{2}$.

□

[18] Factoring by grouping could also be used here.

2.4 Exercises

1. Explain why the Rational Zeros Theorem does not guarantee finding zeros of a polynomial function.

2. If synthetic division reveals a zero, why should we try that value again as a possible solution?

In Exercises 3 – 12, for the given polynomial, use the Rational Zeros Theorem to make a list of possible rational zeros.

3. $f(x) = x^3 - 2x^2 - 5x + 6$
4. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
5. $f(x) = x^4 - 9x^2 - 4x + 12$
6. $f(x) = x^3 + 4x^2 - 11x + 6$
7. $f(x) = x^3 - 7x^2 + x - 7$
8. $f(x) = -2x^3 + 19x^2 - 49x + 20$
9. $f(x) = -17x^3 + 5x^2 + 34x - 10$
10. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
11. $f(x) = 3x^3 + 3x^2 - 11x - 10$
12. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$

In Exercises 13 – 32, find the real zeros of the polynomial. State the multiplicity of each real zero.

13. $f(x) = x^3 - 2x^2 - 5x + 6$
14. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
15. $f(x) = x^4 - 9x^2 - 4x + 12$
16. $f(x) = x^3 + 4x^2 - 11x + 6$
17. $f(x) = x^3 - 7x^2 + x - 7$
18. $f(x) = -2x^3 + 19x^2 - 49x + 20$
19. $f(x) = -17x^3 + 5x^2 + 34x - 10$
20. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
21. $f(x) = 3x^3 + 3x^2 - 11x - 10$
22. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$
23. $f(x) = 9x^3 - 5x^2 - x$
24. $f(x) = 6x^4 - 5x^3 - 9x^2$
25. $f(x) = x^4 + 2x^2 - 15$
26. $f(x) = x^4 - 9x^2 + 14$
27. $f(x) = 3x^4 - 14x^2 - 5$
28. $f(x) = 2x^4 - 7x^2 + 6$
29. $f(x) = x^6 - 3x^3 - 10$
30. $f(x) = 2x^6 - 9x^3 + 10$
31. $f(x) = x^5 - 2x^4 - 4x + 8$
32. $f(x) = 2x^5 + 3x^4 - 18x - 27$

In Exercises 33 – 42, find the real solutions of the polynomial equation.

33. $9x^3 = 5x^2 + x$

34. $9x^2 + 5x^3 = 6x^4$

35. $x^3 + 6 = 2x^2 + 5x$

36. $x^4 + 2x^3 = 12x^2 + 40x + 32$

37. $x^3 - 7x^2 = 7 - x$

38. $2x^3 = 19x^2 - 49x + 20$

39. $x^3 + x^2 = \frac{11x + 10}{3}$

40. $x^4 + 2x^2 = 15$

41. $14x^2 + 5 = 3x^4$

42. $2x^5 + 3x^4 = 18x + 27$

43. Let $f(x) = 5x^7 - 33x^6 + 3x^5 - 71x^4 - 597x^3 + 2097x^2 - 1971x + 567$. With the help of your classmates, find the *x*- and *y*-intercepts of the graph of f. Find the intervals on which the graph is above the *x*-axis and the intervals on which the graph is below the *x*-axis. Sketch a rough graph of f.

2.5 Complex Zeros of Polynomials

Learning Objectives

- Perform operations on complex numbers.
- Find all complex zeros of a polynomial.
- Factor a polynomial to linear and irreducible quadratic factors.
- Use the conjugate of a complex zero to identify an additional zero.
- Create a polynomial given information that includes complex zeros.

In **Section 2.4**, we focused on finding the real zeros of a polynomial function. In this section, we expand our horizons and look for the non-real zeros as well.

Complex Numbers

Consider the polynomial $p(x)=x^2+1$. The zeros of p are the solutions to $x^2+1=0$, or $x^2=-1$. This equation has no real solutions, but x is a quantity whose square is -1. We write such a quantity as i, or $\sqrt{-1}$, and refer to it as the **imaginary unit**.

The number i, while not a real number, plays along well with real numbers and acts very much like any other radical expression. For instance, $3(2i)=6i$, $7i-3i=4i$, $(2-7i)+(3+4i)=5-3i$, and so forth. The key properties which distinguish i from the real numbers are listed below.

Definition 2.5. The imaginary unit i satisfies the two following properties.

1. $i^2=-1$
2. If c is a real number with $c\geq 0$ then $\sqrt{-c}=\left(\sqrt{c}\right)\cdot i$

Property 1 in **Definition 2.5** establishes that i does act as a square root[19] of -1, and property 2 establishes what we mean by the 'principal square root' of a negative real number. In property 2, it is important to remember the restriction on c. For example, it is perfectly acceptable to say

$$\sqrt{-4}=i\sqrt{4}=i(2)=2i$$

[19] Note the use of the indefinite article 'a' and that, while i is a square root of -1, $-i$ is the other square root of -1.

However, $\sqrt{-(-4)} \neq i\sqrt{-4}$ or we would get

$$2=\sqrt{4}=\sqrt{-(-4)}=i\sqrt{-4}=i(2i)=2i^2=2(-1)=-2$$

which is unacceptable.[20] We are now in the position to define the **complex numbers**.

> **Definition 2.6.** A **complex number** is a number of the form $a+bi$ where *a* and *b* are real numbers and i is the imaginary unit.

Complex numbers include things you'd normally expect[21]; for example, $3+2i$ and $\frac{2}{5}-i\sqrt{3}$. However, don't forget that *a* or *b* could be zero, which means numbers like $3i$ and 6 are also complex numbers. In other words, don't forget that the complex numbers include the real numbers, so 0 and $\pi-\sqrt{21}$ are both considered complex numbers. The arithmetic of complex numbers is as you would expect. The only things you need to remember are the two properties in **Definition 2.5**.

Example 2.5.1. Perform the indicated operations. Write your answer in the form $a+bi$.[22]

1. $(1-2i)-(3+4i)$
2. $(1-2i)(3+4i)$
3. $\dfrac{1-2i}{3-4i}$
4. $\sqrt{-3}\sqrt{-12}$
5. $\sqrt{(-3)(-12)}$
6. $(x-(1+2i))(x-(1-2i))$

Solution.

1. As mentioned earlier, we treat expressions involving i as we would any other radical. We combine like terms to get $(1-2i)-(3+4i)=1-2i-3-4i=-2-6i$.

2. Using the distributive property, we get

$$\begin{aligned}(1-2i)(3+4i)&=(1)(3)+(1)(4i)-(2i)(3)-(2i)(4i)\\&=3+4i-6i-8i^2\\&=3-2i-8(-1)\\&=11-2i\end{aligned}$$

---

[20] We want to enlarge the number system so we can solve things like $x^2=-1$, but not at the cost of the established rules already set in place. For that reason, the general properties of radicals simply do not apply for even roots of negative quantities.

[21] The convention is to write roots to the right of *i* while displaying integers and fractions/decimals to the left of *i*.

[22] Okay, we'll accept things like $3-2i$ even though it can be written as $3+(-2)i$.

3. To simplify $\frac{1-2i}{3-4i}$, we deal with the denominator $3-4i$ as we would any other denominator containing a radical, and multiply both numerator and denominator by $3+4i$ (the conjugate of $3-4i$).[23]

$$\begin{aligned}\frac{1-2i}{3-4i}\cdot\frac{3+4i}{3+4i} &= \frac{(1-2i)(3+4i)}{(3-4i)(3+4i)}\\ &= \frac{11-2i}{25}\\ &= \frac{11}{25}-\frac{2}{25}i\end{aligned}$$

4. We use property 2 of **Definition 2.5** first, then apply the rules of radicals applicable to real radicals.

$$\begin{aligned}\sqrt{-3}\sqrt{-12} &= \left(i\sqrt{3}\right)\left(i\sqrt{12}\right)\\ &= i^2\sqrt{3\cdot 12}\\ &= (-1)\sqrt{36}\\ &= -6\end{aligned}$$

5. We adhere to the order of operations here and perform the multiplication before the radical to get $\sqrt{(-3)(-12)}=\sqrt{36}=6$.

6. We can use the distributive property to multiply.

$$\begin{aligned}\left(x-(1+2i)\right)\left(x-(1-2i)\right) &= x^2 - x(1-2i) - x(1+2i) + (1+2i)(1-2i)\\ &= x^2 - x + 2ix - x - 2ix + 1 - 2i + 2i - 4i^2\\ &= x^2 - 2x + 1 - 4(-1)\\ &= x^2 - 2x + 5\end{aligned}$$

□

A couple of remarks are in order. First, the **conjugate** of a complex number $a+bi$ is the number $a-bi$. For example, the conjugate of $3+2i$ is $3-2i$, the conjugate of $3-2i$ is $3+2i$ and the conjugate of $4i$ is $-4i$. Second, a commonly used notation for conjugation is a 'bar': $\overline{a+bi}=a-bi$. Thus, we can write $\overline{4i}=-4i$.

[23] We will talk more about this in a moment.

Zeros of Polynomials

We now return to the business of zeros. Suppose we wish to find the zeros of $f(x)=x^2-2x+5$. To solve the equation $x^2-2x+5=0$, we note that the quadratic doesn't factor nicely, so we resort to the Quadratic Formula.

$$\begin{aligned} x &= \frac{-(-2)\pm\sqrt{(-2)^2-4(1)(5)}}{2(1)} \\ &= \frac{2\pm\sqrt{-16}}{2} \\ &= \frac{2\pm 4i}{2} \\ &= 1\pm 2i \end{aligned}$$

Two things are important to note. First, the zeros $1+2i$ and $1-2i$ are complex conjugates. If ever we obtain non-real zeros to a quadratic function with *real* coefficients, the zeros will be a complex conjugate pair. (Do you see why?) Next, recall that we found $(x-(1+2i))(x-(1-2i))=x^2-2x+5$ in **Example 2.5.1, part 6**. This demonstrates that the Factor Theorem holds even for non-real zeros; i.e. $x=1+2i$ is a zero of f and, sure enough, $(x-(1+2i))$ is a factor of $f(x)$.

It turns out that polynomial division works the same way for all complex numbers, real and non-real alike, so the Factor and Remainder Theorems hold as well. But how do we know if a general polynomial has any complex zeros at all? We have many examples of polynomials with no real zeros. Can there be polynomials with no zeros whatsoever? The answer to that last question is 'No', as long as the degree of the polynomial is at least one, and the theorem which provides that answer is the Fundamental Theorem of Algebra. While the proof of the Fundamental Theorem of Algebra is saved for future math classes, we use this important theorem, along with the Factor Theorem, to arrive at results involving complex factorization, included in the following theorem.

Theorem 2.7. The Fundamental Theorem of Algebra and Complex Factorization: Suppose f is a polynomial function with degree $n\geq 1$. Then f has at least one complex zero.

In actuality, f has exactly *n* zeros, counting multiplicities. If z_1, z_2, …, z_k are the distinct zeros of f, with multiplicities m_1, m_2, …, m_k, respectively, then $f(x)=a(x-z_1)^{m_1}(x-z_2)^{m_2}\cdots(x-z_k)^{m_k}$.

Note that the value a in **Theorem 2.7** is the leading coefficient of $f(x)$ (Can you see why?) and as such, we see that a polynomial is completely determined by its zeros, their multiplicities, and its leading coefficient. The following example demonstrates the results of **Theorem 2.7**.

Example 2.5.2. For the following polynomials, find the degree, the zeros, the multiplicity of each zero, and write the polynomial in factored form.

1. $f(x)=x-2$
2. $g(x)=x^2-4x+4$
3. $h(x)=3x^3+12x$
4. $j(x)=x^4-16$

Solution.

1. We note that $f(x)=x-2$ has degree 1, and cannot be further factored. By setting $x-2=0$, we find that f has a zero of $x=2$, and its multiplicity is 1.

2. The polynomial $g(x)=x^2-4x+4$ has degree 2. We may rewrite g as $g(x)=(x-2)^2$, from which we find one zero, $x=2$, of multiplicity 2.

3. We see that $h(x)=3x^3+12x$ has degree 3. After factoring, we have $h(x)=3x(x^2+4)$, for zeros of $x=0$, $x=2i$ and $x=-2i$. Each zero has multiplicity 1.

   We say $h(x)=3x(x^2+4)$ is factored **over the real numbers**. While not required to do so, we note that, by applying **Theorem 2.7**, we could further factor h as $h(x)=3(x-0)(x-(2i))(x-(-2i))$. In this case, we would say that h is factored **over the complex numbers**.

4. For $j(x)=x^4-16$, the degree is 4 and the polynomial factors as follows.

$$\begin{aligned} j(x) &= x^4-16 \\ &= (x^2-4)(x^2+4) \\ &= (x-2)(x+2)(x^2+4) \end{aligned}$$

   The zeros are $x=2$, $x=-2$, $x=2i$ and $x=-2i$, each of multiplicity 1.

   We note that $j(x)=(x-2)(x+2)(x^2+4)$ is factored over the real numbers. As in part 3, we could further factor j over the complex numbers as $j(x)=(x-2)(x-(-2))(x-(2i))(x-(-2i))$.

□

Complex Zeros

The last two results of this section will will show us how to identify both real and complex zeros.

Theorem 2.8. Conjugate Pairs Theorem: If f is a polynomial function with real number coefficients and $a+bi$ is a zero of f, then its complex conjugate, $a-bi$, is also a zero of f; or vice versa.

The proof of the Conjugate Pairs Theorem uses properties of conjugates, along with the assumption that $f(a+bi)=0$, to show that $f(a-bi)=0$. We leave this proof to the enthused reader for now.

Since nonreal zeros of a polynomial f come in the conjugate pairs $a+bi$ and $a-bi$, the Factor Theorem kicks in to give us both $(x-(a+bi))$ and $(x-(a-bi))$ as factors of $f(x)$. This means that $(x-(a+bi))(x-(a-bi))=x^2-2ax+(a^2+b^2)$ is an irreducible quadratic factor of f. As a result, we have our last theorem of the section.

Theorem 2.9. Real Factorization Theorem: Suppose f is a polynomial function with real number coefficients. Then $f(x)$ can be factored into a product of linear factors corresponding to the real zeros of f and irreducible quadratic factors which give the nonreal zeros of f.

We now present an example that pulls together the major ideas of this section.

Example 2.5.3. Let $f(x)=x^5-3x^4+3x^3-5x^2+12$.

1. Find all of the zeros of f and state their multiplicities.

2. Factor f to linear and irreducible quadratic factors.

Solution.

1. Since f is a fifth degree polynomial, we know that we need to perform at least three successful divisions to get the quotient down to a quadratic function. At that point, we can find the remaining zeros using the Quadratic Formula, if necessary. Using the techniques developed in **Section 2.4**, we get

$$\begin{array}{r|rrrrrr} -1 & 1 & -3 & 3 & -5 & 0 & 12 \\ & \downarrow & -1 & 4 & -7 & 12 & -12 \\ \hline & 1 & -4 & 7 & -12 & 12 & 0 \end{array}$$

We thus have the zero $x=-1$ and we can factor f as $f(x)=(x+1)(x^4-4x^3+7x^2-12x+12)$.

We continue the search for rational zeros of $x^4-4x^3+7x^2-12x+12$.

$$\begin{array}{r|rrrrr} 2 & 1 & -4 & 7 & -12 & 12 \\ & \downarrow & 2 & -4 & 6 & -12 \\ \hline 2 & 1 & -2 & 3 & -6 & 0 \\ & \downarrow & 2 & 0 & 6 & \\ \hline & 1 & 0 & 3 & 0 & \end{array}$$

We have the zero $x=2$ of multiplicity 2, and the quotient x^2+3. We find the solutions to $x^2+3=0$ to be $x=\pm i\sqrt{3}$. From **Theorem 2.7**, we know f has exactly 5 zeros, counting multiplicities, and as such we have the zero $x=2$ of multiplicity 2, and the zeros $x=-1$, $x=i\sqrt{3}$ and $x=-i\sqrt{3}$, each of multiplicity 1.

2. Following the synthetic division in part 1, we factor f as follows.

$$\begin{aligned} f(x) &= x^5-3x^4+3x^3-5x^2+12 \\ &= (x+1)(x^4-4x^3+7x^2-12x+12) \\ &= (x+1)(x-2)^2(x^2+3) \end{aligned}$$

Since x^2+3 cannot be factored over the real numbers, it is an **irreducible quadratic** factor. Our final answer is $f(x)=(x+1)(x-2)^2(x^2+3)$.

□

Our next example uses **Theorem 2.8** to find missing zeros, and then uses these zeros to factor the polynomial in accordance with **Theorem 2.9**.

Example 2.5.4. Let $f(x)=x^4-x^3-2x^2-4x-24$.

1. Given that $x=2i$ is a zero of f, find the remaining zeros of f.
2. Completely factor f to linear and irreducible quadratic factors.

Solution.

1. Since f is a polynomial with real number coefficients and $x=2i$ is a zero of f, $x=-2i$ is also a zero by **Theorem 2.8**. Thus, the following is an irreducible quadratic factor of f:

$$\begin{aligned}(x-(2i))(x-(-2i)) &= (x-2i)(x+2i)\\ &= x^2+2ix-2ix-4i^2\\ &= x^2+4\end{aligned}$$

To find the remaining zeros, we make use of the factor x^2+4. Noting that $f(x)=(x^2+4)\cdot g(x)$ for some polynomial g, we have $g(x)=f(x)\div(x^2+4)$. Using polynomial division, we find that $f(x)\div(x^2+4)=x^2-x-6$, from which

$$\begin{aligned}f(x) &= (x^2+4)\cdot g(x)\\ &= (x^2+4)(x^2-x-6)\end{aligned}$$

Setting $x^2-x-6=0$ gives us zeros of $x=-2$ and $x=3$, in addition to the zeros of $x=2i$ and $x=-2i$.

2. After part 1, there is not much factoring left to do. We already have $f(x)=(x^2+4)(x^2-x-6)$. Factoring x^2-x-6 gives us a final answer of $f(x)=(x^2+4)(x-3)(x+2)$.

□

Our last example turns the tables and ask us to manufacture a polynomial with certain properties.

Example 2.5.5. Find a polynomial p of lowest degree that has integer coefficients and satisfies all of the following criteria.

- The graph of $y=p(x)$ touches, but does not cross, the *x*-axis at $(2,0)$.
- $x=3i$ is a zero of p.
- As $x\to-\infty$, $p(x)\to-\infty$.
- As $x\to\infty$, $p(x)\to-\infty$.

Solution. Since the graph of p touches the *x*-axis at $(2,0)$, we know $x=2$ is a zero of even multiplicity. We are looking for a polynomial of lowest degree, so we need $x=2$ to have multiplicity of exactly 2. The Factor Theorem now tells us $(x-2)^2$ is a factor of $p(x)$.

Since $x=3i$ is a zero and our final answer is to have integer (real) coefficients, $x=-3i$ is also a zero. The Factor Theorem kicks in again to give us $(x-3i)$ and $(x+3i)$ as factors of $p(x)$.

We are given no further information about zeros or intercepts so we conclude, by the Complex Factorization Theorem, that $p(x)=a(x-2)^2(x-3i)(x+3i)$ for some real number *a*.

Our last concern is end behavior. We expand p as follows.

$$\begin{aligned} p(x) &= a(x-2)^2(x-3i)(x+3i) \\ &= a\left(x^2-4x+4\right)\left(x^2+9\right) \\ &= a\left(x^4+9x^2-4x^3-36x+4x^2+36\right) \\ &= a\left(x^4-4x^3+13x^2-36x+36\right) \\ &= ax^4-4ax^3+13ax^2-36ax+36a \end{aligned}$$

In order to obtain integer coefficients, *a* can be any integer. Our last concern is end behavior. Since the leading coefficient of $p(x)$ is ax^4, we need $a<0$ to get $p(x)\to-\infty$ as $x\to\pm\infty$. Hence, if we choose $a=-1$, we get $p(x)=-x^4+4x^3-13x^2+36x-36$.

□

The observant reader will note that we did not give any examples of applications that involve complex numbers. Students often wonder if complex numbers are used in 'real-world' applications. After all, didn't we call *i* the *imaginary* unit? It turns out that complex numbers are useful in many applied fields such as fluid dynamics, electromagnetism and quantum mechanics, but most of the applications require mathematics well beyond College Algebra for a full understanding.

We invite you to find a few examples of complex number applications. A simple Internet search with the phrase 'complex numbers in real life' should get you started. Basic electronics classes are another place to look, but keep in mind that they might use the letter *j* where we have used *i*. For the remainder of the text, we will restrict our attention to real numbers. We do this primarily because the first Calculus sequence you will take, ostensibly the one that this text is preparing you for, studies only functions of real variables. We believe it makes more sense pedagogically to concentrate on concepts such as rational, exponential and logarithmic functions.

2.5 Exercises

1. If $p(x)$ is a polynomial with real number coefficients and $-2i$ is a zero of $p(x)$, what do we know about the factors of $p(x)$?

2. If $p(x)$ is a polynomial with real number coefficients and no real zeros, can any conclusions be drawn about the end behavior of p?

In Exercises 3 – 12, use the given complex numbers *z* and *w* to find and simplify the following. Write your answers in the form $a+bi$.

(a) $z+w$ (b) $z \cdot w$ (c) z^2

(d) $\frac{z}{w}$ (e) the conjugate of *z* (f) $z \cdot (\text{the conjugate of } z)$

3. $z=2+3i$, $w=4i$
4. $z=1+i$, $w=-i$
5. $z=i$, $w=-1+2i$
6. $z=4i$, $w=2-2i$
7. $z=3-5i$, $w=2+7i$
8. $z=-5+i$, $w=4+2i$
9. $z=\sqrt{2}-i\sqrt{2}$, $w=\sqrt{2}+i\sqrt{2}$
10. $z=1-i\sqrt{3}$, $w=-1-i\sqrt{3}$
11. $z=\frac{1}{2}+\frac{\sqrt{3}}{2}i$, $w=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$
12. $z=-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$, $w=-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$

In Exercises 13 – 20, simplify the quantity.

13. $\sqrt{-49}$
14. $\sqrt{-9}$
15. $\sqrt{-25}\sqrt{-4}$
16. $\sqrt{(-25)(-4)}$
17. $\sqrt{-9}\sqrt{-16}$
18. $\sqrt{(-9)(-16)}$
19. $\sqrt{-(-9)}$
20. $-\sqrt{-9}$

We know that $i^2=-1$ which means $i^3=i^2\cdot i=(-1)i=-i$ and $i^4=i^2\cdot i^2=(-1)(-1)=1$. In Exercises 21 – 28, use this information to simplify the given power of *i*.

21. i^5
22. i^6
23. i^7
24. i^8
25. i^{15}
26. i^{26}
27. i^{117}
28. i^{304}

In Exercises 29 – 34, find all of the zeros of the polynomial and then completely factor it to linear and irreducible quadratic factors.

29. $f(x)=x^3+3x^2+4x+12$

30. $f(x)=x^3-2x^2+9x-18$

31. $f(x)=3x^3-13x^2+43x-13$

32. $f(x)=x^3+6x^2+6x+5$

33. $f(x)=4x^4-4x^3+13x^2-12x+3$

34. $f(x)=2x^4-7x^3+14x^2-15x+6$

In Exercises 35 – 50, find all of the zeros of the polynomial

35. $f(x)=x^2-4x+13$

36. $f(x)=3x^2+2x+10$

37. $f(x)=x^2-2x+5$

38. $f(x)=9x^3+2x+1$

39. $f(x)=x^3+7x^2+9x-2$

40. $f(x)=4x^3-6x^2-8x+15$

41. $f(x)=x^4+x^3+7x^2+9x-18$

42. $f(x)=6x^4+17x^3-55x^2+16x+12$

43. $f(x)=-3x^4-8x^3-12x^2-12x-5$

44. $f(x)=8x^4+50x^3+43x^2+2x-4$

45. $f(x)=x^4+9x^2+20$

46. $f(x)=x^4+5x^2-24$

47. $f(x)=x^5-x^4+7x^3-7x^2+12x-12$

48. $f(x)=x^6-64$

49. $f(x)=x^4-2x^3+27x^2-2x+26$ (Hint: $x=i$ is one of the zeros.)

50. $f(x)=2x^4+5x^3+13x^2+7x+5$ (Hint: $x=-1+2i$ is a zero.)

In Exercises 51 – 55, create a polynomial f with real number coefficients which has all of the desired characteristics. You may leave the polynomial in factored form.

51.
- The zeros of f are $c=\pm 1$ and $c=\pm i$.
- The leading term of $f(x)$ is $42x^4$.

52.
- $c=2i$ is a zero.
- The point $(-1,0)$ is a local minimum on the graph of $y=f(x)$.
- The leading term of $f(x)$ is $117x^4$.

53. • The solutions to $f(x)=0$ are $x=\pm 2$ and $x=\pm 7i$.

• The leading term of $f(x)$ is $-3x^5$.

• The point $(2,0)$ is a local maximum on the graph of $y=f(x)$.

54. • f is degree 5.

• $x=6$, $x=i$ and $x=1-3i$ are zeros of f.

• As $x\to-\infty$, $f(x)\to\infty$.

55. • The leading term of $f(x)$ is $-2x^3$.

• $c=2i$ is a zero.

• $f(0)=-16$.

2.6 Polynomial Inequalities

Learning Objectives

- Solve polynomial inequalities graphically
- Solve polynomial inequalities analytically

In this section, we develop techniques for solving polynomial inequalities. We determine solutions analytically, and look at them graphically. This first example motivates the core ideas.

Example 2.6.1. Let $f(x)=2x-1$ and $g(x)=5$.

1. Solve $f(x)=g(x)$.
2. Solve $f(x)<g(x)$.
3. Solve $f(x)>g(x)$.
4. Graph $y=f(x)$ and $y=g(x)$ on the same set of axes and interpret your solutions to parts 1 through 3 above.

Solution.

1. To solve $f(x)=g(x)$, we replace $f(x)$ with $2x-1$ and $g(x)$ with 5 to get $2x-1=5$. Solving for *x*, we get $x=3$.

2. The inequality $f(x)<g(x)$ is equivalent to $2x-1<5$. Solving gives $x<3$, or $(-\infty,3)$.

3. To find where $f(x)>g(x)$, we solve $2x-1>5$. We get $x>3$, or $(3,\infty)$.

4. To graph $y=f(x)$, we plot $y=2x-1$, which is a line with a *y*-intercept of $(0,-1)$ and a slope of 2. The graph of $y=g(x)$ is $y=5$, which is a horizontal line through $(0,5)$.

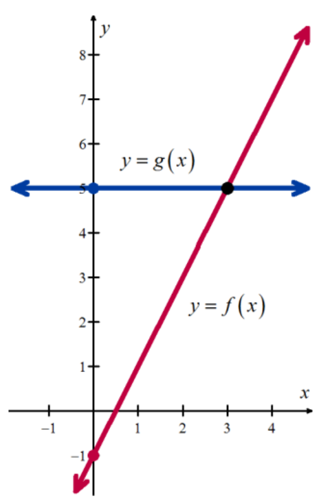
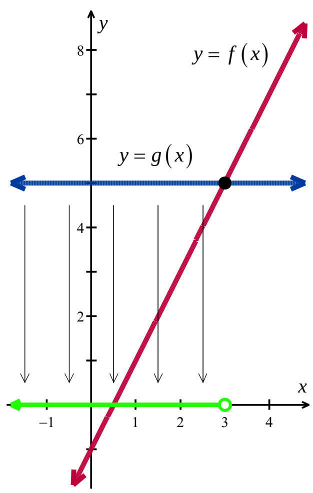


Figure 2.6. 1

A generic point on the graph of $y=f(x)$ is $(x, f(x))$ and a generic point on the graph of $y=g(x)$ is $(x, g(x))$. When we seek solutions to $f(x)=g(x)$, we are looking for *x* values whose *y* values on the graphs of f and g are the same. In part 1, we found $x=3$ is the solution to $f(x)=g(x)$. Sure enough, $f(3)=5$ and $g(3)=5$ so that the point $(3,5)$ is on both graphs. In other words, the graphs of f and g intersect at $(3,5)$.

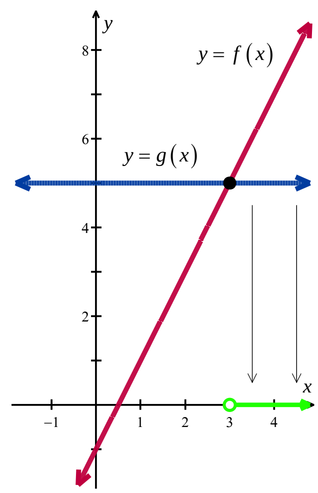
In part 2, we set $f(x)<g(x)$ and solved to find $x<3$. For $x<3$, the point $(x, f(x))$ is below $(x, g(x))$ since the *y* values on the graph of f are less than the *y* values on the graph of g. In part 3, analogously, we solved $f(x)>g(x)$ and found $x>3$. For $x>3$, note that the graph of f is above the graph of g since the *y* values on the graph of f are greater than the *y* values on the graph of g.

Figure 2.6. 2



$f(x)<g(x)$ on $(-\infty,3)$

Figure 2.6. 3



$f(x)>g(x)$ on $(3,\infty)$

□

Solving Polynomial Inequalities Graphically

The preceding example demonstrates the following.

Graphical Interpretation of Equations and Inequalities

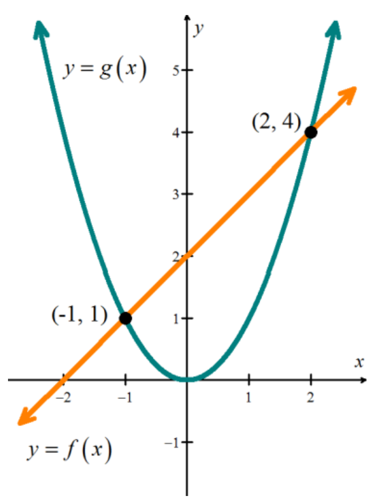
Suppose f and g are functions.

1. The solutions to $f(x)=g(x)$ are the x values where the graphs of $y=f(x)$ and $y=g(x)$ intersect.
2. The solution to $f(x)<g(x)$ is the set of x values where the graph of $y=f(x)$ is below the graph of $y=g(x)$.
3. The solution to $f(x)>g(x)$ is the set of x values where the graph of $y=f(x)$ is above the graph of $y=g(x)$.

The next example turns the tables and furnishes the graphs of two functions from which we are asked to determine solutions to a corresponding equation and inequalities.

Example 2.6.2. The graphs of $f(x)=x+2$ and $g(x)=x^2$ are displayed below. Use these graphs to answer the following and explain what is being represented algebraically by the equation or inequality.

Figure 2.6. 4



1. Solve $f(x)=g(x)$. 2. Solve $f(x)>g(x)$. 3. Solve $f(x)\le g(x)$.

Solution.

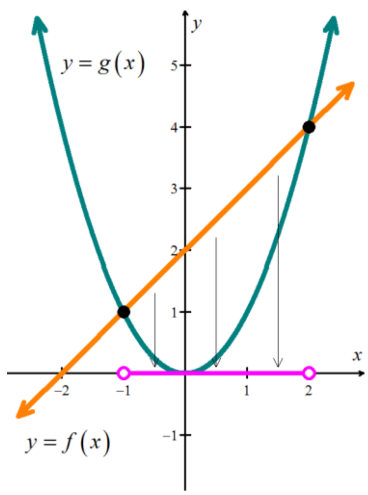
1. To solve $f(x)=g(x)$, we look for where the graphs of f and g intersect. This appears to be at the points $(-1,1)$ and $(2,4)$, in which case our solutions to $f(x)=g(x)$ are $x=-1$ and $x=2$.

The equation represented by $f(x) = g(x)$ is $x + 2 = x^2$ which, following this example, will used to verify analytically that our graphical solution is correct.

2. To solve $f(x) > g(x)$, we look for where the graph of f is above the graph of g. This appears to happen for the *x* values between −1 and 2, for a solution of $(-1, 2)$.

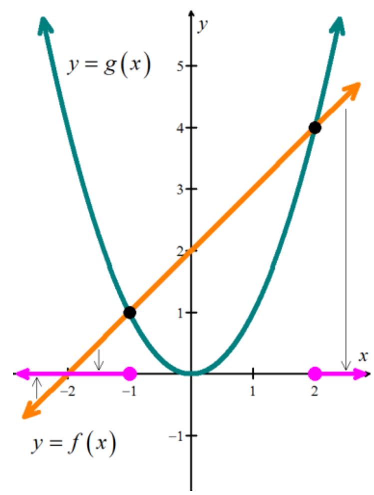
   The inequality being represented by $f(x) > g(x)$ is $x + 2 > x^2$. We will verify shortly that the solution to this inequality is indeed $(-1, 2)$.

Figure 2.6. 5



$f(x) > g(x)$

Figure 2.6. 6



$f(x) \le g(x)$

3. To solve $f(x) \le g(x)$, we look for solutions to $f(x) = g(x)$ as well as $f(x) < g(x)$. In part 1, we found the solution to the former equation to be $x = -1$ and $x = 2$. To solve $f(x) < g(x)$, we look for where the graph of f is below the graph of g. This appears to happen for *x* values less than −1 and greater than 2. Hence, our solution to $f(x) \le g(x)$ is $(-\infty, -1] \cup [2, \infty)$.

   This will be verified analytically, beginning with the observation that the inequality $f(x) \le g(x)$ represents $x + 2 \le x^2$.

□

Solving Polynomial Inequalities Analytically

We now look towards formulating a general analytic procedure for solving all polynomial inequalities.

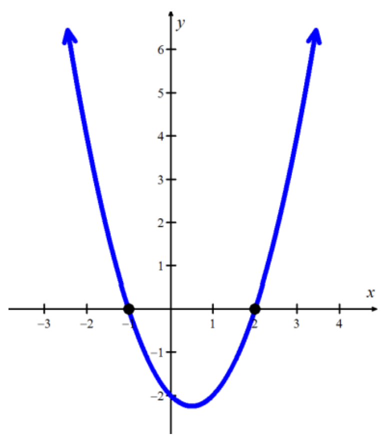
In **Example 2.6.2**, for $f(x)=x+2$ and $g(x)=x^2$, we found a graphical solution to $f(x)=g(x)$. We now proceed to solve this equation algebraically.

$$\begin{aligned} x+2&=x^2 \\ 0&=x^2-x-2 \\ 0&=(x-2)(x+1) \end{aligned}$$

This algebraic result confirms our graphical solutions of $x=-1$ and $x=2$.

To analytically determine the solution in **Example 2.6.2** to $f(x)>g(x)$, or $x+2>x^2$, we note that $x+2>x^2$ is equivalent to $x^2-x-2<0$. Solving $x^2-x-2<0$ corresponds graphically to finding the values of *x* for which the graph of $y=x^2-x-2$ (the parabola in the following illustration) is below the graph of $y=0$ (the *x*-axis).

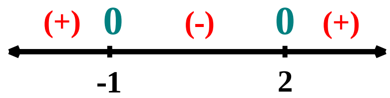
Figure 2.6. 7



$y=x^2-x-2$

We can see that the graph of $y=x^2-x-2$ does dip below the *x*-axis between its two *x*-intercepts, at $x=-1$ and $x=2$. In this case the *x*-intercepts divide the domain (the *x*-axis) into three intervals: $(-\infty,-1)$, $(-1,2)$ and $(2,\infty)$. For every value of *x* in $(-\infty,-1)$, the graph of $y=x^2-x-2$ is above the *x*-axis; in other words, $x^2-x-2>0$ for all *x* in $(-\infty,-1)$. Similarly, $x^2-x-2<0$ for all *x* in $(-1,2)$, and $x^2-x-2>0$ for all *x* in $(2,\infty)$. We can schematically represent this with the following **sign diagram**.

Figure 2.6. 8



Here, the (+) above a portion of the number line indicates $x^2 - x - 2 > 0$ for those values of *x*; the (−) indicates $x^2 - x - 2 < 0$ for the corresponding values of *x*. The numbers labeled on the number line are the zeros of $y = x^2 - x - 2$, so we place 0 above them. We see at once that the solution to $x^2 - x - 2 < 0$ is $(-1, 2)$.

Similarly, in solving $f(x) \le g(x)$, which represents $x + 2 \le x^2$, and is equivalent to $x^2 - x - 2 \ge 0$, we see from the number line that the solution, which includes the zeros, is $(-\infty, -1] \cup [2, \infty)$.

Our next goal is to establish a procedure by which we can generate the sign diagram without graphing the function.

Steps for Solving a Polynomial Inequality

1. Rewrite the inequality, if necessary, as a polynomial function $f(x)$ on one side of the inequality and 0 on the other.
2. Find the zeros of f and place them on the number line with the number 0 above them.
3. Choose a real number, called a **test value**, in each of the intervals determined in step 2.
4. Determine the sign of $f(x)$ for each test value in step 3, and write that sign above the corresponding interval.
5. Choose the interval(s) that correspond to the correct sign to solve the inequality.

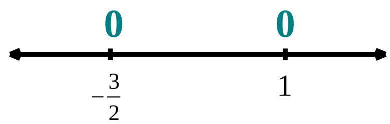
Example 2.6.3. Solve the following inequalities analytically using sign diagrams. Verify your answer graphically.

1. $2x^2 \le 3 - x$
2. $x^2 - 2x > 1$
3. $x^2 + 1 \le 2x$
4. $2x^3 - 19x^2 + 49x - 20 < 0$

Solution.

1. To solve $2x^2 \le 3 - x$, we first get 0 on one side of the inequality, which yields $2x^2 + x - 3 \le 0$. We find the zeros of $f(x) = 2x^2 + x - 3$ by solving $2x^2 + x - 3 = 0$ for *x*. Factoring gives $(2x+3)(x-1) = 0$, for zeros of $x = -\frac{3}{2}$ and $x = 1$.

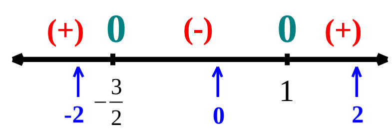
Figure 2.6. 9



Next, we choose a test value in each of the resulting intervals: $\left(-\infty, -\frac{3}{2}\right)$, $\left(-\frac{3}{2}, 1\right)$ and $(1, \infty)$. For the interval $\left(-\infty, -\frac{3}{2}\right)$, we choose[24] $x = -2$; for $\left(-\frac{3}{2}, 1\right)$ we pick $x = 0$; and for $(1, \infty)$, $x = 2$ is our test value. Evaluating the function at the three test values gives the following.

$$f(-2) = 3 \quad > 0$$
$$f(0) = -3 \quad < 0$$
$$f(2) = 7 \quad > 0$$

Figure 2.6. 10



Since we are solving $2x^2 + x - 3 \le 0$, we look for solutions to $2x^2 + x - 3 < 0$ as well as $2x^2 + x - 3 = 0$. For $2x^2 + x - 3 < 0$, we need the intervals which have a $(-)$. Checking the sign diagram, we see this is $\left(-\frac{3}{2}, 1\right)$. We know $2x^2 + x - 3 = 0$ when $x = -\frac{3}{2}$ and $x = 1$, so our final answer is $\left[-\frac{3}{2}, 1\right]$.

We next view our solution graphically, while noting that a graphical solution is only an estimate. In graphing, we refer to the original inequality, $2x^2 \le 3 - x$, letting $g(x) = 2x^2$ and $h(x) = 3 - x$. We are looking for the *x* values where the graph of g is below that of h for the solutions to $g(x) < h(x)$. Additionally, we include the points of intersection to give us the solutions to $g(x) = h(x)$.

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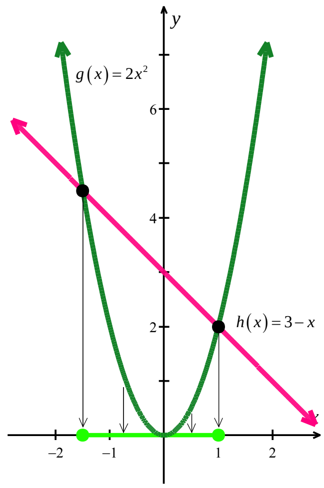
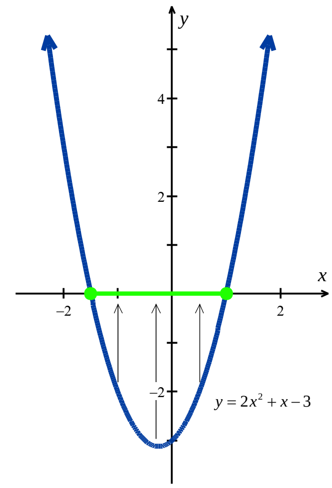
[24] We have to choose something in each interval, but any number in the interval will result in the same sign chart.

Figure 2.6. 11



$g(x) \le h(x)$ for x values in $\left[-\frac{3}{2}, 1\right]$

Figure 2.6. 12



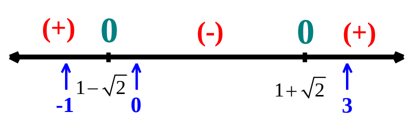
$2x^2 + x - 3 \le 0$ for x values in $\left[-\frac{3}{2}, 1\right]$

The graph on the right shows us that $y = 2x^2 + x - 3$ lies below or intersects with $y = 0$ on the same interval that the graph of $y = g(x)$ lies below or intersects with that of $y = h(x)$.

2. Once again, we re-write $x^2 - 2x > 1$ as $x^2 - 2x - 1 > 0$ and we identify $f(x) = x^2 - 2x - 1$. When we go to find zeros of f, we find that the quadratic $x^2 - 2x - 1$ doesn't factor nicely. Hence, we resort to the quadratic formula to solve $x^2 - 2x - 1 = 0$ and arrive at $x = 1 \pm \sqrt{2}$. As before, these zeros divide the number line into three pieces. To help us decide on test values, we approximate $1 - \sqrt{2} \approx -0.4$ and $1 + \sqrt{2} \approx 2.4$. We choose $x = -1$, $x = 0$ and $x = 3$ as our test values.

$$\begin{aligned} f(-1) &= 2 \quad > 0 \\ f(0) &= -1 \quad < 0 \\ f(3) &= 2 \quad > 0 \end{aligned}$$

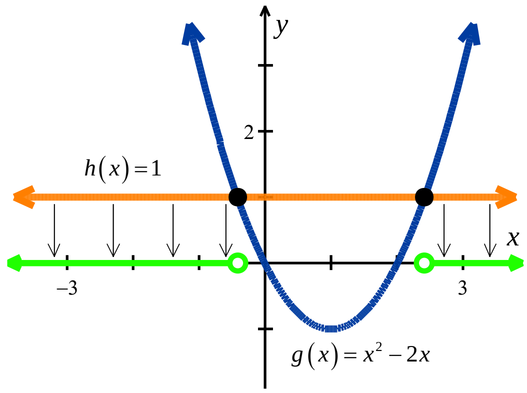
Figure 2.6. 13



Our solution to $x^2 - 2x - 1 > 0$ is where we have $(+)$, which is $\left(-\infty, 1 - \sqrt{2}\right) \cup \left(1 + \sqrt{2}, \infty\right)$.

To check the inequality $x^2 - 2x > 1$ graphically, we set $g(x) = x^2 - 2x$ and $h(x) = 1$. We are looking for the x values where the graph of g is above the graph of h.

Figure 2.6. 14

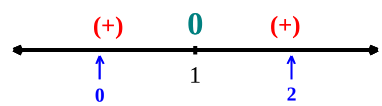


$g(x)>h(x)$ for x values in $\left(-\infty,1-\sqrt{2}\right)\cup\left(1+\sqrt{2},\infty\right)$

3. To solve $x^2+1\le 2x$, as before, we solve $x^2-2x+1\le 0$. Setting $f(x)=x^2-2x+1=0$, we find the only zero of f to be $x=1$. This one zero divides the number line into two intervals, from which we choose $x=0$ and $x=2$ as test values.

Figure 2.6. 15

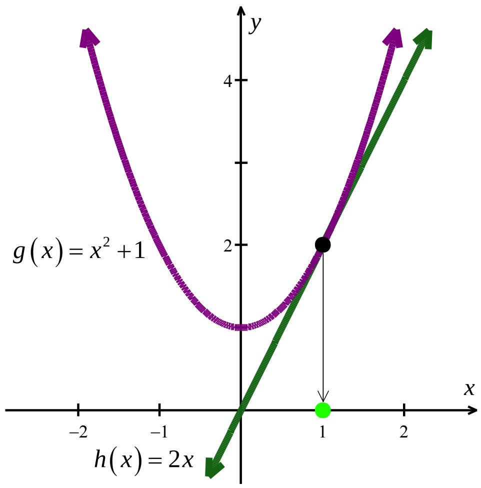
$$f(0)=1 \quad >0$$
$$f(2)=1 \quad >0$$



Since we are looking for solutions to $x^2-2x+1\le 0$, we are looking for x values where $x^2-2x+1<0$ as well as where $x^2-2x+1=0$. Looking at our sign diagram, there are no places where $x^2-2x+1<0$ since there is no $(-)$. Our only solution is $x=1$, where $x^2-2x+1=0$. We write this solution as $\{1\}$.

Graphically, we solve $x^2+1\le 2x$ by graphing $g(x)=x^2+1$ and $h(x)=2x$. We are looking for the x values where the graph of g is below the graph of h for solutions to $x^2+1<2x$ and points where the two graphs intersect for solutions to $x^2+1=2x$.

Figure 2.6. 16



$g(x) \leq h(x)$ when $x = 1$

Notice that the line and the parabola touch at $(1,2)$, but the parabola is always above the line otherwise.[25]

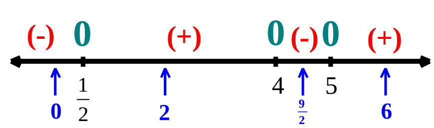
4. To solve our last inequality, $2x^3 - 19x^2 + 49x - 20 < 0$, we set $f(x) = 2x^3 - 19x^2 + 49x - 20 = 0$ and use methods from **Section 2.4** to find the zeros. We find that $x = 4$ is a zero through synthetic division.

$$\begin{array}{r|rrrr} 4 & 2 & -19 & 49 & -20 \\ & \downarrow & 8 & -44 & 20 \\ \hline & 2 & -11 & 5 & 0 \end{array}$$

Thus, $f(x) = (x-4)(2x^2 - 11x + 5) = (x-4)(2x-1)(x-5)$ for additional zeros of $x = \dfrac{1}{2}$ and $x = 5$. In the resulting intervals of $\left(-\infty, \dfrac{1}{2}\right)$, $\left(\dfrac{1}{2}, 4\right)$, $(4,5)$ and $(5,\infty)$, we choose test values of $x = 0$, $x = 2$, $x = \dfrac{9}{2}$ and $x = 6$, respectively.

$$\begin{aligned} f(0) &= -20 & < 0 \\ f(2) &= 18 & > 0 \\ f\left(\frac{9}{2}\right) &= -2 & < 0 \\ f(6) &= 22 & > 0 \end{aligned}$$

Figure 2.6. 17

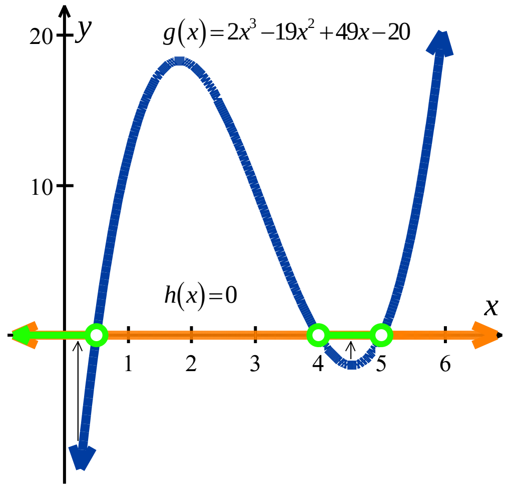


[25] In this case, we say the line $y = 2x$ is **tangent** to $y = x^2 + 1$ at $(1,2)$. Finding tangent lines to arbitrary functions is a fundamental problem solved, in general, with Calculus.

Our solution to $2x^3 - 19x^2 + 49x - 20 < 0$ is where we have $(-)$, which is $\left(-\infty, \frac{1}{2}\right) \cup (4,5)$.

Graphically, we solve $2x^3 - 19x^2 + 49x - 20 < 0$ by graphing $g(x) = 2x^3 - 19x^2 + 49x - 20$ and $h(x) = 0$. We are looking for x values where the graph of g is below the graph of h.

Figure 2.6. 18



$g(x) < h(x)$ for x values in $\left(-\infty, \frac{1}{2}\right) \cup (4,5)$

□

2.6 Exercises

1. In creating a sign chart, how many test values must be used to determine the solution of an inequality whose highest power term is of degree 3?
2. If a polynomial inequality has no solutions, what can be said about the graphs of the two sides of the inequality?

In Exercises 3 – 22, solve the inequality. Write your answer using interval notation.

3. $x^2+2x-3\geq 0$
4. $16x^2+8x+1>0$
5. $x^2+9<6x$
6. $9x^2+16\geq 24x$
7. $x^2+4\leq 4x$
8. $x^2+1<0$
9. $3x^2\leq 11x+4$
10. $x>x^2$
11. $2x^2-4x-1>0$
12. $5x+4\leq 3x^2$
13. $x^2+x+1\geq 0$
14. $x^4-9x^2\leq 4x-12$
15. $(x-1)^2\geq 4$
16. $4x^3\geq 3x+1$
17. $x^4\leq 16+4x-x^3$
18. $3x^2+2x<x^4$
19. $\dfrac{x^3+2x^2}{2}<x+2$
20. $\dfrac{x^3+20x}{8}\geq x^2+2$
21. $2x^4>5x^2+3$
22. $x^6+x^3\geq 6$

23. The profit, in dollars, made by selling x bottles of 100% All-Natural Certified Free-Trade Organic Sasquatch Tonic is given by $P(x)=-x^2+25x-100$, for $0\leq x\leq 35$. How many bottles of tonic must be sold to make at least \$50 in profit?

24. Suppose $C(x)=x^2-10x+27$, $x\geq 0$, represents the cost in hundreds of dollars to produce x thousand pens. Find the number of pens that can be produced for no more than \$1100.

25. The temperature T, in degrees Fahrenheit, t hours after 6 AM, is given by $T(t)=-\frac{1}{2}t^2+8t+32$ for $0\leq t\leq 12$. When is it warmer than 42° Fahrenheit?

26. The height h in feet of a model rocket above the ground t seconds after lift-off is given by $h(t)=-5t^2+100t$ for $0\leq t\leq 20$. When is the rocket at least 250 feet off the ground? Round your answer to two decimal places.

Key Equations

General Form of a Quadratic Function:
$f(x) = ax^2 + bx + c$

Standard Form of a Quadratic Function:
$f(x) = a(x-h)^2 + k$

Quadratic Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Vertex of a Quadratic Function:
$\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$

Polynomial Function:

$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$,
where $a_n \neq 0$ and n is a nonnegative integer

Imaginary Unit: $i = \sqrt{-1}$

Key Terms

Axis of Symmetry: The vertical line through the vertex of a parabola

Complex Number: A number of the form $a + bi$ where a and b are real numbers and i is the imaginary unit.

Conjugate Pairs Theorem: If *f* is a polynomial with real number coefficients and $a + bi$ is a zero of *f*, then it's conjugate, $a - bi$, is also a zero of *f*.

Constant Term: Term with no variable

Constraint: Equation that places limits on the variables used in the objective function

Continuous: A function is continuous if the graph is a single, unbroken curve

Degree of a Polynomial: The highest power of the variable in the polynomial

Discriminant: $b^2 - 4ac$ in a quadratic function

Dividend: A number or polynomial being divided

Divisor: A number or polynomial by which another is to be divided

End Behavior: Describes what is happening to function values as $x \to \infty$ or $x \to -\infty$

Factor Theorem: The real number *c* is a factor of a polynomial *p* if and only if $(x-c)$ is a factor of $p(x)$.

Factor: When the remainder is zero, the divisor is a factor of the dividend

Fundamental Theorem of Algebra: If *f* is a polynomial with degree $n \geq 1$, then f has at least one complex zero.

Intermediate Value Theorem: If f is a continuous function on an interval containing $x = a$ and $x = b$, and $f(a)$ and $f(b)$ have different signs, then f has at least one zero between $x = a$ and $x = b$

Leading Coefficient: Coefficient of the leading term

Leading Term: Term containing the highest power of the variable

Multiplicity of a Zero: How many times the factor occurs in the polynomial

Objective Function: Function to be maximized or minimized

Parabola: Graph of a quadratic function

Quotient: The answer to a division problem.

Rational Zeros Theorem: If *r* is a zero of a polynomial function *f*, then *r* is of the form $\dfrac{p}{q}$, where *p* is a factor of the constant term and *q* is a factor of the leading coefficient.

Real Factorization Theorem: If *f* is a polynomial function with real coefficients, then *f* can be factored into a product of linear and irreducible quadratic factors.

Remainder Theorem: If p is a polynomial and c is a real number, when $p(x)$ is divided by c, the remainder is $p(c)$.

Smooth: A function that is continuous and without corners or cusps over its domain

Synthetic Division: A method of dividing polynomials by a linear expression

Vertex: Minimum or maximum value of a parabola

CHAPTER 3
RATIONAL FUNCTIONS

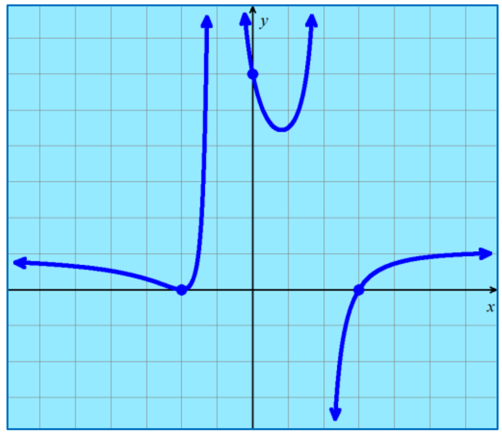


Figure 3.0. 1

Chapter Outline

Introduction

In this chapter, you will explore rational functions numerically, analytically, and graphically. A primary focus throughout the chapter will be the domain of a rational function, what is happening at values near any exclusion(s), and/or what is happening as the inputs go to positive or negative infinity. The goal is for you to make sense of rational functions numerically, analytically and graphically. You will begin by evaluating the function near exclusions and for very large positive or negative values of x to get a sense of what the function is doing. On this foundation, you will develop an understanding of vertical asymptotes, holes, and horizontal (end behavior) asymptotes. Using domain, x- and y-intercepts, vertical asymptotes, holes, end behavior (or horizontal asymptote), you will sketch graphs of rational functions. In the last section, you will use both graphical and analytical methods to solve rational equalities and inequalities.

Section 3.1 is devoted to introducing you to rational functions and helping you understand their behavior near restrictions in the domain and at positive and negative infinity. The section starts with a definition of a rational function. Next there is a review of how to determine the domain of a rational function (here you're revisiting ideas first introduced in 1.1). Using numeric strategies, we then explore how some rational functions behave near their exclusions; this leads to the idea of a vertical asymptote. A similar numeric process for large positive and negative input values is employed to develop the idea of end behavior/horizontal asymptotes. You also learn how to find *x*- and *y*- intercepts. You will continue to develop skills throughout the section with progressively more ‘complicated’ functions. The end goal is for you to be able to describe behavior of rational functions as a step towards sketching them in the next couple of sections. You will also rely on these understandings for analytic purposes such as solving equations or inequalities later on.

Section 3.2 builds from 3.1 by asking you to use your understanding of domain, vertical and horizontal asymptotes, and *x*- and *y*- intercepts to graph rational functions. The section provides several examples of functions with more than one vertical asymptote and a variety of horizontal asymptotes. Throughout the section, you are required to analyze and synthesize your understanding in order to create accurate facsimiles of rational function graphs (the graph of the functions in this section do NOT include oblique asymptotes or holes.)

Section 3.3 introduces both oblique asymptotes and holes to further your understanding of behavior and graphing of rational functions. You will see, for the first time, rational functions where a factor in the denominator reduces with one in the numerator (commonly referred to as ‘canceling’.) You will learn that while you must determine the domain of a rational function before you ‘cancel,’ you find intercepts, asymptotes, and proceed with graphing the function after cancelling. Using the same numeric methods you used to explore domain restriction in 3.1, you explore values close to the domain restriction that ‘canceled’ out of expressions here and learn that exclusions in the domain that ‘cancel out’ create ‘holes’ in the graph of the function (not asymptotes). This section also has rational functions where the degree of the numerator is larger than that of the denominator. Again, you will use the strategy of evaluating functions for very large positive and negative values of *x* to determine end behavior (oblique asymptotes.)

Section 3.4 is devoted to solving rational equations and inequalities. The section starts by making visual sense of a rational equation or inequality by graphing the left and right side of the equation or inequality, to help you ‘see’ the solution or solution region. You then build on that understanding to learn algebraic methods for solving.

3.1 Introduction to Rational Functions

Learning Objectives

- Identify a rational function.
- Determine the domain of a rational function.
- Find the *x*- and *y*-intercepts for a rational function.
- Identify vertical and horizontal asymptotes.
- Graph irreducible rational functions with constant or first degree numerators and denominators of degree one.

In this chapter, we study **rational functions** – functions that are ratios of polynomial functions.

Functions that are Rational

Definition 3.1. A **rational function** is a function that is the ratio of polynomial functions. Said differently, r is a rational function if it is of the form $r(x)=\frac{p(x)}{q(x)}$, where p and q are polynomial functions.[1]

Example 3.1.1. Determine if the following functions are rational functions. Explain your reasoning.

1. $f(x)=\frac{2x-1}{x+1}$
2. $g(x)=\frac{2\sqrt{x}-4x+3}{3x^2-5x+2}$

Solution.

1. $f(x)=\frac{2x-1}{x+1}$ is of the form $f(x)=\frac{p(x)}{q(x)}$ where $p(x)=2x-1$ and $q(x)=x+1$ are polynomial functions. So, f is a ratio of two polynomial functions and hence a rational function.

2. $g(x)=\frac{2\sqrt{x}-4x+3}{3x^2-5x+2}$ is of the form $g(x)=\frac{p(x)}{q(x)}$, but $p(x)=2\sqrt{x}-4x+3$ is not a polynomial since *x* is raised to a non-integer power in the term $2\sqrt{x}$. Thus, g is not a rational function.

□

[1] According to this definition, all polynomial functions are also rational functions. (Take $q(x)=1$.)

Domains of Rational Functions

We recall from **Section 1.1** that a fraction is defined only if its denominator is not zero. To find the domain of the function $f(x)=\frac{2x-1}{x+1}$, from the previous example, we proceed as we did in **Section 1.1**: start with the set of real numbers, then find the zeros of the denominator and exclude them from the set of real numbers. Setting $x+1=0$ results in $x=-1$. Hence, our domain is $(-\infty,-1)\cup(-1,\infty)$, also stated as $\{x \mid x\neq -1\}$. In the following example, we determine the domain of three rational functions that appear similar on first glance, but each have unique characteristics.

Example 3.1.2. Find the domain of the following rational functions.

1. $F(x)=\frac{x+1}{x^2-9}$
2. $G(x)=\frac{x+1}{x^2+9}$
3. $H(x)=\frac{x+3}{x^2-9}$

Solution.

1. To find the domain of $F(x)=\frac{x+1}{x^2-9}$, we begin by setting the denominator equal to zero.

$$x^2-9=0$$
$$(x-3)(x+3)=0$$

   We find that $x=\pm 3$ results in a denominator of zero, so our domain is $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$, or $\{x \mid x\neq \pm 3\}$.

2. Proceeding as before, we determine the domain of $G(x)=\frac{x+1}{x^2+9}$ by first setting the denominator equal to zero. Since there are no real solutions to $x^2+9=0$, there are no values of x that must be excluded from the denominator. Thus, the domain is all real numbers, $(-\infty,\infty)$.

3. The domain of $H(x)=\frac{x+3}{x^2-9}$ is identical to the domain of $F(x)=\frac{x+1}{x^2-9}$ from part 1 of this example. It is $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$. Since the denominators are the same, the values which must be excluded from the domain are the same.

□

In part 3 of the preceeding example, it may be tempting to simplify H as $H(x)=\frac{x+3}{(x+3)(x-3)}=\frac{1}{x-3}$. Without noting that this can only be done if $x+3\neq 0$, such a simplification will result in an incorrect domain. We will look more closely at rational functions that contain common factors in the numerator and denominator when we get to **Section 3.3**.

Finding *x*- and *y*-Intercepts

For any function, an *x*-intercept occurs at a value of *x* for which the output function value is zero. For rational functions, the output function value is zero only when the numerator is zero, with the added requirement that an *x*-intercept must be in the domain of the rational function. The *y*-intercept for a rational function occurs when $x = 0$, if that function is defined at zero. If the function is not defined at zero, there is no *y*-intercept.

Example 3.1.3. Find the *x*- and *y*-intercepts for the following three functions.

1. $F(x) = \dfrac{x+1}{x^2-9}$
2. $K(x) = \dfrac{x+3}{x^2+9}$
3. $J(x) = \dfrac{-2}{x^2-9}$

Solution.

1. To find *x*-intercepts for $y = F(x) = \dfrac{x+1}{x^2-9}$, we set $y = 0$ and recall that a rational function equals zero when the numerator is zero.

$$\begin{aligned} \frac{x+1}{x^2-9} &= 0 \\ x+1 &= 0 \\ x &= -1 \end{aligned}$$

In **Example 3.1.2**, we found that the domain of F is $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$, which includes $x = -1$. Thus, the *x*-intercept is the point $(-1,0)$. To find the *y*-intercept, we set $x = 0$ and solve for *y*:

$$y = \frac{0+1}{0^2-9} = -\frac{1}{9}$$

The *y*-intercept is the point $\left(0,-\dfrac{1}{9}\right)$.

2. For $y = K(x) = \dfrac{x+3}{x^2+9}$, *x*-intercepts are found by setting $y = 0$:

$$\begin{aligned} 0 &= \frac{x+3}{x^2+9} \\ x+3 &= 0 \\ x &= -3 \end{aligned}$$

The domain of K, which is all real numbers, includes $x=-3$. Thus, we have an *x*-intercept at the point $(-3,0)$. We find the *y*-intercept by setting $x=0$:

$$y=\frac{0+3}{0^2+9}=\frac{3}{9}=\frac{1}{3}$$

The *y*-intercept is the point $\left(0,\ \frac{1}{3}\right)$.

3. Since the numerator of $J(x)=\frac{-2}{x^2-9}$ is −2, which is never equal to zero, J does not have any *x*-intercepts. Setting $x=0$ results in $y=J(0)=\frac{-2}{0^2-9}=\frac{2}{9}$, so the *y*-intercept of J is the point $\left(0,\ \frac{2}{9}\right)$.

□

Identifying Vertical Asymptotes

Consider the function $f(x)=\frac{2x-1}{x+1}$ from **Example 3.1.1**. We found that $x=-1$ is not in the domain of f, which means $f(-1)$ is undefined. To find out more about the **local behavior** of f near $x=-1$, we can make two tables that show corresponding function values when *x* is close to −1.

We first choose values a little less than −1; for example, $x=-1.1$, $x=-1.01$, $x=-1.001$, and so on. These values are to the left of −1 on the real number line, so we say they 'approach −1 from the left.'

x	−1.1	−1.01	−1.001	−1.0001
$f(x)=\frac{2x-1}{x+1}$	32	302	3002	30002

As the *x* values approach −1 from the left, the function values become larger and larger positive numbers. We express this symbolically by stating

$$\text{as } x\to -1^-,\ f(x)\to\infty$$

Similarly, the values $x=-0.9$, $x=-0.99$, $x=-0.999$, etc., are to the right of −1 on the real number line, so we say these values 'approach −1 from the right.'

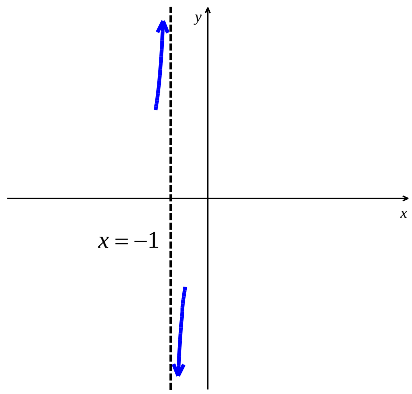
x	-0.9	-0.99	-0.999	-0.9999
$f(x)=\dfrac{2x-1}{x+1}$	-28	-298	-2998	-29998

As the x values approach -1 from the right, the function values become very large negative numbers, expressed symbolically by

$$\text{as } x \to -1^{+},\ f(x) \to -\infty$$

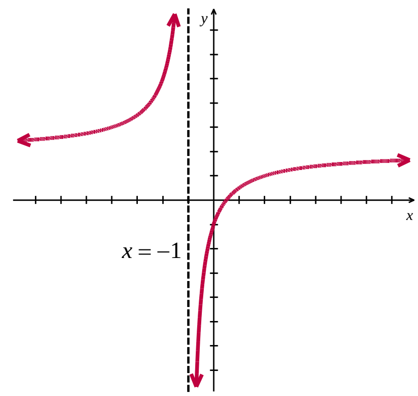
These numerical results are confirmed by the following graph of $y = f(x)$. For this type of unbounded behavior, we say the graph has a **vertical asymptote** of $x = -1$, which is identified by a dashed line.

Figure 3.1. 1



Behavior near $x = -1$:
as $x \to -1^{-}$, $f(x) \to \infty$; as $x \to -1^{+}$, $f(x) \to -\infty$

Figure 3.1. 2



$$y = f(x) = \frac{2x-1}{x+1}$$

Definition 3.2. The line $x = c$ is called a **vertical asymptote** of the graph of a function $y = f(x)$ if, as $x \to c^{-}$ or as $x \to c^{+}$, either $f(x) \to \infty$ or $f(x) \to -\infty$.

The following steps may be helpful in identifying vertical asymptotes.

Identifying Vertical Asymptotes

1. Reduce the fraction, if possible. To reduce the fraction, factor the numerator and denominator and cancel factors appearing in both numerator and denominator.[2]
2. Set the denominator equal to zero and solve the resulting equation.
3. If c is a zero of the denominator, then $x = c$ is a vertical asymptote.[3]

[2] Keep in mind that the domain should be carefully noted before canceling any common factors.
[3] There may be one, more than one, or no vertical asymptotes.

For practice in finding vertical asymptotes, we return to the functions from **Example 3.1.2**.

Example 3.1.4. Identify the vertical asymptotes of the graphs of the following rational functions.

1. $F(x)=\dfrac{x+1}{x^2-9}$ 2. $G(x)=\dfrac{x+1}{x^2+9}$ 3. $H(x)=\dfrac{x+3}{x^2-9}$

Solution.

1. To reduce the fraction, we first factor the numerator and denominator of $F(x)=\dfrac{x+1}{x^2-9}$ to get $F(x)=\dfrac{x+1}{(x+3)(x-3)}$. Since there are no common factors, this fraction cannot be reduced. We next set the denominator equal to zero to arrive at $x+3=0$ and $x-3=0$, from which $x=\pm 3$. Hence, the lines $x=-3$ and $x=3$ are vertical asymptotes.

2. Since the fraction $G(x)=\dfrac{x+1}{x^2+9}$ is irreducible, we proceed to set the denominator equal to zero. As noted in **Example 3.1.2**, there are no real solutions to $x^2+9=0$, so the graph of $y=G(x)$ does not have any vertical asymptotes.

3. To reduce the fraction $H(x)=\dfrac{x+3}{x^2-9}$, we factor the numerator and the denominator, and then cancel the common factor:[4]

$$\frac{x+3}{x^2-9}=\frac{x+3}{(x-3)(x+3)}=\frac{1}{x-3}$$

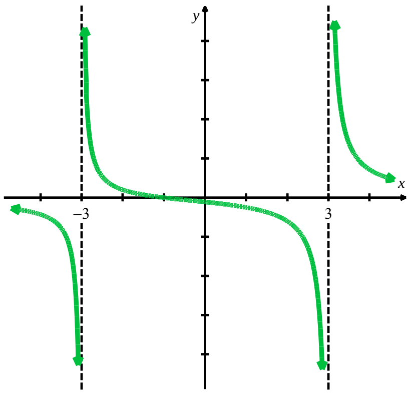
Setting the denominator equal to zero, we have $x-3=0$ and get $x=3$. The vertical asymptote is the line $x=3$.

□

Using graphing technology[5] enables us to visualize the results from **Example 3.1.4**.

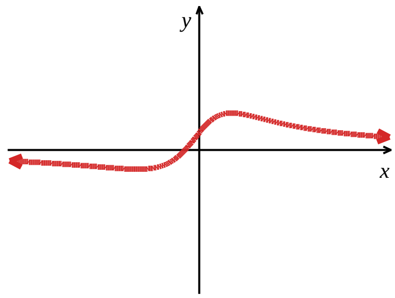
[4] We will look at the effect that this factor, $x+3$, has on the graph when we get to **Section 3.3**.
[5] Try Wolfram Alpha, Desmos or a free online graphing calculator.

Figure 3.1. 3



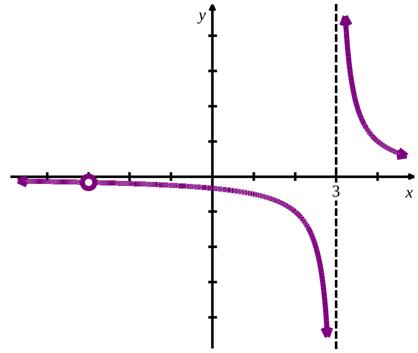
$$y = F(x) = \frac{x+1}{x^2-9}$$

Figure 3.1. 4



$$y = G(x) = \frac{x+1}{x^2+9}$$

Figure 3.1. 5



$$y = H(x) = \frac{x+3}{x^2-9}$$

Identifying Horizontal Asymptotes

Now, let's consider the graph of the function $f(x) = \frac{2x-1}{x+1}$ for large positive and negative *x* values, or the end behavior of the graph. As we discussed in **Section 2.2**, the **end behavior** of a function is its behavior as *x* attains larger[6] and larger negative values without bound, denoted $x \to -\infty$, and as *x* becomes large without bound, written as $x \to \infty$. We again refer to tables.

x	-10	-100	-1000	-10000
$f(x) = \frac{2x-1}{x+1}$	≈2.333	≈2.0303	≈2.0030	≈2.0003

Values of $f(x)$ as *x* approaches $-\infty$

x	10	100	1000	10000
$f(x) = \frac{2x-1}{x+1}$	≈1.7273	≈1.9703	≈1.9970	≈1.9997

Values of $f(x)$ as *x* approaches ∞

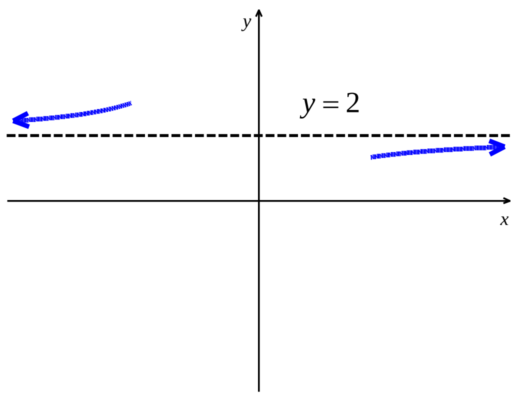
From these tables, it appears that the value of $f(x)$ approaches 2 for both large positive and negative *x* values. We express these symbolically by

$$\text{as } x \to -\infty,\ f(x) \to 2, \text{ and as } x \to \infty,\ f(x) \to 2$$

[6] Here, the word 'larger' means larger in absolute value.

This means that, as shown below, the graph of $f(x)=\dfrac{2x-1}{x+1}$ 'levels off' and approaches the horizontal line $y=2$ on both the left and right hand sides of the graph. In this case, we say the graph of $y=f(x)$ has a **horizontal asymptote** of $y=2$, identified as follows by a dashed line.

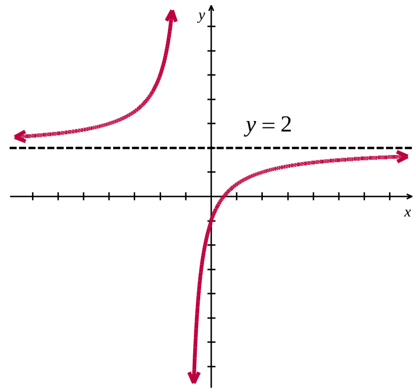
Figure 3.1. 6



End Behavior:

as $x\to-\infty,\ f(x)\to 2$; as $x\to\infty,\ f(x)\to 2$

Figure 3.1. 7



$y=f(x)=\dfrac{2x-1}{x+1}$

Definition 3.3. The line $y=c$ is called a **horizontal asymptote** of the graph of a function $y=f(x)$ if, as $x\to-\infty$ or $x\to\infty$, $f(x)\to c$.

Horizontal asymptotes for rational functions occur when the degree of the numerator is less than, or equal to, the degree of the denominator. If the degree of the numerator is greater than the degree of the denominator, the graph will not have a horizontal asymptote.[7] We would like to develop a way of finding horizontal asymptotes without forming tables, as we did above. To develop a general rule, we use the fact that a polynomial's end behavior is governed by its leading term, as found in **Chapter 2**. Let's consider the following two cases.

- We start with the case where the degree of the numerator is less than the degree of the denominator. Consider $f(x)=\dfrac{4x+2}{x^2+4x-5}$. By plugging in large positive or negative *x* values, such as $f(100)\approx 0.039$ or $f(-100)\approx -0.041$, we find that $f(x)\to 0$ as $x\to\pm\infty$. A simple explanation, without plugging in *x* values, is that for large positive or negative *x* values, $4x+2$ behaves like $4x$

[7] We will talk more about rational functions without horizontal asymptotes in **Section 3.3**.

while x^2+4x-5 behaves like x^2. It follows that $f(x)=\dfrac{4x+2}{x^2+4x-5}$ behaves like $y=\dfrac{4x}{x^2}=\dfrac{4}{x}$ and, as $x\to\pm\infty$, $y=\dfrac{4}{x}\to 0$. Thus, the graph of $y=f(x)$ has a horizontal asymptote of $y=0$.

- Now consider the case where the degree of the numerator is the same as the degree of the denominator. Earlier we saw that the horizontal asymptote of the graph of $f(x)=\dfrac{2x-1}{x+1}$ is $y=2$. A simple explanation, without plugging in *x* values, is that for large positive or negative *x* values, $2x+1$ behaves like $2x$ while $x+1$ behaves like *x*. Thus, $f(x)=\dfrac{2x-1}{x+1}$ behaves like $y=\dfrac{2x}{x}=2$ for all *x* values, including when $x\to\pm\infty$. Notice that 2 is just the ratio of the leading coefficients of numerator and denominator.

This idea works in general, when there is a horizontal asymptote, and leads to the following result for identifying horizontal asymptotes.

Identifying Horizontal Asymptotes

Suppose $r(x)=\dfrac{p(x)}{q(x)}$ is a rational function where p and q are polynomial functions. If the degree of p is less than or equal to the degree of q, then r has a horizontal asymptote that may be determined as follows.

1. If the degree of p is less than the degree of q, then the horizontal asymptote is the line $y=0$.
2. If the degree of p is the same as the degree of q, then the horizontal asymptote is the line $y=\dfrac{\text{leading coefficient of } p}{\text{leading coefficient of } q}$.

Example 3.1.5. Identify the horizontal asymptote of the graph for each of the following rational functions.

1. $f(x)=\dfrac{5x}{x^2+1}$
2. $g(x)=\dfrac{x^2-4}{x+1}$
3. $h(x)=\dfrac{6x^3-3x+1}{5-2x^3}$

Solution.

1. The numerator of $f(x)=\dfrac{5x}{x^2+1}$ is $5x$, which has degree 1. The denominator of $f(x)$ is x^2+1, which has degree 2. Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is $y=0$.

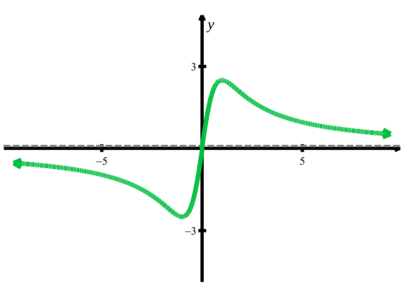
2. The numerator of $g(x)=\dfrac{x^2-4}{x+1}$ is x^2-4, which has degree 2, while the denominator, $x+1$, has degree 1. With the degree of the numerator being greater than the degree of the denominator, we do not have a horizontal asymptote. We will find in **Section 3.3** that the graph of $y=g(x)$ does have what will be referred to as a slant, or oblique, asymptote.

3. The degree of the numerator and denominator of $h(x)=\dfrac{6x^3-3x+1}{5-2x^3}$ are both three, so the graph of h has a horizontal asymptote of $y=\dfrac{6}{-2}$, which simplifies to $y=-3$.

□

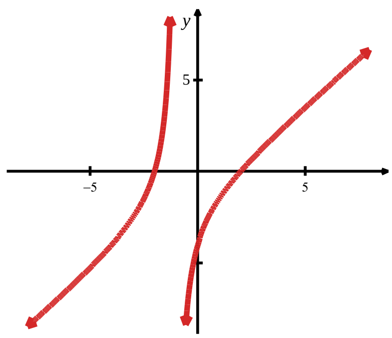
Once again, we employ graphing technology to visualize these results.

Figure 3.1. 8



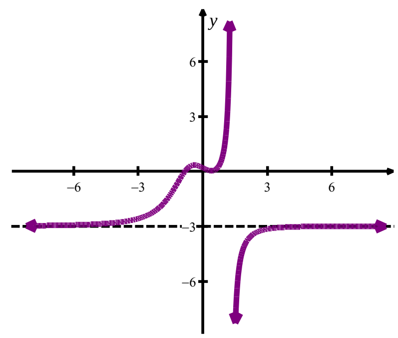
$$y=f(x)=\frac{5x}{x^2+1}$$

Figure 3.1. 9



$$y=g(x)=\frac{x^2-4}{x+1}$$

Figure 3.1. 10



$$y=h(x)=\frac{6x^3-3x+1}{5-2x^3}$$

Graphing

In this section, we limit our graphing of rational functions to those having denominators of degree one, and numerators also of degree one or constant. A general procedure for graphing rational functions follows.

Steps for Graphing Rational Functions (specific to Section 3.1)

Suppose r is a rational function with a constant, or first degree, numerator and a denominator of degree one.

1. Find the domain of r. After recording the domain, reduce r to lowest terms, if possible.
2. Find the *x*- and *y*-intercepts, if any exist.
3. Find the vertical asymptote.
4. Find the horizontal asymptote.
5. Identify additional points, as needed, to see how the graph approaches the asymptotes.
6. Plot the intercepts and use dashed lines to sketch the asymptotes, adding additional points if desired. Sketch the graph, using smooth curves that pass through the intercepts and approach the asymptotes.

Example 3.1.6. Graph the rational function $g(x)=\dfrac{1}{x+2}$.

Solution.

1. We first note that the domain of g excludes $x=-2$, and is therefore $(-\infty,-2)\cup(-2,\infty)$. Since g cannot be reduced, we move on to step 2.

2. Since setting $y=g(x)=\dfrac{1}{x+2}=0$ has no real solution, there are no *x*-intercepts. Setting $x=0$ results in $y=\dfrac{1}{0+2}$, for a *y*-intercept at $\left(0,\ \dfrac{1}{2}\right)$.

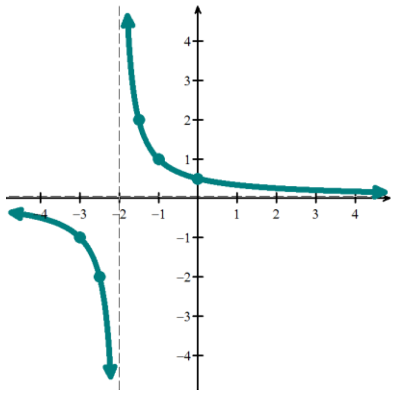
3. Setting the single factor in the denominator equal to zero, we find a vertical asymptote of $x=-2$.

4. Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is $y=0$.

5. To see how the graph approaches the asymptotes, we identify points near the vertical asymptote, $x=-2$, as follows: $(-3,-1)$, $(-2.5,-2)$, $(-1.5,\ 2)$ and $(-1,\ 1)$.

6. After plotting the *y*-intercept and marking the asymptotes with dashed lines, we plot the additional points from step 5 and sketch a smooth curve, passing through all points and approaching both asymptotes.

Figure 3.1. 11



$$y = g(x) = \frac{1}{x+2}$$

Note that we could have skipped plotting additional points to the right of $x = -2$ since the *y*-intercept is above the *x*-axis and the curve has no *x*-intercepts so cannot cross the *x*-axis. Additionally, plotting a single point to the left of the vertical asymptote would have been sufficient since, again, there are no *x*-intercepts.

□

We next graph a function similar to $g(x)$, but with both numerator and denominator having degree one.

Example 3.1.7. Graph the rational function $h(x) = \dfrac{x+1}{x+2}$.

Solution.

1. Like $g(x)$ in **Example 3.1.6**, the domain of $h(x) = \dfrac{x+1}{x+2}$ is $(-\infty,-2)\cup(-2,\infty)$. Also like g, the function h cannot be reduced.

2. Setting $y = h(x) = 0$ gives us $\dfrac{x+1}{x+2} = 0$, from which we have $x+1=0$, resulting in an *x*-intercept of $(-1,0)$. To find the *y*-intercept, we set $x=0$ to get $y = \dfrac{0+1}{0+2}$, for a *y*-intercept of $\left(0, \dfrac{1}{2}\right)$.

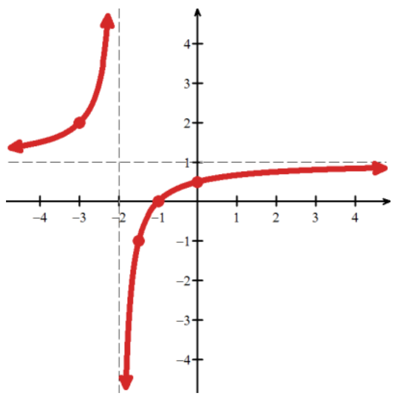
3. After setting the denominator equal to zero, we find that h has a vertical asymptote of $x = -2$.

4. The degree of the numerator and denominator are the same, so the horizontal asymptote is the line $y = \dfrac{1}{1}$, or $y = 1$.

5. We determine how the graph approaches the asymptotes by plotting additional points. Since there are no x-intercepts to the left of $x=-2$, plotting the single point $(-3,\ 2)$ is sufficient. To the right of $x=-2$, there is an x-intercept of $(-1,0)$. The y-intercept at $\left(0,\ \frac{1}{2}\right)$ tells us the curve is above the x-axis to the right of the x-intercept. We find an additional point, $(-1.5,-1)$, to determine the curve's behavior between the vertical asymptote and the x-intercept.

6. After plotting intercepts, marking asymptotes with dashed lines, and adding additional points, we sketch the graph of $y=h(x)$ with smooth curves that pass through points and approach asymptotes.

Figure 3.1. 12



$$y=h(x)=\frac{x+1}{x+2}$$

□

3.1 Exercises

1. Can the graph of a rational function have no vertical asymptotes? Explain.
2. Can the graph of a rational function have no x-intercepts? Explain.

In Exercises 3 – 6, determine if the given function is a rational function. Explain your reasoning.

3. $f(x)=\dfrac{6}{x}$
4. $f(x)=\sqrt{x}-6$
5. $f(x)=\dfrac{1-3x^2}{4x^{\pi}+x^2-1}$
6. $f(x)=\dfrac{2x^6-30}{x^2+5x+3}$

In Exercises 7 – 12, find the domain of the rational function.

7. $f(x)=\dfrac{x-1}{x+2}$
8. $f(x)=\dfrac{x+1}{x^2-1}$
9. $f(x)=\dfrac{x^2+4}{x^2-2x-8}$
10. $f(x)=\dfrac{x^2+4x-3}{x^4-5x^2+4}$
11. $f(x)=\dfrac{x+1}{x^2+25}$
12. $f(x)=\dfrac{x+5}{x^2+25}$

In Exercises 13 – 18, find the x- and y-intercepts for the rational function.

13. $f(x)=\dfrac{x+5}{x^2+4}$
14. $f(x)=\dfrac{x}{x^2-x}$
15. $f(x)=\dfrac{x^2+8x+7}{x^2+11x+30}$
16. $f(x)=\dfrac{x^2+x+6}{x^2-10x+24}$
17. $f(x)=\dfrac{94-2x^2}{3x^2-12}$
18. $f(x)=\dfrac{x+5}{x^2-25}$

In Exercises 19 – 24, identify the vertical asymptotes for the rational function and complete the following statement by filling in the blanks.

As $x\to-\infty$, $f(x)\to$ ________, and as $x\to\infty$, $f(x)\to$ ________.

19. $f(x)=\dfrac{4}{x-1}$
20. $f(x)=\dfrac{x-4}{x-6}$

21. $f(x)=\dfrac{x}{x^2-9}$

22. $f(x)=\dfrac{x}{x^2+5x-36}$

23. $f(x)=\dfrac{3x^2+2}{4x^2-1}$

24. $f(x)=\dfrac{3x-4}{x^3-16x}$

In Exercises 25 – 30, identify the vertical and horizontal asymptotes for the rational function.

25. $f(x)=\dfrac{x^2-1}{x^3+9x^2+14x}$

26. $f(x)=\dfrac{x+5}{x^2+25}$

27. $f(x)=\dfrac{2}{5x+2}$

28. $f(x)=\dfrac{3+x}{x^3-27}$

29. $f(x)=\dfrac{4-2x}{3x-1}$

30. $f(x)=\dfrac{2x^2-8}{2x^2-4x+2}$

In Exercises 31 – 38, graph the rational function. Be sure to draw any asymptotes as dashed lines.

31. $f(x)=\dfrac{1}{x-2}$

32. $f(x)=\dfrac{4}{x+2}$

33. $f(x)=\dfrac{2x-3}{x+4}$

34. $f(x)=\dfrac{x-5}{3x-1}$

35. $f(x)=\dfrac{x-3}{x}$

36. $f(x)=\dfrac{5x}{6-2x}$

37. $f(x)=\dfrac{x}{3x-6}$

38. $f(x)=\dfrac{3+7x}{5-2x}$

39. The cost C in dollars to remove $p\%$ of the invasive species of Ippizuti fish from Sasquatch Pond is given by $C(p)=\dfrac{1770p}{100-p}$, $0\le p<100$.

(a) Find and interpret $C(25)$ and $C(95)$.

(b) What does the vertical asymptote at $x=100$ mean within the context of the problem?

(c) What percentage of the Ippizuti fish can you remove for $40,000?

40. The population of Sasquatch in Portage County can be modeled by the function $P(t)=\dfrac{150t}{t+15}$ where $t=0$ represents the year 1803. Find the horizontal asymptote of the graph of $y=P(t)$ and explain what it means.

3.2 Graphing Rational Functions

Learning Objectives

- Graph irreducible rational functions with denominators of degree greater than one and numerators having the same or a lesser degree.

In this section, we continue graphing rational functions by focusing on functions that have a denominator of degree greater than one. As in **Section 3.1**, we limit our functions to those with a numerator that has the same degree, or a smaller degree, than the denominator. We follow the general procedure outlined in **Section 3.1**, noting that we may have more than one vertical asymptote, due to having a denominator of degree two or higher. While it would also be possible to have no vertical asymptotes, that particular scenario will be addressed in **Section 3.3**.

Example 3.2.1. Graph the rational function $f(x)=\dfrac{3x}{x^2-4}$.

Solution.

1. To determine the domain of $f(x)=\dfrac{3x}{x^2-4}$, we set the denominator equal to zero to get $x^2-4=0$. We find $x=\pm 2$ so our domain is $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$. We next factor $f(x)$ as $f(x)=\dfrac{3x}{(x-2)(x+2)}$ and observe that no cancellation of common factors is possible, so f is in lowest terms.

2. To find the *x*-intercepts of the graph of $y=f(x)$, we set $y=f(x)=0$. Solving $\dfrac{3x}{(x-2)(x+2)}=0$ results in $3x=0$, from which we get $x=0$. Since $x=0$ is in our domain, $(0,0)$ is the *x*-intercept. To find the *y*-intercept, we set $x=0$ and find $y=f(0)=0$ so that $(0,0)$ is our *y*-intercept as well.[8]

3. We find the vertical asymptotes by setting the denominator of $f(x)=\dfrac{3x}{(x-2)(x+2)}$ equal to zero to get $x=\pm 2$. Thus, the vertical asymptotes are the lines $x=-2$ and $x=2$.

[8] Since functions can have at most one *y*-intercept, once we find that $(0,0)$ is on the graph, we know it is the *y*-intercept.

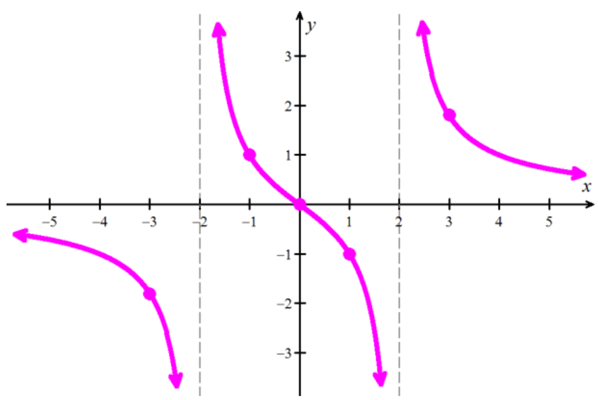
4. We next identify the horizontal asymptote. Since the degree of the numerator of $f(x)=\dfrac{3x}{x^2-4}$ is 1 and the degree of the denominator is 2, the degree of the numerator is less than the degree of the denominator and so $y=0$ is the horizontal asymptote.

5. We find additional points to help determine how the graph approaches the asymptotes. Looking for intervals where the graph is above or below the *x*-axis, we note that function values may only change sign at vertical asymptotes and *x*-intercepts. Thus, we choose points in intervals separated by vertical asymptotes and *x*-intercepts.

x	-3	-1	1	3
$f(x)$	-1.8	1	-1	1.8

6. To graph $y=f(x)$, we mark the vertical asymptotes with dashed lines, note that the horizontal asymptote is the *x*-axis, and plot the intercept, along with additional points. We sketch the graph with smooth curves that pass through the intercept and approach the asymptotes, using additional points as guides.

Figure 3.2. 1



$$y=f(x)=\frac{3x}{x^2-4}$$

□

A few notes are in order.

- First, the vertical asymptotes, $x=-2$ and $x=2$, result from zeros in the denominator of $f(x)=\dfrac{3x}{(x-2)(x+2)}$. Each of these zeros is of multiplicity one and, due to this odd multiplicity, we find that the function changes sign at $x=-2$ and at $x=2$. Making this observation in advance would allow us to plot fewer points when determining the graph's behavior near asymptotes.

- Next, the graph of $y = f(x)$ certainly seems to possess symmetry with respect to the origin. In fact, we can check that $f(-x) = -f(x)$ to see that f is an odd function.
- We see that the graph of f crosses the *x*-axis at $(0,0)$, thus crossing the horizontal asymptote of $y = 0$. While the graph of a rational function may cross its horizontal asymptote, the graph will never cross a vertical asymptote. We will see another case of a graph crossing its horizontal asymptote in the next example.

Example 3.2.2. Graph the rational function $g(x) = \dfrac{2-x}{x^3 - 6x^2 + 9x}$.

Solution.

1. To determine the domain, we set $x^3 - 6x^2 + 9x = 0$ and factor to get $x(x-3)^2 = 0$, from which we find $x = 0$ and $x = 3$. Our domain is $(-\infty,0)\cup(0,3)\cup(3,\infty)$. Factoring $g(x)$ results in $g(x) = \dfrac{2-x}{x(x-3)^2}$. No further reducing of terms is possible.

2. To find *x*-intercepts, we set $y = g(x) = 0$ and proceed to solve $2 - x = 0$, from which $x = 2$. The resulting *x*-intercept is $(2,0)$. The function $g(x)$ is undefined when $x = 0$, so there is no *y*-intercept.

3. Setting the denominator of $g(x) = \dfrac{2-x}{x(x-3)^2}$ equal to zero, we find vertical asymptotes of $x = 0$ and $x = 3$.

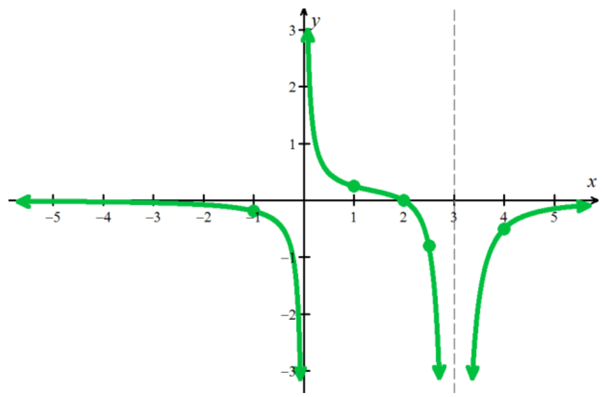
4. Since the degree of the numerator of $g(x) = \dfrac{2-x}{x^3 - 6x^2 + 9x}$ is 1 and the degree of the denominator is 3, the degree of the numerator is less than the degree of the denominator, so we have a horizontal asymptote of $y = 0$.

5. We look for additional points in intervals separated by vertical asymptotes and *x*-intercepts.

x	-1	1	2.5	4
$g(x)$	-0.1875	0.25	-0.8	-0.5

6. To graph $y = g(x)$, we note that the *y*-axis is the vertical asymptote $x = 0$; we mark the vertical asymptote $x = 3$ with a dashed line; we note that the *x*-axis is the horizontal asymptote $y = 0$. After plotting the intercept and additional points, we draw smooth curves that pass through the intercept and approach asymptotes.

Figure 3.2. 2



$$y = g(x) = \frac{2-x}{x^3 - 6x^2 + 9x}$$

□

Before moving on, we look at the denominator of $g(x) = \dfrac{2-x}{x(x-3)^2}$ to see that the vertical asymptote $x = 0$ results from a zero of odd multiplicity one. As expected, the function values change in sign at $x = 0$. The vertical asymptote $x = 3$ results from a zero of even multiplicity two. We note that the function does not change in sign at $x = 3$, as is the case with any vertical asymptote corresponding to a zero of even multiplicity.

Example 3.2.3. Graph the rational function $h(x) = \dfrac{2x^2 - 3x - 5}{x^2 - x - 6}$.

Solution.

1. To determine the domain of $h(x) = \dfrac{2x^2 - 3x - 5}{x^2 - x - 6}$, we set $x^2 - x - 6 = 0$ to get $x = -2$ and $x = 3$. It follows that the domain is $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$. Factoring the numerator and denominator of $h(x)$ results in $h(x) = \dfrac{(2x-5)(x+1)}{(x-3)(x+2)}$. After observing that no cancellation of like terms is possible, we conclude that $h(x)$ is in lowest terms.

2. To locate x-intercepts, we set $y = h(x) = 0$. Using the factored form of $h(x)$ above, we find the zeros to be the solutions of $(2x-5)(x+1) = 0$. We obtain $x = \frac{5}{2}$ and $x = -1$. Since both of these numbers are in the domain of h, we have the two x-intercepts $\left(\frac{5}{2}, 0\right)$ and $(-1, 0)$. To find the y-intercept, we set $x = 0$ and find $y = h(0) = \frac{5}{6}$, so our y-intercept is $\left(0, \frac{5}{6}\right)$.

3. To identify vertical asymptotes, we look for values of x that cause the denominator of $h(x) = \frac{(2x-5)(x+1)}{(x-3)(x+2)}$ to be zero. The resulting vertical asymptotes are $x = -2$ and $x = 3$.

4. With the numerator and denominator both having degree 2, we use the leading coefficient of each to determine that the horizontal asymptote of $y = h(x) = \frac{2x^2 - 3x - 5}{x^2 - x - 6}$ is $y = \frac{2}{1} = 2$.

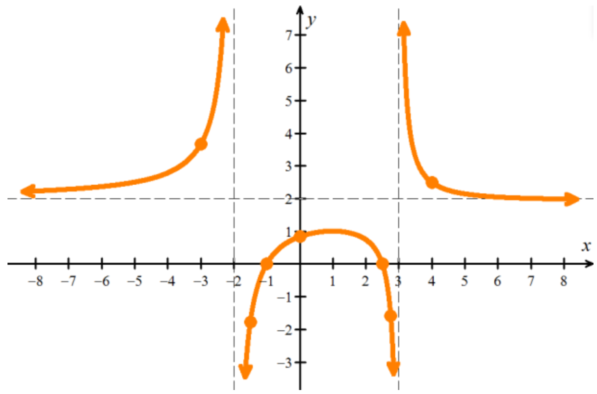
5. We find additional points in intervals separated by the x-intercepts and vertical asymptotes.

x	-3	-1.5	0	2.75	4
$h(x)$	≈ 3.667	-1.778	≈ 0.833	-1.579	2.5

Noting that the point $\left(0, \frac{5}{6}\right)$ was already determined in it’s role as the y-intercept, we also note that fewer points could be plotted by using odd multiplicities to indicate sign changes at the vertical asymptotes. We will explore this technique at the end of the section.

6. We mark the asymptotes with dashed lines, plot intercepts and additional points, and use smooth curves to sketch $y = h(x)$, passing through intercepts and approaching asymptotes.

Figure 3.2. 3



$$y = h(x) = \frac{2x^2 - 3x - 5}{x^2 - x - 6}$$

□

In the next example, we graph a function without plotting additional points.

Example 3.2.4. Graph the rational function $j(x) = \dfrac{(x+2)^2(x-3)}{(x+1)^2(x-2)}$.

Solution. We begin with the first four steps of the general graphing procedure from **Section 3.1**, after which we diverge into a more intuitive approach.

1. Setting the denominator, $(x+1)^2(x-2)$ equal to zero, we get $x = -1$ and $x = 2$ for a domain of $(-\infty,-1)\cup(-1,2)\cup(2,\infty)$. Neither the numerator nor the denominator can be further factored, and $j(x)$ cannot be reduced through elimination of common factors.

2. To find *x*-intercepts, we set $j(x) = 0$, from which $(x+2)^2(x-3) = 0$ results in *x*-intercepts $(-2,0)$ and $(3,0)$. Setting $x = 0$ gives us $y = j(0) = \dfrac{(0+2)^2(0-3)}{(0+1)^2(0-2)} = \dfrac{-12}{-2}$, for a y-intercept of $(0,6)$.

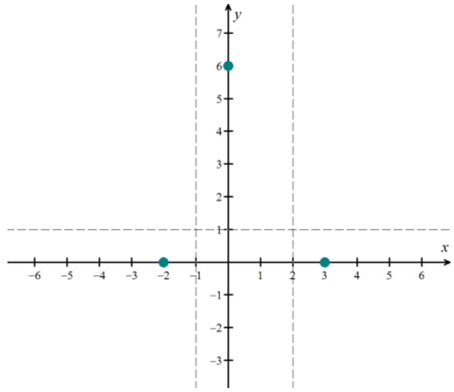
3. From the denominator, we find vertical asymptotes of $x = -1$ and $x = 2$.

4. With the degree of the numerator and denominator being 3, we look at leading coefficients to determine the horizontal asymptote, and find it to be $y = 1$.

The remaining steps follow the thought process we might use to graph this function without finding additional points.

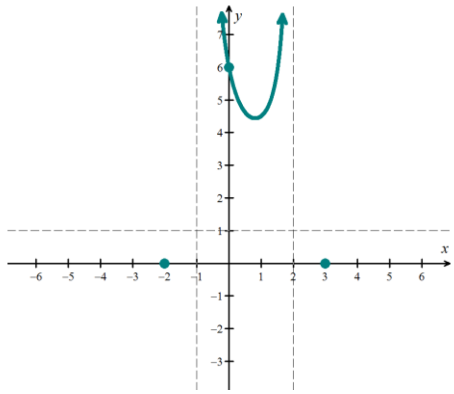
- We plot the information we have so far, including intercepts at $(-2,0)$, $(3,0)$ and $(0,6)$, and asymptotes of $x=-1$, $x=2$ and $y=1$.

Figure 3.2. 4



- The graph cannot cross the *x*-axis between the vertical asymptotes since there is no *x*-intercept between $x=-1$ and $x=2$. Since the *y*-intercept is above the *x*-axis, the graph will stay above the *x*-axis, approaching the vertical asymptotes as follows.[9]

Figure 3.2. 5

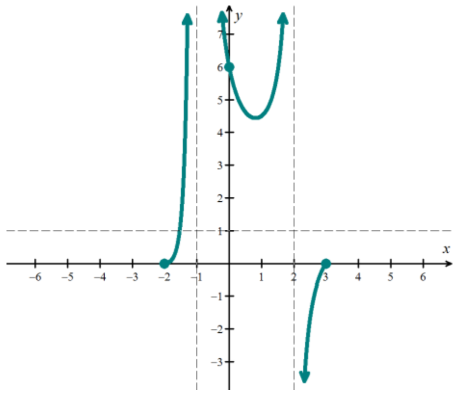


While not part of the intuitive process, we can verify our assumptions by finding a couple of points, like $(-0.25,\ 7.8642)$ and $(1.5,\ 5.88)$.

- The vertical asymptote of $x=-1$ is a result of the factor $(x+1)^2$ in the denominator. The even multiplicity tells us the function will not have a sign change at $x=-1$. The vertical asymptote $x=2$ comes from the factor $(x-2)$. An odd multiplicity of one indicates a sign change at $x=2$.

[9] Without graphing additional points it is not possible to know how far down the graph dips. As we have noted, it is even possible that this graph could cross the horizontal asymptote.

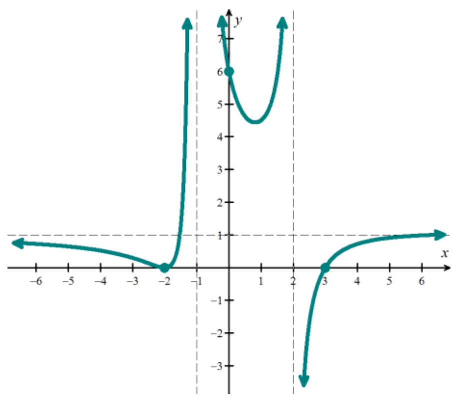
Figure 3.2. 6



We can again check our results by identifying points; for example $(-1.5,\ 1.2857)$ and $(2.5,\ -1.653)$.

- The *x*-intercept of $(-2,0)$ is a result of $(x+2)^2$ in the numerator. As we discussed when graphing polynomials, the even power prevents a sign change of our function at $x=-2$, and results in the graph merely touching, not crossing, the *x*-axis at that point. The *x*-intercept of $(3,0)$ is a result of the factor $(x-3)$, and the odd power of this factor causes a sign change for the function. Thus, the graph crosses the *x*-axis at $(3,0)$. We complete the graph, drawing smooth curves that include appropriate behavior at intercepts and near asymptotes.

Figure 3.2. 7



$$y=j(x)=\frac{(x+2)^2(x-3)}{(x+1)^2(x-2)}$$

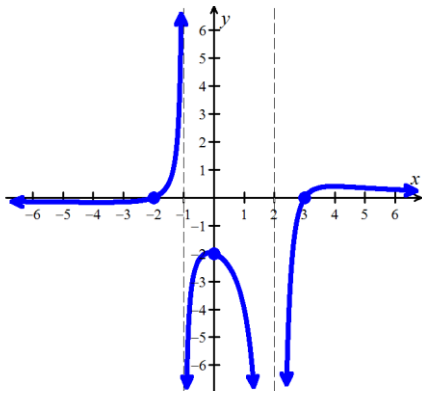
Once again, we can test our graph by calculating points such as $(-3,\ 0.3)$ and $(4,\ 0.72)$.

□

Using intuition to save calculation time when graphing is helpful, although in some cases plotting at least one additional point can prove useful. We end this section with an example in which we determine an equation of a rational function from its graph.

Example 3.2.5. Write an equation for the rational function shown below.

Figure 3.2. 8



Solution. The graph appears to have *x*-intercepts at $x = -2$ and $x = 3$. At both points, the graph passes through the intercept, suggesting factors with odd multiplicities. For convenience, we'll assume multiplicity of one, giving us factors of $(x+2)$ and $(x-3)$ in the numerator.

The graph has two vertical asymptotes. At $x = -1$, the function appears to have a sign change, so we can assume the corresponding factor has odd multiplicity of one. There is no sign change at $x = 2$, so the corresponding factor must have an even multiplicity, and we'll assume that multiplicity is two. Thus, we have the factors $(x+1)$ and $(x-2)^2$ in the denominator.

Putting together what we know so far, the function looks like $f(x) = a \cdot \frac{(x+2)(x-3)}{(x+1)(x-2)^2}$ for some constant *a* that is yet unknown. We do have one additional piece of information, and that is the *y*-intercept at $(0,-2)$. We plug in 0 for *x* and −2 for $y = f(x)$ to solve for *a*.

$$-2 = a \cdot \frac{(0+2)(0-3)}{(0+1)(0-2)^2}$$

$$-2 = a \cdot \frac{-6}{4}$$

$$-2 \cdot \frac{4}{-6} = a$$

$$a = \frac{4}{3}$$

This gives us a final function of $f(x) = \frac{4(x+2)(x-3)}{3(x+1)(x-2)^2}$. As a last check, we confirm that the degree of the numerator is less than the degree of the denominator, verifying that our function has the required horizontal asymptote of $y = 0$.

□

3.2 Exercises

1. How can multiplicities be used in graphing rational functions?

2. What property of a rational function results in its graph crossing the *x*-axis at an intercept?

In Exercises 3 – 14, graph the rational function. Be sure to draw any asymptotes as dashed lines.

3. $f(x)=\frac{1}{x^2}$

4. $f(x)=\frac{1}{x^2+x-12}$

5. $f(x)=\frac{x}{x^2+x-12}$

6. $f(x)=\frac{4x}{x^2-4}$

7. $f(x)=\frac{3x^2-5x-2}{x^2-9}$

8. $f(x)=\frac{x+7}{x^2+6x+9}$

9. $f(x)=\frac{4}{x^2-4x+4}$

10. $f(x)=\frac{5}{x^2+2x+1}$

11. $f(x)=\frac{3x^2-14x-5}{3x^2+8x-16}$

12. $f(x)=\frac{2x^2+7x-15}{3x^2-14x+15}$

13. $f(x)=\frac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)}$

14. $f(x)=\frac{(x+2)^2(x-5)}{(x-3)(x+1)(x+4)}$

In Exercises 15 – 20, write an equation for a rational function with the given characteristics.

15. Vertical asymptotes at $x=5$ and $x=-5$; *x*-intercepts at $(2,0)$ and $(-1,0)$; *y*-intercept at $(0,4)$

16. Vertical asymptotes at $x=-4$ and $x=-1$; *x*-intercepts at $(1,0)$ and $(5,0)$; *y*-intercept at $(0,7)$

17. Vertical asymptotes at $x=-4$ and $x=-5$; *x*-intercepts at $(4,0)$ and $(-6,0)$; horizontal asymptote at $y=7$

18. Vertical asymptotes at $x=-3$ and $x=6$; *x*-intercepts at $(-2,0)$ and $(1,0)$; horizontal asymptote at $y=-2$

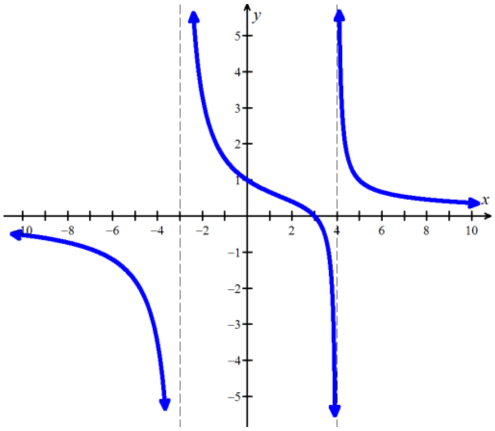
19. Vertical asymptote at $x=-1$; graph touches but does not cross *x*-axis at $(2,0)$; *y*-intercept at $(0,2)$

20. Vertical asymptote at $x=3$; graph touches but does not cross *x*-axis at $(1,0)$; *y*-intercept at $(0,4)$

In Exercises 21 – 26, use the graph to write an equation for the function.

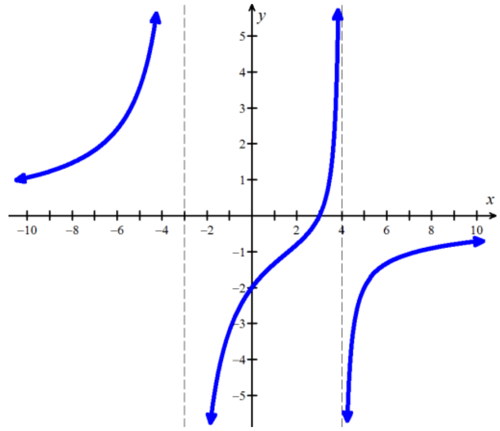
21.

Figure 3.2. 9



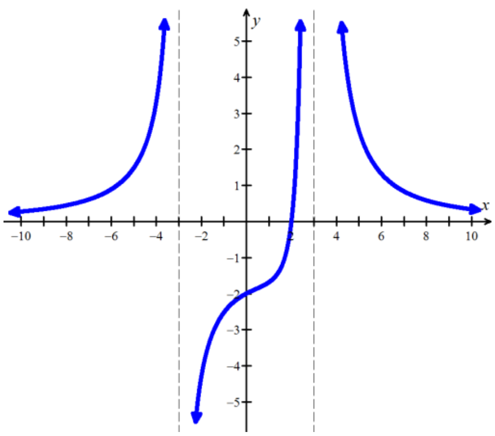
22.

Figure 3.2. 10



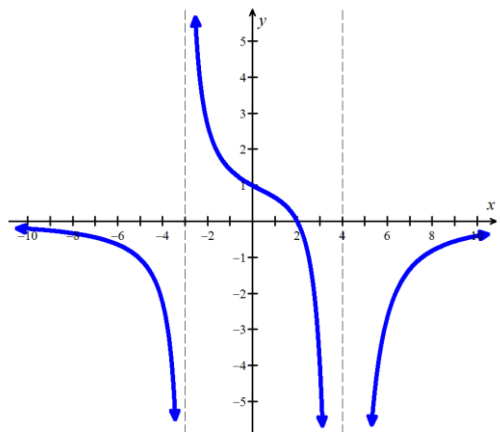
23.

Figure 3.2. 11



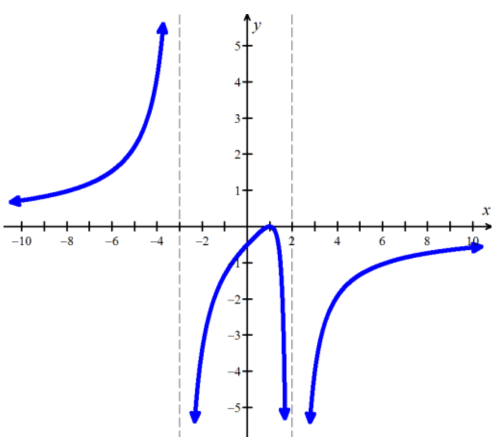
24.

Figure 3.2. 12



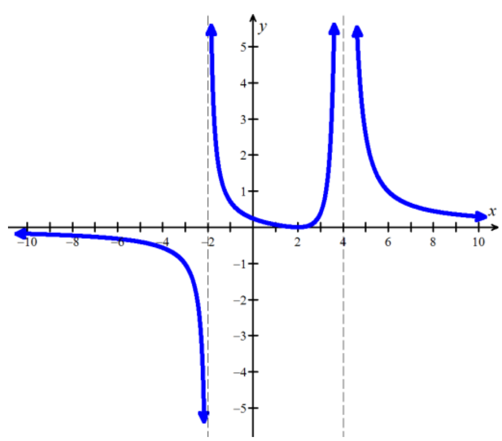
25.

Figure 3.2. 13



26.

Figure 3.2. 14



3.3 Graphs with Holes and Variations on Asymptotes

Learning Objectives

- Identify holes in the graph of a rational function.
- Graph rational functions without vertical asymptotes.
- Find slant (oblique) asymptotes.
- Graph rational functions having slant asymptotes.

In this section, we look at rational functions whose graphs have holes, do not have vertical asymptotes, or have slant asymptotes.

Holes in Graphs of Rational Functions

Graphs of rational functions do not contain any points on their vertical asymptotes since rational functions are not defined for x values where vertical asymptotes occur. Additionally, it is possible for the graph of a rational function to exclude a point which is not on a vertical asymptote. Such a point is referred to as a **hole** and is graphed as an open circle. The function $H(x)=\dfrac{x+3}{x^2-9}$, from **Example 3.1.2**, provides us with an example of a rational function whose graph contains a hole, and we graph this function in the following example.

Example 3.3.1. Graph the rational function $H(x)=\dfrac{x+3}{x^2-9}$.

Solution. We follow the steps outlined in **Section 3.1**.

1. As we found in **Example 3.1.2**, $x=\pm 3$ results in a denominator of zero, so our domain is $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$. We proceed with factoring before reducing the function to lowest terms.

$$\begin{aligned} H(x) &= \frac{x+3}{(x-3)(x+3)} \\ &= \frac{1}{x-3},\ x\neq -3 \end{aligned}$$

Since $x=-3$ is not in the domain of H, and is thus excluded as a value for x in the simplified version of $H(x)$, the point having an x coordinate of −3 cannot be included in the graph of the function. We will indicate this missing point on the graph by replacing it with an open circle. This means that the graph of $y = H(x)$ is the graph of $y = K(x) = \dfrac{1}{x-3}$ with the point $\left(-3, \dfrac{1}{-3-3}\right) = \left(-3, -\dfrac{1}{6}\right)$ removed. So, from here we proceed to graph $K(x) = \dfrac{1}{x-3}$.

2. We next locate the intercepts. We observe that there are no *x*-intercepts since $\dfrac{1}{x-3} \neq 0$ for any value of *x*. There is a *y*-intercept when $y = \dfrac{1}{0-3}$, at the point $\left(0, -\dfrac{1}{3}\right)$.

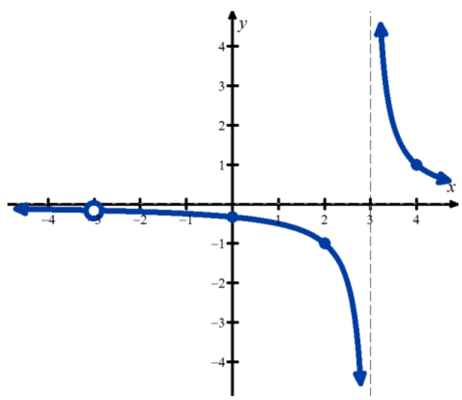
3. The vertical asymptote for the graph of $y = K(x)$, or $y = H(x)$, is the line $x = 3$.

4. The degree of the numerator is less than the degree of the denominator, so the horizontal asymptote is $y = 0$.

5. Two additional points on the graph are $(2, -1)$ and $(4, 1)$. The point we should remove is $\left(-3, -\dfrac{1}{6}\right)$.

6. Using the above information, we sketch the graph.

Figure 3.3. 1



$$y = H(x) = \frac{x+3}{x^2 - 9}$$

□

Graphing Rational Functions without Vertical Asymptotes

We have already seen a rational function with the domain of all real numbers. The function $G(x) = \dfrac{x+1}{x^2+9}$ in **Example 3.1.2** looked similar to the other functions in that example, but its denominator of $x^2 + 9$ was never zero. We found the domain of G to be $(-\infty, \infty)$. In the next example, we follow the general graphing procedure from **Section 3.1** in graphing the function G.

Example 3.3.2. Graph the rational function $G(x)=\dfrac{x+1}{x^2+9}$.

Solution.

1. The denominator x^2+9 is never zero, so the domain of G is $(-\infty,\infty)$. No factoring or further simplification of G is possible.

2. To find *x*-intercepts, we set $G(x)=0$, from which $x+1=0$, resulting in an *x*-intercept of $(-1,0)$. Setting $x=0$ gives us the *y*-intercept of $\left(0,\ \dfrac{1}{9}\right)$.

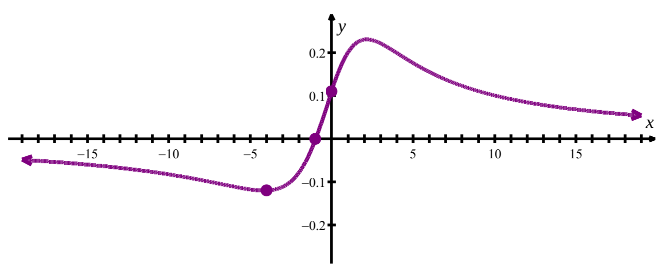
3. Since no values of *x* cause the denominator to be zero, there are no vertical asymptotes.

4. With the degree of the numerator being 1 and the degree of the denominator being 2, the degree of the numerator is less that the degree of the denominator and so the horizontal asymptote is $y=0$.

5. There are no vertical asymptotes, only the *x*-intercept of $(-1,0)$ that might separate positive from negative function values. To the right of $(-1,0)$, the *y*-intercept of $\left(0,\ \dfrac{1}{9}\right)$ tells us the graph is above the *x*-axis. To the left, we find the point $(-4,-0.12)$.

6. The horizontal asymptote, $y=0$, is the *x*-axis. We plot the intercepts and additional point, then sketch the graph of $y=G(x)$ with smooth curves that approach the horizontal asymptote as $x\to-\infty$ and as $x\to\infty$.

Figure 3.3. 2



$$y=G(x)=\frac{x+1}{x^2+9}$$

□

Identifying Slant Asymptotes

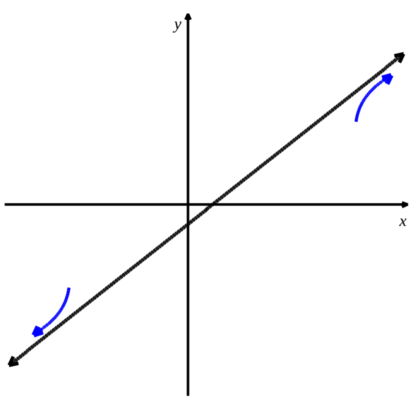
We finish this section, and our graphing of rational functions, with the third (and final!) kind of asymptote that can be associated with the graphs of rational functions. If we perform long division on the function $f(x)=\frac{x^2-4}{x+1}$, we get $f(x)=x-1-\frac{3}{x+1}$. Since the term $\frac{3}{x+1}\to 0$ as $x\to\pm\infty$, it stands to reason that for large positive and negative *x* values, the function values $f(x)=x-1-\frac{3}{x+1}\approx x-1$. Geometrically, this means that the graph of $y=f(x)$ should be close to the line $y=x-1$ as $x\to\pm\infty$. We see this play out numerically and graphically below.

x	$f(x)$	$x-1$
-10	≈ -10.6667	-11
-100	≈ -100.9697	-101
-1000	≈ -1000.9970	-1001
-10000	≈ -10000.9997	-10001

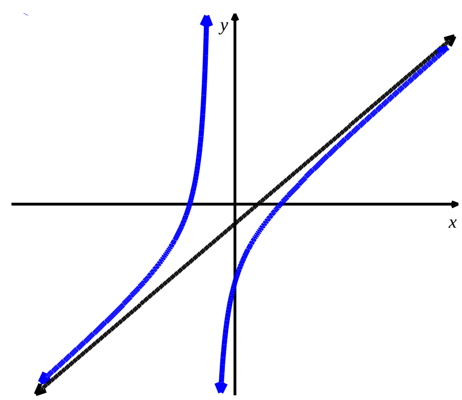
Figure 3.3. 3

x	$f(x)$	$x-1$
10	≈ 8.7273	9
100	≈ 98.9703	99
1000	≈ 998.9970	999
10000	≈ 9998.9997	9999

Figure 3.3. 4



$y=x-1$ and behavior of $f(x)$ as $x\to\pm\infty$



$y=x-1$ and $y=f(x)$

In this case, we say the line $y=x-1$ is a **slant asymptote**[10] to the graph of $y=f(x)$. Informally, we say the nonhorizontal (slant) line $y=x-1$ is the slant asymptote of the graph of $y=f(x)$ if, as $x\to-\infty$ or as $x\to\infty$, $f(x)\approx x-1$, or the graph of $y=f(x)$ resembles that of $y=x-1$. Formally, this can be

[10] Also called an **oblique asymptote** in some, ostensibly higher class (and more expensive), texts.

stated as: $y = x - 1$ is the slant asymptote of the graph of $y = f(x)$ if, as $x \to -\infty$ or as $x \to \infty$, $\left[f(x) - (x-1)\right] \to 0$.

Definition 3.4. The line $y = mx + b$, where $m \neq 0$, is called a **slant asymptote** of the graph of a function $y = f(x)$ if as $x \to -\infty$ or as $x \to \infty$, $\left[f(x) - (mx+b)\right] \to 0$.

This means that if the line $y = mx + b$, where $m \neq 0$, is a slant asymptote of the graph of a function $y = f(x)$, then as $x \to -\infty$ or as $x \to \infty$, $f(x) \approx mx + b$ or the graph of $y = f(x)$ resembles that of $y = mx + b$.

Our next task is to determine the conditions under which the graph of a rational function has a slant asymptote, and if it does, how to find it. In the case of $f(x) = \dfrac{x^2 - 4}{x+1}$, the degree of the numerator $x^2 - 4$ is 2, which is exactly one more than the degree of its denominator $x + 1$, which is 1. This results in a linear quotient polynomial, and it is this quotient polynomial that is the slant asymptote. Generalizing this situation gives us the following theorem.[11]

Theorem 3.1. Determination of Slant Asymptotes: Suppose r is a rational function and $r(x) = \dfrac{p(x)}{q(x)}$, where the degree of p is exactly one more than the degree of q. Then the graph of $y = r(x)$ has the slant asymptote $y = L(x)$ where $L(x)$ is the quotient obtained by dividing $p(x)$ by $q(x)$.

Unlike the shortcut we have been using to find horizontal asymptotes, there is no recourse in finding slant asymptotes but to use long division. We will demonstrate this in the first problem of **Example 3.3.3**.

Example 3.3.3. Identify the slant asymptote of the graph of the following rational functions, if one exists.

1. $f(x) = \dfrac{x^2 - 4x + 2}{1 - x}$
2. $g(x) = \dfrac{x^2 - 4}{x - 2}$
3. $h(x) = \dfrac{x^3 + 1}{x^2 - 4}$

---

[11] This theorem is brought to you courtesy of **Theorem 2.4** and Calculus.

Solution.

1. For $f(x)=\dfrac{x^2-4x+2}{1-x}$, the degree of the numerator is 2 and the degree of the denominator is 1, so **Theorem 3.1** guarantees us a slant asymptote. To find it, we divide $1-x=-x+1$ into x^2-4x+2.

$$\begin{array}{r} -x+3 \\ -x+1\overline{\smash{\big)}\,x^2-4x+2} \\ -\underline{\left(x^2-x\right)} \\ -3x+2 \\ -\underline{\left(-3x+3\right)} \\ -1 \end{array}$$

The result is a quotient of $-x+3$ with remainder -1. The slant asymptote is given by the quotient of this long division. That is, the slant asymptote is the line $y=-x+3$. Notice that, as opposed to the case for horizontal asymptotes, the ratio of leading terms of the numerator and denominator, $\dfrac{x^2}{-x}=-x$, does not give the correct result for the slant asymptote since it is not the quotient of the long division. So, as stated above, there is no recourse in finding slant asymptotes but to use long division.

2. As with the previous example, the degree of the numerator of $g(x)=\dfrac{x^2-4}{x-2}$ is 2 and the degree of the denominator is 1, so **Theorem 3.1** applies. In this case,

$$\begin{aligned} g(x) &= \frac{x^2-4}{x-2} \\ &= \frac{(x+2)(x-2)}{(x-2)} \\ &= x+2, \quad x\neq 2 \end{aligned}$$

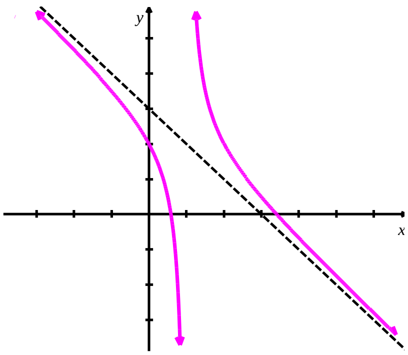
We see that when we divide $x-2$ into x^2-4, we get a quotient of $x+2$ (with remainder 0), and so we have the slant asymptote $y=x+2$. We note that the slant asymptote is identical to the graph of $y=g(x)$ except at $x=2$, where the latter has a hole at $(2,4)$.

3. For $h(x)=\dfrac{x^3+1}{x^2-4}$, the degree of the numerator is 3 and the degree of the denominator is 2, so again, we are guaranteed the existence of a slant asymptote. The long division $\left(x^3+1\right)\div\left(x^2-4\right)$ gives a quotient of just *x*, so our slant asymptote is the line $y=x$.

□

The graphs of the three functions from **Example 3.3.3** appear below, with slant asymptotes represented by dashed lines. Following these three graphs, we will proceed with our own graphing of rational functions that have slant asymptotes.

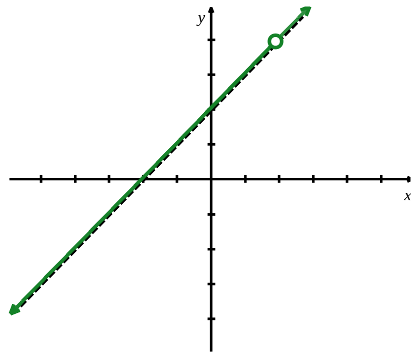
Figure 3.3. 5



$$y = f(x) = \frac{x^2 - 4x + 2}{1 - x}$$

slant asymptote $y = -x + 3$

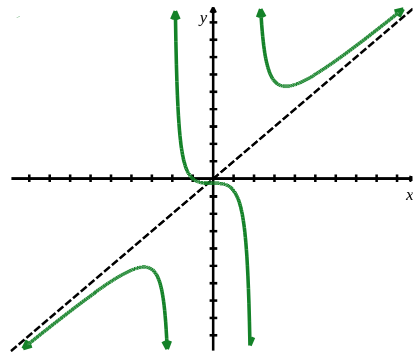
Figure 3.3. 6



$$y = g(x) = \frac{x^2 - 4}{x - 2}$$

slant asymptote $y = x + 2$

Figure 3.3. 7



$$y = h(x) = \frac{x^3 + 1}{x^2 - 4}$$

slant asymptote $y = x$

A rational function may have a horizontal asymptote or a slant asymptote, but not both. We note that the method for finding the slant asymptote also works for finding the horizontal asymptote. In **Section 3.1**, we found that the horizontal asymptote of $f(x) = \frac{2x-1}{x+1}$ is $y = 2$. Using long division, we see that $f(x) = 2 - \frac{3}{x+1}$, so 2 is the quotient and hence $y = 2$ is the horizontal asymptote. Similarly, $y = 0$ is the horizontal asymptote of the graph of $f(x) = \frac{4x+2}{x^2+4x-5}$ since the numerator has smaller degree than the denominator. Using long division, we find $f(x) = 0 + \frac{4x+2}{x^2+4x-5}$. Since the quotient is zero, the horizontal asymptote is $y = 0$. Although mathematically horizontal and slant asymptotes are the same, for ease of calculation we have described different ways of finding them.

Graphing Rational Functions that have Slant Asymptotes

Before moving on to graphing functions with slant asymptotes, we revisit the steps for graphing rational functions, changing the wording a bit to accommodate information added in **Section 3.2** and **Section 3.3**.

Steps for Graphing Rational Functions

Suppose r is a rational function.

1. **1.** Find the domain of r. After recording the domain, identify the location of any holes and reduce r to lowest terms, if possible. From this point, proceed with the reduced function but with the domain of r.
2. **2.** Find the x- and y-intercepts, if any exist.
3. **3.** Find the vertical asymptotes, if any exist.
4. **4.** Find the horizontal or slant asymptote, if one exists.
5. **5.** Plot additional points, as needed, to see how the graph approaches the asymptotes and to identify the location of holes, if any exist.
6. **6.** Plot the intercepts and holes, use dashed lined to sketch the asymptotes, and add additional points if desired. Sketch the graph, using smooth curves that pass through the intercepts and approach the asymptotes.

In the following two examples, we demonstrate these steps for graphing rational functions.

Example 3.3.4. Graph the rational function $f(x) = \dfrac{3x^2 - 2x + 1}{x - 1}$.

Solution. We follow the steps listed above.

1. Setting the denominator equal to zero, we get $x = 1$, for a domain of $(-\infty, 1) \cup (1, \infty)$. The numerator is not factorable, so $f(x)$ cannot be written in lower terms, and there are subsequently no holes.

2. Setting $y = f(x) = 0$, we look for x-intercepts where $3x^2 - 2x + 1 = 0$. After applying the quadratic formula, we find there are no real solutions, and conclude that there are no x-intercepts. To find the y-intercept, we have $y = f(0) = \dfrac{3(0)^2 - 2(0) + 1}{0 - 1}$, for a y-intercept of $(0, -1)$.

3. Setting the denominator of $x - 1$ equal to zero results in a vertical asymptote of $x = 1$.

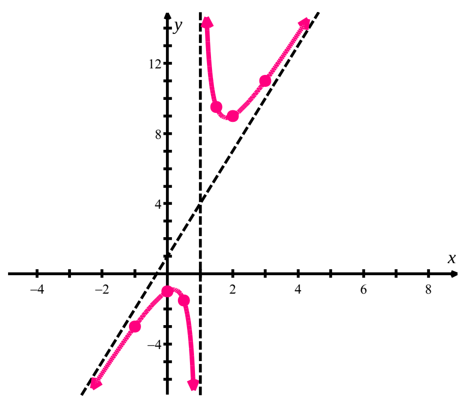
4. The degree of the numerator is 2 and the degree of the denominator is 1, so we have a slant asymptote. After using long division[12], we have $\left(3x^2-2x+1\right)\div\left(x-1\right)=3x+1+\dfrac{2}{x-1}$, from which we find that the slant asymptote is $y=3x+1$.

5. We find additional points on both sides of the vertical asymptote, noting that the function can only change sign at $x=1$.

x	-1	0.5	1.5	2	3
$f\left(x\right)$	-3	-1.5	9.5	9	11

6. To graph $y=f\left(x\right)$, we mark the asymptotes with dashed lines, plot the intercept and extra points, then use smooth curves to draw the graph, passing through the intercept and using the points to guide us in approaching both the vertical and slant asymptote.

Figure 3.3. 8



$$y=f\left(x\right)=\frac{3x^2-2x+1}{x-1}$$

□

Example 3.3.5. Graph the rational function $g\left(x\right)=\dfrac{2x^3+5x^2+4x+1}{x^2+3x+2}$.

Solution.

1. To determine the domain, we set the denominator equal to zero to find values of $x=-1$ and $x=-2$ that must be excluded. Thus, our domain is $\left(-\infty,-2\right)\cup\left(-2,-1\right)\cup\left(-1,\infty\right)$. We next factor the numerator and denominator to determine any holes and then reduce g to simplest terms.

---

[12] Synthetic division works here.

To factor $2x^3+5x^2+4x+1$, we use the Rational Zeros Theorem to identify potential rational zeros of ±1 and $\pm\frac{1}{2}$. We then follow with synthetic division that verifies $x=-1$ is a zero.

$$\begin{array}{r|rrrr} -1 & 2 & 5 & 4 & 1 \\ & \downarrow & -2 & -3 & -1 \\ \hline & 2 & 3 & 1 & 0 \end{array}$$

We use the results to factor $g(x)=\frac{(x+1)(2x^2+3x+1)}{x^2+3x+2}=\frac{(x+1)(x+1)(2x+1)}{(x+1)(x+2)}$. After noting that there is a hole in the graph at $x=-1$, we write g as $g(x)=\frac{(x+1)(2x+1)}{x+2}$ but remember that its domain is still $(-\infty,-2)\cup(-2,-1)\cup(-1,\infty)$, which does not include $x=-1$.

2. To find *x*-intercepts, we set $g(x)=\frac{(x+1)(2x+1)}{x+2}=0$ to get $x=-1$ and $x=-\frac{1}{2}$. Since there is a hole in the graph at $x=-1$, this point will not be a *x*-intercept, but it will be important to include a hole at $(-1,0)$ when sketching the graph. We set *x* equal to zero to find a *y*-intercept at $y=\frac{(0+1)(2\cdot0+1)}{0+2}=\frac{1}{2}$. Thus, our intercepts are $\left(-\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$.

3. Setting $x+2=0$, the single remaining factor in the denominator yields a vertical asymptote of $x=-2$.

4. In $g(x)=\frac{(x+1)(2x+1)}{x+2}=\frac{2x^2+3x+1}{x+2}$, the degree of the numerator is 2 and the degree of the denominator is 1. We have a slant asymptote since the degree of the numerator is greater than the degree of the denominator by one. With a divisor of the form $x-c$, we use synthetic division to find the slant asymptote.

$$\begin{array}{r|rrr} -2 & 2 & 3 & 1 \\ & \downarrow & -4 & 2 \\ \hline & 2 & -1 & 3 \end{array}$$

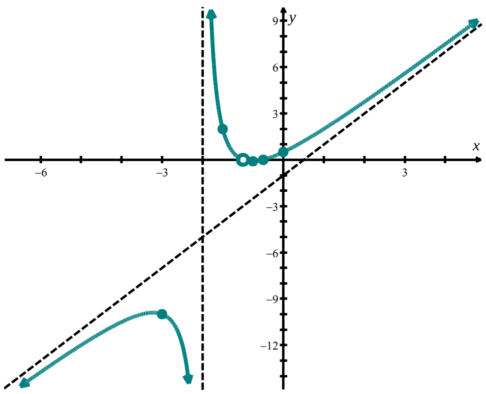
So $(2x^2+3x+1)\div(x+2)=2x-1+\frac{3}{x+2}$ and the slant asymptote is $y=2x-1$.

5. We find additional points in intervals separated by the vertical asymptote, the *x*-intercept, and, in this case, the hole that occurs at the point $(-1,0)$ since this could also be a location where the function changes sign.

x	-3	-1.5	-0.75
$g(x)$	-10	2	-0.1

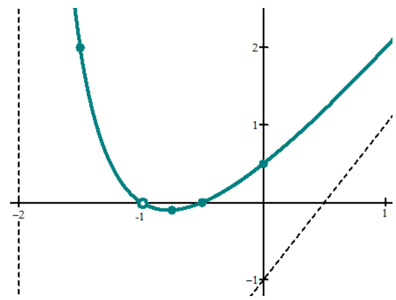
6. After marking the asymptotes with dashed lines and plotting the intercepts, hole, and additional points, we complete the graph with smooth curves that approach both asymptotes.

Figure 3.3. 9



$$y = g(x) = \frac{2x^3 + 5x^2 + 4x + 1}{x^2 + 3x + 2}$$

Figure 3.3. 10



a closer look at the interval $(-2,1)$

□

3.3 Exercises

1. What characteristics of a rational function indicate the absence of vertical asymptotes?

2. What characteristics of a rational function indicate the presence of a slant asymptote?

In Exercises 3 – 8, identify the slant asymptote of the graph of the rational function, if one exists.

3. $f(x)=\frac{24x^2+6x}{2x+1}$

4. $f(x)=\frac{4x^2-10}{2x-4}$

5. $f(x)=\frac{81x^2-18}{3x-2}$

6. $f(x)=\frac{6x^3-5x}{3x^2+4}$

7. $f(x)=\frac{x^3}{1-x}$

8. $f(x)=\frac{x^2+5x+4}{x-1}$

In Exercises 9 – 26, graph the rational function. Be sure to draw any asymptotes as dashed lines.

9. $f(x)=\frac{x^2+2x-3}{x^2-1}$

10. $r(x)=\frac{x^2-x-6}{x^2-4}$

11. $h(x)=\frac{2x^2+x-1}{x-4}$

12. $g(x)=\frac{2x^2-3x-20}{x-5}$

13. $R(x)=\frac{2x-1}{-2x^2-5x+3}$

14. $j(x)=\frac{4x}{x^2+4}$

15. $G(x)=\frac{x^2-x-12}{x^2+x-6}$

16. $F(x)=\frac{x^2-x-6}{x+1}$

17. $K(x)=\frac{x^2-x}{3-x}$

18. $H(x)=\frac{x^3+2x^2+x}{x^2-x-2}$

19. $s(x)=\frac{x^2-2x+1}{x^3+x^2-2x}$

20. $B(x)=\frac{x^2-1}{x^2-2x-3}$

21. $S(x)=\frac{x-2}{x^2-4}$

22. $k(x)=\frac{x^2-25}{x^3-6x^2+5x}$

23. $w(x)=\frac{x^3+1}{x^2-1}$

24. $Z(x)=\frac{x^3-3x+1}{x^2+1}$

25. $J(x)=\frac{-x^3+4x}{x^2-9}$

26. $q(x)=\frac{18-2x^2}{x^2-9}$

The six-step graphing procedure outlined in **Section 3.3** cannot tell us everything of importance about the graph of a rational function. Without Calculus, we need to use graphing technology to reveal the hidden mysteries of rational function behavior. Working with your classmates, use graphing technology to examine the graphs of the rational functions in **Exercises 27 – 30**. Compare and contrast their features. Which of the features can the six-step process reveal and which features cannot be detected by it?

27. $f(x)=\frac{1}{x^2+1}$

28. $g(x)=\frac{x}{x^2+1}$

29. $h(x)=\frac{x^2}{x^2+1}$

30. $r(x)=\frac{x^3}{x^2+1}$

3.4 Solving Rational Equations and Inequalities

Learning Objectives

- Solve rational equations.
- Solve rational inequalities graphically.
- Solve rational inequalities algebraically.

In this section, we solve equations and inequalities involving rational functions. We begin with rational equations, which you are likely familiar with from prior math classes.

Solving Rational Equations

As with polynomial equations, we use graphs and algebra to solve rational equations.

Example 3.4.1. Solve each of the following equations, graphically and algebraically.

1. $\frac{2x-1}{x+3}=0$
2. $\frac{2x-1}{x+3}=4$

Solution. We begin determining a graphical solution for each equation, letting $f(x)=\frac{2x-1}{x+3}$, $g(x)=0$ and $h(x)=4$. The solution to $\frac{2x-1}{x+3}=0$ is found where $f(x)=g(x)$ and the solution to $\frac{2x-1}{x+3}=4$ occurs when $f(x)=h(x)$.

Figure 3.4. 1

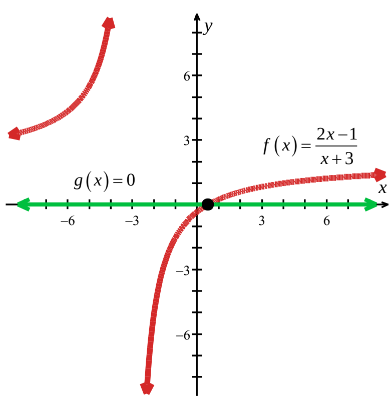
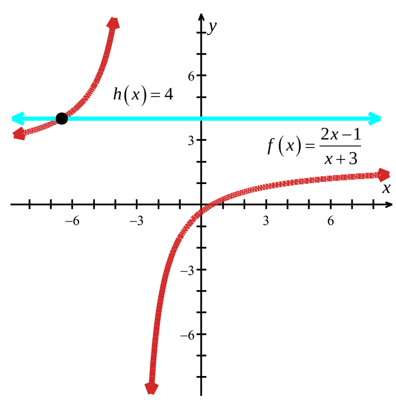


Figure 3.4. 2



While the points of intersection are not easily identifiable from these graphs, we will find through algebra that $f(x) = g(x)$ at the point $\left(\frac{1}{2}, 0\right)$ and that $f(x) = h(x)$ at the point $\left(-\frac{13}{2}, 4\right)$. Thus, the solution to $\frac{2x-1}{x+3} = 0$ is $x = \frac{1}{2}$ and the solution to $\frac{2x-1}{x+3} = 4$ is $x = -\frac{13}{2}$.

To solve the equations algebraically, we see that $\frac{2x-1}{x+3} = 0$ when $2x-1=0$, so our solution is $x = \frac{1}{2}$.

For $\frac{2x-1}{x+3} = 4$, we have

$$\begin{aligned} \frac{2x-1}{x+3} &= 4 \\ 2x-1 &= 4(x+3) \\ 2x-1 &= 4x+12 \\ -2x &= 13 \end{aligned}$$

Thus, our solution to $\frac{2x-1}{x+3} = 4$ is $x = -\frac{13}{2}$.

□

A few observations are in order before moving on.

- The solution to $\frac{2x-1}{x+3} = 0$ is the same as the x-intercept of $f(x) = \frac{2x-1}{x+3}$.
- The denominator of the function $f(x) = \frac{2x-1}{x+3}$ gives us a domain restriction when $x+3=0$; this is the location of the vertical asymptote $x = -3$.
- Each equation from the previous example has a single solution, as indicated by their graphs.

Example 3.4.2. Solve $\frac{x+1}{x^2-1} = 0$.

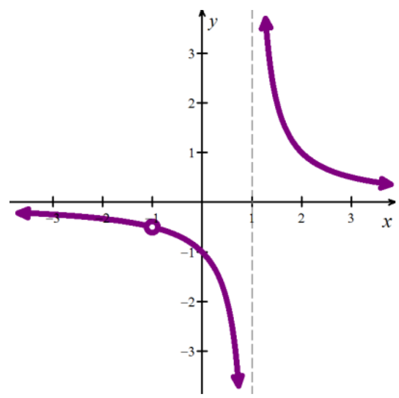
Solution. Here, we begin by solving the equation algebraically, factoring as follows.

$$\begin{aligned} \frac{x+1}{x^2-1} &= 0 \\ \frac{x+1}{(x-1)(x+1)} &= 0 \end{aligned}$$

While the only possible solution occurs when the numerator is zero, in this case that potential solution of $x = -1$ is not allowed due to the domain restriction imposed by the denominator. Thus, the equation

$\frac{x+1}{x^2-1}=0$ does not have any solutions. From the following graph of $y=\frac{x+1}{x^2-1}$, we see that this function is never equal to zero, confirming the conclusion that there is no solution.

Figure 3.4. 3



$$y=\frac{x+1}{x^2-1}$$

□

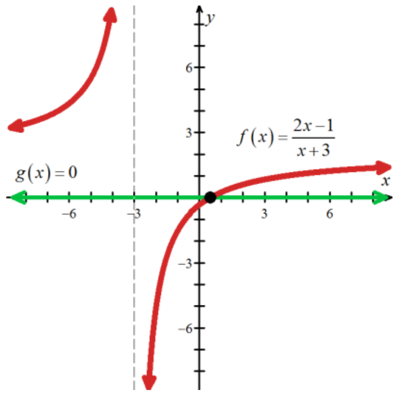
We move on to solving rational inequalities.

Solving Rational Inequalities Graphically

Example 3.4.3. Solve $\frac{2x-1}{x+3}\geq 0$ graphically.

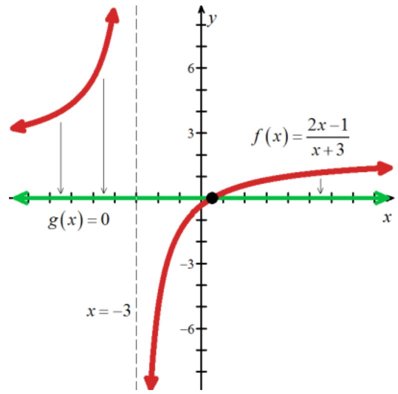
Solution. As in **Example 3.4.1**, we set $f(x)=\frac{2x-1}{x+3}$ and $g(x)=0$. The solutions to $f(x)=g(x)$ are the *x*-coordinates of the points where the graphs of $y=f(x)$ and $y=g(x)$ intersect. The solution to $f(x)\geq g(x)$ represents not only where the graphs meet, but the intervals over which the graph of $y=f(x)$ is above, $>$, the graph of $y=g(x)$. We show solutions graphically below.

Figure 3.4. 4



Graphs intersect at $\left(\frac{1}{2},0\right)$

Figure 3.4. 5



f is above g on $(-\infty,-3)$ and $\left(\frac{1}{2},\infty\right)$

We found in **Example 3.4.1** that the two graphs intersect at the point where $x=\frac{1}{2}$. Looking at the graphs of the two functions, we see that the graph of $y=f(x)$ is above the graph of $y=g(x)$ on $(-\infty,-3)$, as well as $\left(\frac{1}{2},\infty\right)$. Putting these results together, our solutions are *x* values less than −3 and *x* values greater than or equal to $\frac{1}{2}$; this is the solution set $(-\infty,-3)\cup\left[\frac{1}{2},\infty\right)$.

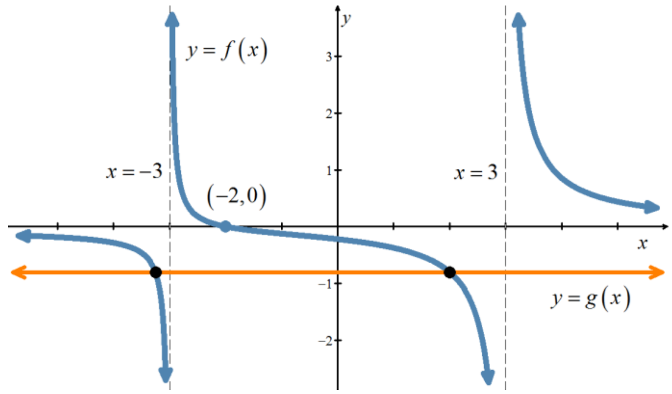
□

Before moving on to our next example, we note that the solution point $x=\frac{1}{2}$ was determined algebraically in **Example 3.4.1**. While geometric interpretations of solutions to equations and inequalities are useful, in most cases some algebraic computation is required to verify these solutions.

Example 3.4.4. Solve $\frac{x+2}{x^2-9}\geq -\frac{4}{5}$ graphically.

Solution. Let $f(x)=\frac{x+2}{x^2-9}$ and $g(x)=-\frac{4}{5}$. We use techniques from **Section 3.2** to graph the function $f(x)=\frac{x+2}{x^2-9}$, with *x*-intercept $(-2,0)$ and vertical asymptotes $x=-3$ and $x=3$. The graph of $g(x)=-\frac{4}{5}$ is a horizontal line.

Figure 3.4. 6



$$y = f(x) = \frac{x+2}{x^2-9} \text{ and } y = g(x) = -\frac{4}{5}$$

To solve $f(x) \geq g(x)$, we find where the graphs of f and g intersect, $f(x) = g(x)$, and where the graph of f is above the graph of g, $f(x) > g(x)$.

$$f(x) = g(x)$$
$$\frac{x+2}{x^2-9} = -\frac{4}{5}$$
$$5(x+2) = -4(x^2-9)$$
$$5x+10 = -4x^2+36$$
$$4x^2+5x-26=0$$
$$(4x+13)(x-2)=0$$

After setting each factor equal to zero, we find $x = -\frac{13}{4}$ and $x = 2$. Both potential solutions are acceptable since neither makes the denominator of the rational equation, x^2-9, equal to zero. Of course, we can also see that there are two solutions when we look at the graph. We next check for solutions where the graph of f is above the graph of g. This includes x values to the left of $-\frac{13}{4}$, x values between the vertical asymptote $x=-3$ and 2, and x values to the right of the vertical asymptote $x=3$. The final solution set is $\left(-\infty, -\frac{13}{4}\right] \cup (-3, 2] \cup (3, \infty)$.

□

Steps for Solving a Rational Inequality Graphically

Consider the rational inequality $f(x)<g(x)$, $f(x)\le g(x)$, $f(x)>g(x)$ or $f(x)\ge g(x)$.

1. Graph $y=f(x)$ and $y=g(x)$.
2. Find their point(s) of intersection algebraically by solving $f(x)=g(x)$.
3. (a) The solution to $f(x)<g(x)$ is the set of *x* values where the graph of $y=f(x)$ is below the graph of $y=g(x)$.

   (b) The solution to $f(x)\le g(x)$ is the set of *x* values where the graph of $y=f(x)$ intersects or is below the graph of $y=g(x)$.

   (c) The solution to $f(x)>g(x)$ is the set of *x* values where the graph of $y=f(x)$ is above the graph of $y=g(x)$.

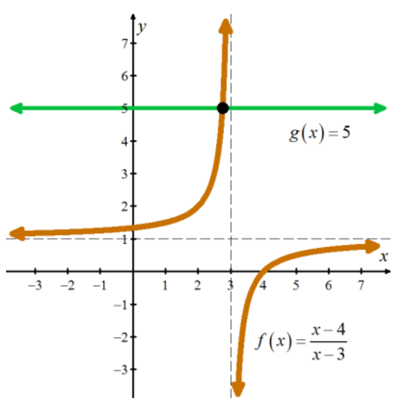
   (d) The solution to $f(x)\ge g(x)$ is the set of *x* values where the graph of $y=f(x)$ intersects or is above the graph of $y=g(x)$.

Example 3.4.5. Solve $\frac{x-4}{x-3}<5$ graphically.

Solution.

1. We graph $f(x)=\frac{x-4}{x-3}$ and $g(x)=5$, following the usual steps for graphing a rational function and a line.

Figure 3.4. 7



2. To find the point(s) of intersection, we solve $f(x)=g(x)$, or $\frac{x-4}{x-3}=5$.

$$\begin{aligned}\frac{x-4}{x-3}&=5\\x-4&=5(x-3)\\x-4&=5x-15\\-4x&=-11\\x&=\frac{11}{4}\end{aligned}$$

3. The solution of $f(x)<g(x)$ is the set of x values where the graph of $y=f(x)$ is below the graph of $y=g(x)$. Since the graph of $y=f(x)$ is below the graph of $y=g(x)$ to the left of the point of intersection, $\left(\frac{11}{4},5\right)$, and to the right of the vertical asymptote, $x=3$, we find the solution set to be $\left(-\infty,\frac{11}{4}\right)\cup(3,\infty)$.

□

Just as we have relied on algebra to some extent in solving rational inequalities graphically, we will use our knowledge of graphs of rational functions in solving rational inequalities algebraically.

Solving Rational Inequalities Algebraically

We begin with the algebraic solution to the inequality presented in **Example 3.4.5**.

Example 3.4.6. Solve $\frac{x-4}{x-3}<5$ algebraically.

Solution. For a fully algebraic solution to the inequality $\frac{x-4}{x-3}<5$, we must avoid the temptation to multiply both sides by $(x-3)$, as we did when solving the equation. The problem is that, depending on x, $(x-3)$ may be positive or it may be negative. If $(x-3)$ is positive, multiplying by $(x-3)$ does not affect the inequality, but if $(x-3)$ is negative, multiplying by $(x-3)$ would reverse the inequality. Instead of working with these two separate cases, we collect all of the terms on the left side of the inequality with 0 on the right side, and then determine the sign of the left side.

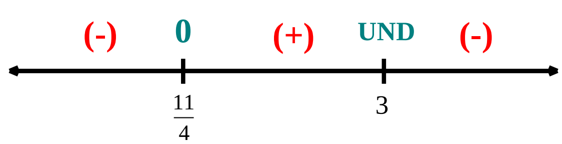
$$\frac{x-4}{x-3}<5$$
$$\frac{x-4}{x-3}-5<0$$
$$\frac{x-4-5(x-3)}{x-3}<0$$
$$\frac{-4x+11}{x-3}<0$$

We let $r(x)=\frac{-4x+11}{x-3}$ represent the left side. The only value excluded from the domain is $x=3$, which is the solution to $x-3=0$. The solution to $-4x+11=0$ gives us the zero of $x=\frac{11}{4}$. To determine the sign of r in intervals bounded by the zero and vertical asymptote, we select test values in each interval.

Interval	Test Value	Function Value	Sign
$\left(-\infty,\frac{11}{4}\right)$	$x=0$	$r(0)=-\frac{11}{3}$	$(-)$
$\left(\frac{11}{4},3\right)$	$x=\frac{23}{8}$	$r\left(\frac{23}{8}\right)=4$	$(+)$
$(3,\infty)$	$x=4$	$r(4)=-5$	$(-)$

We next construct a sign diagram in which we denote zeros with '0'. Vertical asymptotes and other *x* values that are excluded from the domain have the symbol 'UND' for 'undefined'.

Figure 3.4. 8



We find $r(x)<0$, or $(-)$, on the intervals $\left(-\infty,\frac{11}{4}\right)$ and $(3,\infty)$, so our solution set is $\left(-\infty,\frac{11}{4}\right)\cup(3,\infty)$.

□

Steps for Solving a Rational Inequality Algebraically

1. Rewrite the rational inequality as $r(x)<0$, $r(x)\le 0$, $r(x)>0$, or $r(x)\ge 0$, by moving all non-zero terms to the left side and simplifying.
2. Find the values at which $r(x)$ is undefined or zero. Equivalently, find values excluded from the domain of r and *x*-intercepts.
3. Place these values on a real number line. Write 'UND' or '0' above the values at which $r(x)$ is undefined or zero, respectively. This divides the real number line into subintervals in which $r(x)$ has the same sign. Test an *x* value in each subinterval to find the sign of $r(x)$ and record it above each interval as '$(-)$' or '$(+)$'. This determines the sign of $r(x)$ everywhere.
4. Choose the *x* values that correspond to the inequality in step 1 for the final solution.

We often shorten step 3 by just saying 'form a sign diagram for $r(x)$'.

Example 3.4.7. Solve $\frac{x+2}{x^2-9}\ge -\frac{4}{5}$ algebraically.

Solution.

1. We rewrite the inequality to get 0 on the right side.

$$\frac{x+2}{x^2-9}\ge -\frac{4}{5}$$

$$\frac{x+2}{x^2-9}+\frac{4}{5}\ge 0$$

$$\frac{5(x+2)+4(x^2-9)}{5(x^2-9)}\ge 0$$

$$\frac{4x^2+5x-26}{5(x^2-9)}\ge 0$$

$$\frac{(4x+13)(x-2)}{5(x-3)(x+3)}\ge 0$$

2. Letting $r(x)=\frac{(4x+13)(x-2)}{5(x-3)(x+3)}$, we note that $r(x)$ is undefined when $x=-3$ or $x=3$, and that $r(x)=0$ when $x=-\frac{13}{4}$ or when $x=2$.

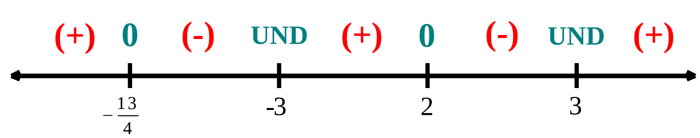
3. We begin by finding the sign in each interval determined by these values.

Interval	Test Value	Function Value	Sign
$\left(-\infty,-\frac{13}{4}\right)$	$x=-5$	$r(-5)=0.6125$	$(+)$
$\left(-\frac{13}{4},-3\right)$	$x=-\frac{25}{8}$	$r\left(-\frac{26}{8}\right)=-0.6694$	$(-)$
$(-3,2)$	$x=1$	$r(1)=0.425$	$(+)$
$(2,3)$	$x=\frac{5}{2}$	$r\left(\frac{5}{2}\right)=-0.8364$	$(-)$
$(3,\infty)$	$x=5$	$r(5)=1.2375$	$(+)$

We could achieve similar results by noting that each of the zeros in the numerator and denominator of $r(x)$ has multiplicity one. Thus, $r(x)$ will change sign across each of these values, so we could determine all of the signs by finding the sign in only one interval.

For the sign diagram, we place the *x* values on the real number line, indicating their role as zero or undefined, and add the appropriate signs to each interval.

Figure 3.4. 9



4. We are interested in where $r(x)\geq 0$. This occurs when $x\leq -\frac{13}{4}$, $-3<x\leq 2$ or $x>3$. Therefore, the solution set is $\left(-\infty,-\frac{13}{4}\right]\cup(-3,2]\cup(3,\infty)$.

□

Our last example has expressions of *x* on both sides of the inequality. We choose an algebraic solution, although the same results may be obtained through a graphical solution.

Example 3.4.8. Solve $\dfrac{2x^2-2x-14}{x-2}\leq x+1$.

Solution.

1. We begin by collecting all non-zero terms on the left side so that we have 0 on the right side.

$$\frac{2x^2-2x-14}{x-2}-x-1\le 0$$

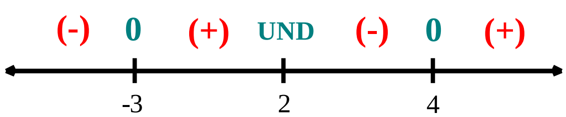
Our next move is to find a common denominator on the left side, and to combine terms so that we have a single, simplified, rational expression.

$$\frac{2x^2-2x-14}{x-2}-x-1\le 0$$
$$\frac{2x^2-2x-14-x(x-2)-1(x-2)}{x-2}\le 0$$
$$\frac{x^2-x-12}{x-2}\le 0$$

2. We let $r(x)=\frac{x^2-x-12}{x-2}=\frac{(x-4)(x+3)}{x-2}$, from which we find the domain excludes $x=2$, and the zeros are $x=-3$ and $x=4$.

3. We compose a sign chart, testing values in each interval.[13]

Figure 3.4. 10



We want to find where $r(x)\le 0$. On the intervals $(-\infty,-3)$ and $(2,4)$, we see that $r(x)<0$. After adding these intervals to the zeros −3 and 4, we get the solution set $(-\infty,-3]\cup(2,4]$.

□

[13] We have not included a table of test values here; try this on your own! As a time saver, note that we are only interested in signs, not precise values so estimating can save a lot of time. Additionally, taking advantage of multiplicities will lessen the number of intervals in which you need to calculate signs.

3.4 Exercises

1. Give an example of a rational equation that does not have a solution.

2. Give an example of a rational inequality that has a single solution.

In Exercises 3 – 8, solve the rational equation. Check domains to eliminate extraneous solutions.

3. $\dfrac{2x-3}{x+4}=0$

4. $\dfrac{2x-3}{x+4}=5$

5. $\dfrac{x}{5x+4}=3$

6. $\dfrac{3x-1}{x^2+1}=1$

7. $\dfrac{2x+17}{x+1}=x+5$

8. $\dfrac{x^2-2x+1}{x^3+x^2-2x}=1$

In Exercises 9 – 26, solve the rational inequality. Express your answer using interval notation.

9. $\dfrac{2}{x+1}>0$

10. $\dfrac{1}{x+2}\geq 0$

11. $\dfrac{4}{2x-3}\leq 0$

12. $\dfrac{2}{(x-1)(x+2)}<0$

13. $\dfrac{x-3}{x+2}\leq 0$

14. $\dfrac{x+2}{(x-1)(x-4)}\geq 0$

15. $\dfrac{(x+3)^2}{(x-1)^2(x+1)}>0$

16. $\dfrac{x}{x^2-1}>0$

17. $\dfrac{4x}{x^2+4}\geq 0$

18. $\dfrac{x^2-x-12}{x^2+x-6}>0$

19. $\dfrac{3x^2-5x-2}{x^2-9}<0$

20. $\dfrac{x^3+2x^2+x}{x^2-x-2}\geq 0$

21. $\dfrac{x^2+5x+6}{x^2-1}>0$

22. $\dfrac{3x-1}{x^2+1}\leq 1$

23. $\dfrac{2x+17}{x+1}>x+5$

24. $\dfrac{1}{x^2+1}<0$

25. $\dfrac{x^4 - 4x^3 + x^2 - 2x - 15}{x^3 - 4x^2} \geq x$

26. $\dfrac{5x^3 - 12x^2 + 9x + 10}{x^2 - 1} \geq 3x - 1$

27. The population of Sasquatch in Salt Lake County was modeled by the function $P(t) = \dfrac{150t}{t+15}$, where $t = 0$ represents the year 1803. When were there fewer than 100 Sasquatch in Salt Lake County?

Key Equations

(none for this chapter)

Key Terms

Rational Function: A function that is a ratio of polynomial functions

Vertical Asymptote: The line $x = c$ is a vertical asymptote of a function *f* if as $x \to c^{+}$ or as $x \to c^{-}$, $f(x) \to \pm\infty$.

Horizontal Asymptote: The line $y = c$ is a horizontal asymptote of a function *f* if as $x \to \pm\infty$, $f(x) \to c$.

Hole: Excluded point in a function that is not on a vertical asymptote; graphed by an open circle

Slant Asymptote: The line $y = mx + b$, where $m \neq 0$, is a slant asymptote of a function f if as $x \to \pm\infty$, $[f(x) - (mx + b)] \to 0$.

CHAPTER 4
EXPONENTIAL AND LOGARITHMIC FUNCTIONS

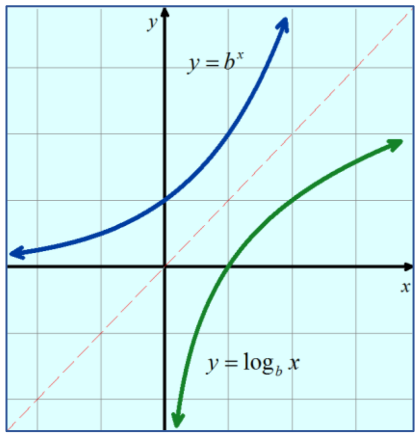


Figure 4.0. 1

Chapter Outline

Introduction

In this chapter, we investigate exponential and logarithmic functions, equations, graphs and applications. The first two sections in the chapter review ideas explored in Intermediate Algebra and lay the foundation for the last three sections where you should solidify your skills in solving, graphing and applying these functions and equations. Other key ideas in this chapter are a) the nature of the relationship between exponential and logarithmic functions (they are inverses of one another); b) how ideas about graphing, properties, and solving equations are interrelated; and c) the prevalence of exponential and logarithmic relationships in the world around us.

In Section 4.1 you are asked to think about the output of an exponential functions at 'non-standard' values, for example evaluating $f(x) = 2^x$ for $x = 3.1$ or $x = \pi$. The inquiry leads to the understanding that for basic exponential functions (in the form of $f(x) = b^x$ and $b > 0$), the domain is all real numbers, the range is $(0,\infty)$, exponential functions are one-to-one, and the *y*-intercept is $(0,1)$. From this, two major ideas are developed a) basic rules of graph transformations apply to exponential functions, and b) the fact that exponential functions are one-to-one allows us to solve many basic equations involving exponents where a common base can be found. The latter then leads to developing an understanding of the equivalence relationship between logarithms and exponentials, how the graphs of the two basic functions ($f(x)=b^x$ and $f(x)=\log_b(x)$) are reflections of one another over the line $y=x$, and finally that the domain of a basic logarithmic function is $(0,\infty)$, while the range is all real numbers (because logarithmic and exponential functions are inverses of each other.)

In Section 4.2, you continue to explore logarithmic functions and equations. The beginning of the section is devoted to building an understanding of logarithmic properties around changes of base. The section then moves to using properties of logarithms to solve equations involving exponents where a common base cannot be found. At the end of the section, you explore how to simplify and expand logarithmic expressions and how this skill applies to solving logarithmic equations.

Section 4.3 solidifies ideas introduced in 4.1 and 4.2 about exponential equations and functions. By the end of this section, you should be able to solve a wide variety of exponential equations. You should also be able to graph a variety of exponential functions involving transformations, and be able to state domain, range (a great deal of emphasis is placed here), and *x*- and *y*-intercepts of the transformed functions.

Section 4.4 also solidifies ideas from 4.1 and 4.2, but focuses on logarithmic functions. By the end of this section, you should be able to solve a wide variety of logarithmic equations. You should also be able to graph logarithmic functions involving an assortment of transformations, state the *x*- and *y*-intercept of the graphs as well as the domain (a great deal of emphasis is placed here), and identify the range.

Section 4.5 focuses on applications of logarithmic and exponential functions such as compounding interest, uninhibited growth and/or decay, and cooling/heating problems. Attention should be paid in this section to using ideas developed in earlier sections. For example, you should understand when using the Exponential Growth or Decay formula $A(t)=A_0e^{kt}$ that if kt is negative, $A(t)$ will be decreasing, or that if kt is 0, then $A(0)=A_0$, the starting condition. You should also have a solid idea of how to use your understanding of properties of logarithmic or exponential functions to manipulate or develop formulas for solving problems.

4.1 Introduction to Exponentials and Logarithms

Learning Objectives

- Evaluate exponential expressions and functions.
- Graph basic exponential functions, including transformations.
- Use the one-to-one property to solve common-base exponential equations.
- Evaluate logarithmic expressions and functions.
- Solve logarithmic equations by conversion to exponential form.
- Graph basic logarithmic functions, including transformations.

Up to this point, we have dealt with functions that involve terms like x^2 or $x^{\frac{2}{3}}$; in other words, terms of the form x^p where the base of the term, x, varies but the exponent of each term, p, remains constant. In this chapter, we study functions of the form $f(x)=b^x$ where the base b is a constant and the exponent x is the variable. This first section introduces us to 'exponentials' and 'logarithms' while the rest of the chapter will explore their properties and applications. We begin with a quick review of exponents, and revisit some basic properties.

Example	General Definition
$2^3=2\cdot 2\cdot 2=8$	$a^n=\underbrace{a\cdot a\cdot a\cdots a}_{n\text{ times}}$
$16^{\frac{1}{4}}=2$ since $2>0$ and $2^4=16$	If n is even, then $a^{\frac{1}{n}}=\sqrt[n]{a}=b$, where $b>0$ and $b^n=a$.
$(-8)^{\frac{1}{3}}=-2$ since $(-2)^3=-8$	If n is odd, then $a^{\frac{1}{n}}=\sqrt[n]{a}=b$ where $b^n=a$.
$8^{\frac{2}{3}}=\left(8^{\frac{1}{3}}\right)^2=2^2=4$	For $\dfrac{m}{n}$ in lowest terms, $a^{\frac{m}{n}}=\left(a^{\frac{1}{n}}\right)^m=(a^m)^{\frac{1}{n}}$ if $a^{\frac{1}{n}}$ is defined.

If we cannot find the exact value of an exponential term, we can estimate it. For example, $85^{\frac{1}{4}}$ is just a bit more than 3 since $3^4=81$. Using a calculator we get $85^{\frac{1}{4}}\approx 3.03637$. What if the exponent is not a rational number, for example 2^{π}? We will not formally define irrational exponents. However, since

$\pi \approx 3.14159$ and, as we will learn shortly, $y = 2^x$ is a continuously increasing function, we know that $2^3 < 2^\pi < 2^4$, or equivalently $8 < 2^\pi < 16$. In fact, by using a calculator to evaluate, we find that $2^\pi \approx 8.82498$.

Knowing that it is possible to evaluate exponentials with either rational or irrational exponents, we move on to discussing exponentials and their applications to our daily lives.

Example 4.1.1. The value of a car can be modeled by $V(x) = 25(0.8)^x$, where $x \geq 0$ is the age of the car in years and $V(x)$ is the value in thousands of dollars. Find and interpret $V(0)$ and $V(7)$.

Solution. To find $V(0)$, we replace *x* with 0 to obtain $V(0) = 25(0.8)^0 = 25$. To find $V(7)$, we replace *x* with 7 and have $V(7) = 25(0.8)^7 = 5.24288$. Since *x* represents the age of the car in years and $V(x)$ is measured in thousands of dollars, $V(0) = 25$ tells us that the purchase price of the car was $25,000, while $V(7) = 5.24288$ tells us that the value of the car after seven years is about $5,243.

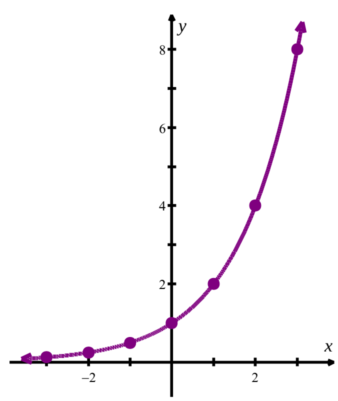
□

Basic Exponential Functions

We start our exploration of exponential functions with a graph of $f(x) = 2^x$. After creating a table of values, we plot and connect the points with a smooth curve.

x	$f(x) = 2^x$	$(x, f(x))$
-3	$2^{-3} = \frac{1}{8}$	$\left(-3, \frac{1}{8}\right)$
-2	$2^{-2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	$2^{-1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	$2^0 = 1$	$(0,1)$
1	$2^1 = 2$	$(1,2)$
2	$2^2 = 4$	$(2,4)$
3	$2^3 = 8$	$(3,8)$

Figure 4.1. 1



$y = f(x) = 2^x$

A few remarks about the graph of $f(x) = 2^x$ are in order.

- The domain of f is $(-\infty, \infty)$, the range is $(0, \infty)$ and the point $(0,1)$ is the *y*-intercept.

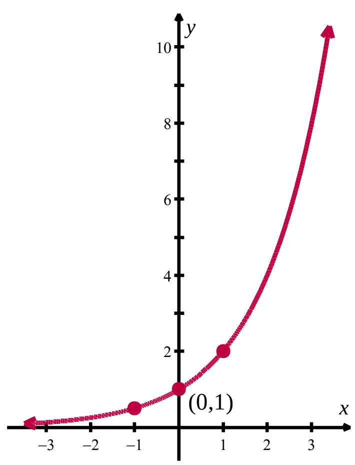
- The line $y=0$, also known as the *x*-axis, is a horizontal asymptote since, as $x \to -\infty$, $f(x) \to 0$. We note that this horizontal asymptote is different from those of rational functions in that the graph approaches the *x*-axis to the left, but not to the right.
- As *x* becomes larger, $f(x)$ grows larger as well, in fact very quickly. This is called exponential growth due to the fact that *x* appears in the exponent.
- The graph of f passes the horizontal line test, which means f is one-to-one. That is, by **Definition 1.11**, if $2^m = 2^n$ then $m=n$. We will use this property in solving exponential equations. Another consequence of $f(x)=2^x$ being one-to-one is that it is invertible (see **Theorem 1.3**). Later in this section we will see that the inverse function is called a logarithm.

We proceed with a definition of the function $f(x)=b^x$. We do not include negative values for *b* since they may result in non-real values for b^x, such as $(-2)^{1/2}=\sqrt{-2}$. We eliminate $b=0$ since 0^0 is undefined, and we also eliminate $b=1$ since $f(x)=1^x=1$ is equivalent to the constant function $y=1$.

Definition 4.1. A function of the form $f(x)=b^x$ where *b* is a positive real number, $b \neq 1$, is called an **exponential function** with **base *b***.

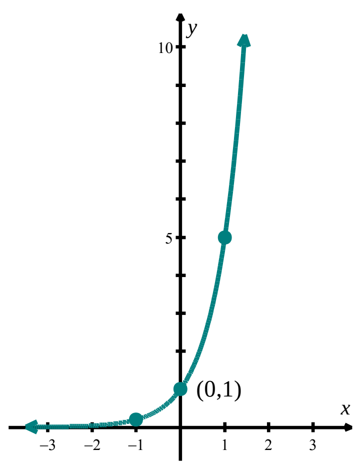
To get a better feel for the shape of exponential functions, we compare the following graphs.

Figure 4.1. 2



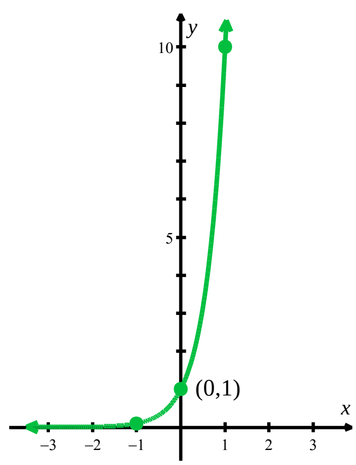
$y=f(x)=2^x$

Figure 4.1. 3



$y=g(x)=5^x$

Figure 4.1. 4



$y=h(x)=10^x$

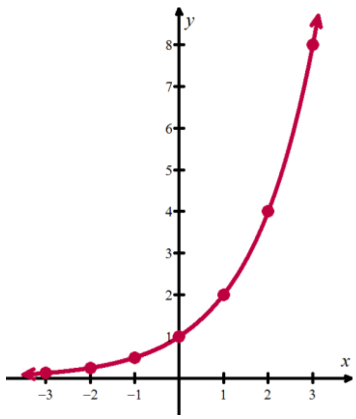
We see that all three graphs share the same domain and range, basic shape, horizontal asymptote, and *y*-intercept of $(0,1)$. We also see that the larger the base value, the faster the graph approaches its horizontal asymptote on the left side and the steeper/faster the growth is on the right side.

What if $0<b<1$? Consider $F(x)=\left(\frac{1}{2}\right)^x$. We could certainly build a table of values and connect the points, or we could take a step back and note that

$$F(x)=\left(\frac{1}{2}\right)^x=\left(2^{-1}\right)^x=2^{-x}=f(-x)$$

where $f(x)=2^x$. Thinking back to **Section 1.3**, the graph of $f(-x)$ is obtained from the graph of $f(x)$ by reflecting it across the y-axis.[1]

Figure 4.1. 5

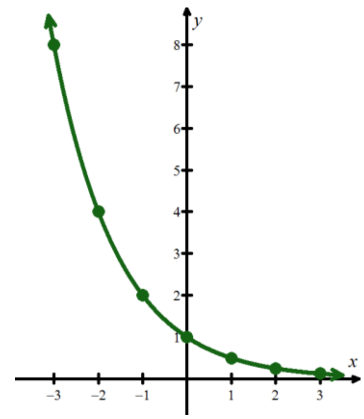


$y=f(x)=2^x$

reflect across y-axis

multiply each x-coordinate by -1

Figure 4.1. 6



$y=F(x)=2^{-x}=\left(\frac{1}{2}\right)^x$

We see that the domain and range of F match that of f, namely $(-\infty,\infty)$ and $(0,\infty)$, respectively. Like f, F is one-to-one, but while f is always increasing, F is always decreasing. We summarize the basic properties of exponential functions, as follows.

[1] Try creating a table of values for $F(x)=\left(\frac{1}{2}\right)^x$ and $G(x)=2^{-x}$ to verify this relationship.

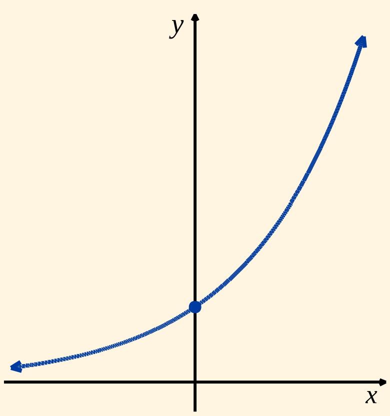
Properties of the Exponential Function $f(x)=b^x$, with $b>0$ and $b\neq 1$

- The domain of f is $(-\infty,\infty)$ and the range is $(0,\infty)$.
- The graph of f has a *y*-intercept at the point $(0,1)$ and its horizontal asymptote is $y=0$.
- f is a one-to-one function. If $b^m=b^n$ then $m=n$.

If $b>1$,

- f is always positive and increasing.
- As $x\to-\infty$, $f(x)\to 0$.
- As $x\to\infty$, $f(x)\to\infty$.

Figure 4.1. 7

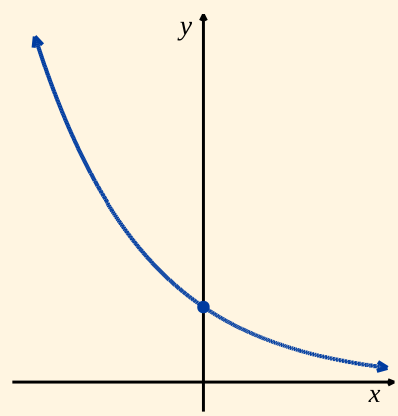


$y=b^x$, $b>1$

If $0<b<1$,

- f is always positive and decreasing.
- As $x\to-\infty$, $f(x)\to\infty$.
- As $x\to\infty$, $f(x)\to 0$.

Figure 4.1. 8



$y=b^x$, $0<b<1$

Graphing Basic Exponential Functions Using Transformations

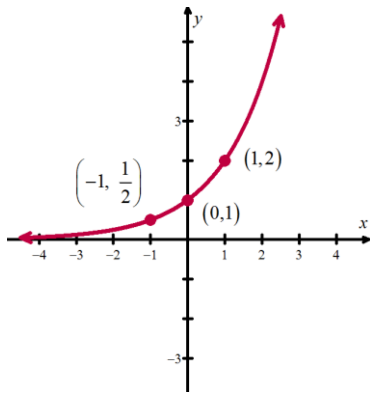
If we think of the exponential function $y=b^x$ as a new toolkit, or parent, function, we can use results from **Section 1.3** to graph transformations of $y=b^x$, as in the next example.

Example 4.1.2. Graph $f(x)=2^{x+1}-3$ by using transformations. State the domain, range and asymptote.

Solution. We graph f through transformations of the function $y=2^x$. We start with a sketch of $y=2^x$ that passes through the points $\left(-1,\frac{1}{2}\right)$, $(0,1)$ and $(1,2)$, and approaches its horizontal asymptote of $y=0$. We then apply a sequence of transformations that result in the graph of $y=f(x)=2^{x+1}-3$. Since the input changes from *x* to $x+1$, we subtract 1 from each of the *x* coordinates on the graph of

$y = 2^x$, which shifts the graph to the left by one unit. The '-3' affects the output, and so we next subtract 3 from each *y* coordinate, resulting in a shift down by 3 units.

Figure 4.1. 9

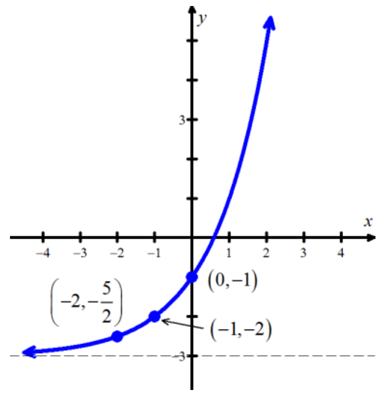


$y = 2^x$

1. Subtract 1 from each *x*-coordinate; shift left 1 unit.

2. Subtract 3 from each *y*-coordinate; shift down 3 units

Figure 4.1. 10



$y = f(x) = 2^{x+1} - 3$

Since the domain of $y = 2^x$ is the set of all real numbers, subtracting 1 from the *x* coordinates does not change the domain. However, both the range, $(0, \infty)$, and the horizontal asymptote, $y = 0$, are changed by shifting the graph down by 3 units. Thus, the domain of $f(x) = 2^{x+1} - 3$ is $(-\infty, \infty)$, its range is $(-3, \infty)$ and its horizontal asymptote is $y = -3$.

□

Solving Exponential Equations with Common Bases Using the One-to-One Property

Suppose, for instance, we wanted to solve the equation $2^x = 128$. After a moment's calculation, we find $128 = 2^7$, so we can write $2^x = 128$ as $2^x = 2^7$. The one-to-one property of exponential functions tells us that, since $2^x = 2^7$, our solution is $x = 7$.

Example 4.1.3. Solve the following equations using the one-to-one property of exponentials.

1. $9^x = \dfrac{1}{27}$
2. $2^{3x} = 16^{1-x}$

Solution.

1. To apply the one-to-one property to $9^x = \dfrac{1}{27}$, we look for a common base. Noting that both 9 and 27 are powers of 3, we will write each side of the equation with base 3.

$$9^x = \frac{1}{27}$$
$$\left(3^2\right)^x = \frac{1}{3^3}$$
$$3^{2x} = 3^{-3}$$

From the one-to-one property of exponential functions, we find $2x = -3$, from which $x = -\frac{3}{2}$.

2. Since 16 is a power of 2, we can rewrite the equation $2^{3x} = 16^{1-x}$ as follows.

$$2^{3x} = \left(2^4\right)^{1-x}$$
$$2^{3x} = 2^{(4)(1-x)}$$

Using the one-to-one property of exponential functions, we get $3x = 4(1-x)$ which gives us $x = \frac{4}{7}$.

□

Definition of Logarithms

We begin with the observation that $2^x = 8$ means *x* is the power to which we raise 2 to get 8, or 'the power of 2 that gives 8'. We adopt a special notation for this *x* value: $\log_2 8$, read as 'logarithm in base 2 of 8'. Since we can see that $2^3 = 8$, we conclude that $\log_2 8 = 3$.

Definition 4.2. For $y > 0$ and b a positive constant other than 1, $\log_b y$ is called a **logarithm in base *b* of *y***, and is the power of b that gives *y*.

A few notes:

- It is common to say 'log' in place of 'logarithm'.
- If $\log_b y = x$ then $y = b^x$.
- Because of the connection between logarithms and exponentials, we must have $b > 0$ and $b \neq 1$ since we have defined the exponential function $f(x) = b^x$ with the same conditions on b.
- Since the range of the exponential function $f(x) = b^x$ is the set of positive numbers, we must have the same condition: $b^x = y > 0$.

Example 4.1.4. Find the exact values of the following logarithms.

1. $\log_3 81$
2. $\log_5\left(\frac{1}{25}\right)$
3. $\log_8 4$

Solution.

1. Since $\log_3 81$ is the power of 3 that gives 81, we check powers of 3: $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$. We see that the answer is 4. That is, $\log_3 81 = 4$ since $3^4 = 81$.

2. For $\log_5\left(\frac{1}{25}\right)$, we look for the power of 5 that gives $\frac{1}{25}$. Since $\frac{1}{25} = \frac{1}{5^2} = 5^{-2}$, we find $\log_5\left(\frac{1}{25}\right) = -2$.

3. To determine the value of $\log_8 4$, we search for the power of 8 that gives 4. This is a bit harder to guess, so we look for the value of *x* for which $\log_8 4 = x$, or equivalently $8^x = 4$. We can then solve this equation using the one-to-one property of exponential functions.

$$\begin{aligned} 8^x &= 4 \\ \left(2^3\right)^x &= 2^2 \\ 2^{3x} &= 2^2 \end{aligned}$$

We find $3x = 2$, from which $x = \frac{2}{3}$. So, $\log_8(4) = \frac{2}{3}$. We saw in the first part of this section that $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = 2^2 = 4$, which verifies this answer.

□

While we cannot find the exact value of every logarithm, as we did in the previous example, we can estimate values of logarithms. For example, $\log_3 85$ is just a bit more that 4, since $3^4 = 81$ and 85 is a bit more than 81. The only logarithms that we can find the exact value of are those of the form $\log_b y$ for which the argument *y* is a power of the base b.

We note that the statements $\log_3 81 = 4$ and $3^4 = 81$ contain the same information. One is stated using a logarithm and the other using an exponential. This equivalency is stated, in general, as follows.

Equivalence Relation Between Logarithms and Exponentials

For $y > 0$ and b a positive constant other than 1, $\log_b y = x \Leftrightarrow b^x = y$.

The statement $\log_b y = x$ is the logarithmic form and the statement $b^x = y$ is the exponential form.

Example 4.1.5. Convert the following from logarithmic form to exponential form, or vice versa.

1. $\log_6 \sqrt{6} = \frac{1}{2}$
2. $10^{-3} = \frac{1}{1000}$

Solution.

1. Using the equivalence relation, $\log_6 \sqrt{6} = \frac{1}{2} \Rightarrow 6^{\frac{1}{2}} = \sqrt{6}$.
2. Using the equivalence relation, $10^{-3} = \frac{1}{1000} \Rightarrow \log_{10}\left(\frac{1}{1000}\right) = -3$.

□

As we will see in the next example, we can use the equivalency between logarithms and exponentials in solving logarithmic equations.

Solving Logarithmic Equations by Conversion to Exponential Form

Example 4.1.6. Solve the following equations by first changing forms.

1. $\log_3 x = 4$
2. $\log_2 (x-1) = 3$

Solution.

1. We convert $\log_3 x = 4$ to exponential form to get $x = 3^4$, from which we have $x = 81$.
2. We start by converting $\log_2 (x-1) = 3$ to the exponential form $x - 1 = 2^3$. Then $x = 2^3 + 1$, for a final answer of $x = 9$.

□

Basic Logarithmic Functions

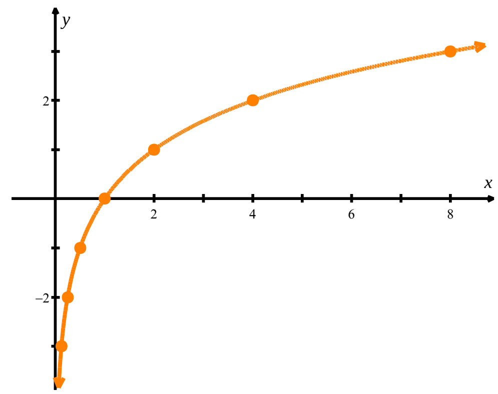
We are now ready for our definition of logarithmic functions.

> **Definition 4.3.** A function of the form $f(x) = \log_b x$ where b is a positive real number, $b \neq 1$, and $x > 0$, is called a **logarithmic function** with **base *b***.

We begin our exploration of logarithmic functions with a graph of $g(x) = \log_2 x$. After creating a table of values, we plot and connect the points with a smooth curve. Since we can easily evaluate the logarithm for powers of the base 2, we choose such numbers for the *x* values.

x	$g(x)=\log_2 x$	$(x,g(x))$
$2^{-3}=\frac{1}{8}$	$\log_2 2^{-3}=-3$	$\left(\frac{1}{8},-3\right)$
$2^{-2}=\frac{1}{4}$	$\log_2 2^{-2}=-2$	$\left(\frac{1}{4},-2\right)$
$2^{-1}=\frac{1}{2}$	$\log_2 2^{-1}=-1$	$\left(\frac{1}{2},-1\right)$
$2^{0}=1$	$\log_2 2^{0}=0$	$(1,0)$
$2^{1}=2$	$\log_2 2^{1}=1$	$(2,1)$
$2^{2}=4$	$\log_2 2^{2}=2$	$(4,2)$
$2^{3}=8$	$\log_2 2^{3}=3$	$(8,3)$

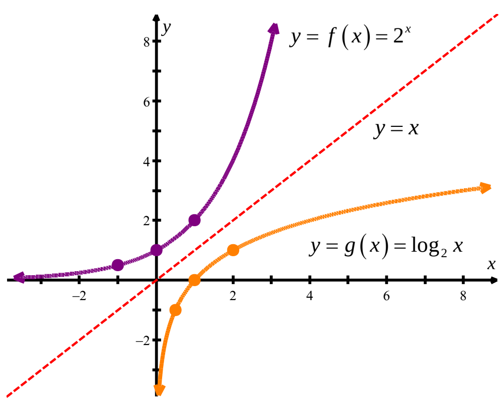
Figure 4.1. 11



$y=g(x)=\log_2 x$

Comparing the graph of $g(x)=\log_2 x$ with the graph of $f(x)=2^x$, shown at the beginning of this section, we see that the two graphs are symmetric about the line $y=x$, as indicated by the fact that the x and y coordinates of their points are interchanged. It follows that f and g are inverses of each other.[2]

Figure 4.1. 12



To show this more formally, consider $(f\circ g)(x)$ and $(g\circ f)(x)$.

[2] Refer to **Section 1.5**.

$$\begin{aligned}(f\circ g)(x) &= f(g(x))\\ &= f(\log_2 x)\\ &= 2^{\log_2 x}\\ &= x\end{aligned}$$

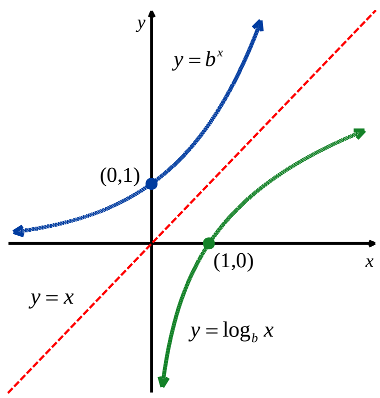
since $\log_2 x$ is the power of 2 that gives x

$$\begin{aligned}(g\circ f)(x) &= g(f(x))\\ &= g(2^x)\\ &= \log_2 2^x\\ &= x\end{aligned}$$

since x is the power of 2 that gives 2^x

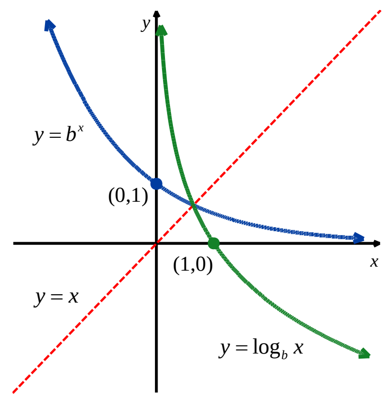
Since $(f\circ g)(x)=x$ and $(g\circ f)(x)=x$, we have shown f and g are inverses of each other.[3] In general, $f(x)=b^x$ and $g(x)=\log_b x$ are inverses of each other.[4] We use this inverse property to graph basic logarithmic functions by reflecting graphs of basic exponential functions across the line $y=x$.

Figure 4.1. 13



$b>1$

Figure 4.1. 14



$0<b<1$

As seen from the graphs, the *x*-intercept of the logarithmic function $f(x)=\log_b x$ is the point $(1,0)$ and its vertical asymptote is the *y*-axis. These and other properties are summarized, as follows.

---

[3] See **Definition 1.10**.
[4] This general case is verified in **Section 4.2**.

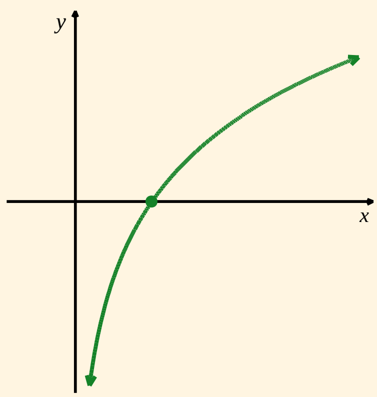
Properties of the Logarithmic Function $f(x) = \log_b x$, with $b > 0$ and $b \neq 1$

- The domain of f is $(0, \infty)$ and the range is $(-\infty, \infty)$.
- The graph of f has an *x*-intercept at the point $(1, 0)$ and its vertical asymptote is $x = 0$.
- f is a one-to-one function. If $\log_b u = \log_b v$ then $u = v$.

If $b > 1$,

- f is always increasing.
- As $x \to 0^+$, $f(x) \to -\infty$.
- As $x \to \infty$, $f(x) \to \infty$.

Figure 4.1. 15

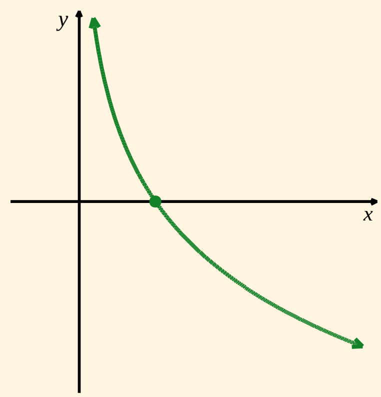


$y = \log_b x, \ b > 1$

If $0 < b < 1$,

- f is always decreasing.
- As $x \to 0^+$, $f(x) \to \infty$.
- As $x \to \infty$, $f(x) \to -\infty$.

Figure 4.1. 16



$y = \log_b x, \ 0 < b < 1$

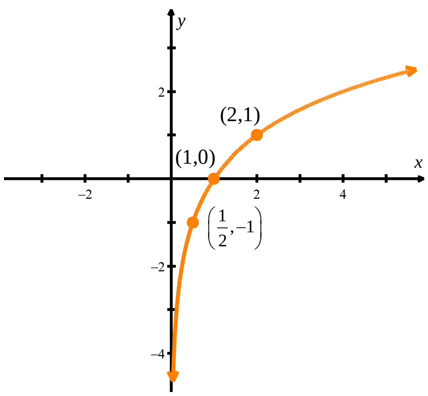
Graphing Basic Logarithmic Functions Using Transformations

If we think of the logarithmic function $f(x) = \log_b x$ as another new toolkit, or parent, function, we can use results from **Section 1.3** to graph transformations of $f(x) = \log_b x$, as in the next two examples.

Example 4.1.7. Graph $f(x) = \log_2(x+3) - 1$. State the domain, range and asymptotes.

Solution. We graph f through transformations of the function $y = \log_2 x$. The change of input from *x* to $x + 3$ means that we need to subtract 3 from each *x* coordinate of points on the graph of $y = \log_2 x$; this shifts the graph to the left by three units. The change of -1 to the output means that we next subtract 1 from each *y* coordinate, or shift the graph down by one unit.

Figure 4.1. 17

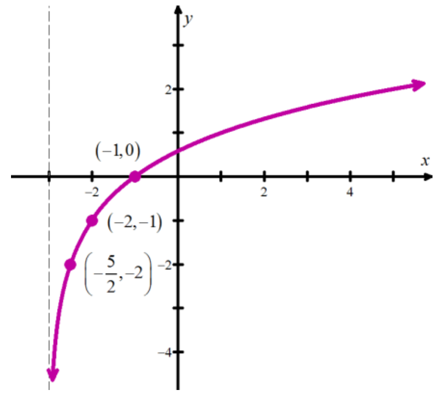


$y = \log_2 x$

1. Shift left 3 units.

2. Shift down 1 unit.

Figure 4.1. 18



$y = f(x) = \log_2(x+3) - 1$

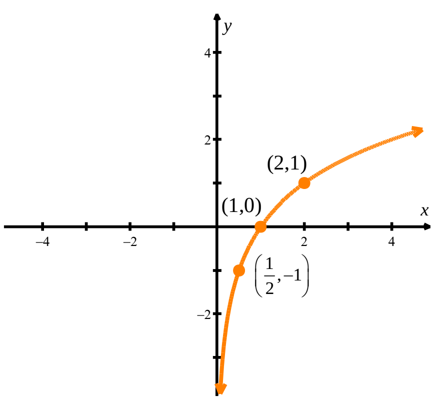
The domain of f is $(-3,\infty)$, its range is $(-\infty,\infty)$, and the vertical asymptote is $x=-3$.

□

Example 4.1.8. Graph $f(x) = 1+\log_2(-x)$. State the domain, range and asymptote.

Solution. We graph f through transformations of the function $y=\log_2 x$. The change of input from *x* to –*x* means that we need to multiply each *x* coordinate of the points on $y=\log_2 x$ by −1; this reflects the graph across the *y*-axis. Adding 1 to the output means that we next need to add one unit to each *y* coordinate; this shifts the graph up by one unit.

Figure 4.1. 19

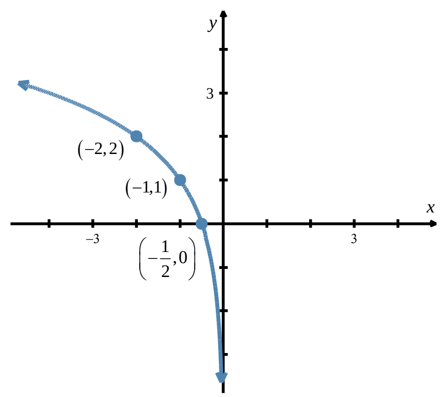


$y = \log_2 x$

1. Reflect across *x* axis.

2. Shift up one unit.

Figure 4.1. 20



$y = f(x) = 1+\log_2(-x)$

The domain of f is $(-\infty,0)$, its range is $(-\infty,\infty)$, and its vertical asymptote is the line $x=0$.

□

4.1 Exercises

1. The inverse of every logarithmic function is an exponential function and vice-versa. What does this tell us about the relationship between the coordinates of the points on the graphs of each?

2. Does the graph of a logarithmic function have a horizontal asymptote? Explain.

In Exercises 3 – 6, rewrite each equation in exponential form.

3. $\log_5 25 = 2$
4. $\log_{25} 5 = \frac{1}{2}$
5. $\log_3\left(\frac{1}{81}\right) = -4$
6. $\log_{\frac{4}{3}}\left(\frac{3}{4}\right) = -1$

In Exercises 7 – 10, rewrite each equation in logarithmic form.

7. $2^3 = 8$
8. $5^{-3} = \frac{1}{125}$
9. $4^{\frac{5}{2}} = 32$
10. $\left(\frac{1}{3}\right)^{-2} = 9$

In Exercises 11 – 28, simplify the expression without using a calculator.

11. $\log_3 27$
12. $\log_6 216$
13. $\log_2 32$
14. $\log_6\left(\frac{1}{36}\right)$
15. $\log_{27} 9$
16. $\log_{36} 216$
17. $\log_{\frac{1}{5}} 625$
18. $\log_{\frac{1}{6}} 216$
19. $\log_{36} 36$
20. $\log_4 8$
21. $\log_6 1$
22. $\log_{13} \sqrt{13}$
23. $\log_{36} \sqrt[4]{36}$
24. $7^{\log_7 3}$
25. $36^{\log_{36} 216}$
26. $\log_{36} 36^{216}$
27. $\log_5 3^{\log_3 5}$
28. $\log_2 3^{\log_3 2}$

In Exercises 29 – 37, solve the equation using the one-to-one property of exponential equations.

29. $2^{4x} = 8$
30. $3^{x-1} = 27$
31. $5^{2x-1} = 125$
32. $4^{2x} = \frac{1}{2}$
33. $8^x = \frac{1}{128}$
34. $2^{x^3-x} = 1$
35. $3^{7x} = 81^{4-2x}$
36. $3^{7x+2} = \left(\frac{1}{9}\right)^{2x}$
37. $\left(\frac{1}{2}\right)^{3x} = 2^{x+4}$

In Exercises 38 – 46, solve the equation by converting the logarithmic equation to exponential form.

38. $\log_3 x = 2$
39. $\log_2 x = 6$
40. $\log_9 x = \frac{1}{2}$
41. $\log_6 x = -3$
42. $\log_2 x = -3$
43. $\log_3 x = 3$

44. $\log_{18} x = 2$

45. $\log_3 (7-2x) = 2$

46. $\log_{\frac{1}{2}} (2x-1) = -3$

In Exercises 47 – 52, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and using transformations. Track at least three points of your choice through the transformations. State the domain, range and asymptote of g.

47. $f(x) = 2^x$; $g(x) = 2^x - 1$

48. $f(x) = 2^x$; $g(x) = 2^{-x}$

49. $f(x) = 2^x$; $g(x) = 2^x + 3$

50. $f(x) = 2^x$; $g(x) = 2^{x-2}$

51. $f(x) = \left(\frac{1}{3}\right)^x$; $g(x) = \left(\frac{1}{3}\right)^{x-1}$

52. $f(x) = 3^x$; $g(x) = 3^{-x} + 2$

In Exercises 53 – 58, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and using transformations. Track at least three points of your choice through the transformations. State the domain, range and asymptote of g.

53. $f(x) = \log_2 x$; $g(x) = \log_2 (x+1)$

54. $f(x) = \log_2 x$; $g(x) = \log_2 (x-2)$

55. $f(x) = \log_{\frac{1}{3}} x$; $g(x) = \log_{\frac{1}{3}} x + 1$

56. $f(x) = \log_3 x$; $g(x) = -\log_3 (x-2)$

57. $f(x) = \log_4 x$; $g(x) = 2\log_4 x$

58. $f(x) = \log_3 x$; $g(x) = 2\log_3 (x+4) - 1$

(Logarithmic Scales) In Exercises 59 – 61, we introduce three widely used measurement scales which involve common logarithms: the Richter scale, the decibel scale and the pH scale. The computations involved in all three scales are nearly identical so pay attention to the subtle differences.

59. Earthquakes are complicated events and it is not our intent to provide a complete discussion of the science involved in them. Instead, we refer the interested reader to a solid course in Geology or the U.S. Geological Survey’s Earthquake Hazards Program. The Richter scale measures the magnitude of an earthquake by comparing the amplitude of the seismic waves of the given earthquake to those of a ‘magnitude 0 event’, which was chosen to be a seismograph reading of 0.001 millimeters recorded on a seismometer 100 kilometers from the earthquake’s epicenter. Specifically, the magnitude of an earthquake is given by

$$M(x) = \log\left(\frac{x}{0.001}\right)$$

where x is the seismograph reading in millimeters of the earthquake recorded 100 kilometers from the epicenter.

(a) Show that $M(0.001) = 0$.

(b) Compute $M(80,000)$.

(c) Show that an earthquake that registered 6.7 on the Richter scale had a seismograph reading ten times larger than one that measured 5.7.

60. While the decibel scale can be used in many disciplines, we shall restrict our attention to its use in acoustics, specifically its use in measuring the intensity level of sound. The Sound Intensity Level *L* (measured in decibels) of a sound intensity *I* (measured in watts per square meter) is given by

$$L(I) = 10\log\left(\frac{I}{10^{-12}}\right).$$

Like the Richter scale, this scale compares *I* to a baseline: $10^{-12}\frac{W}{m^2}$ is the threshold of human hearing.

(a) Compute $L(10^{-6})$.

(b) Damage to your hearing can start with short term exposure to sound levels around 115 decibels. What intensity *I* is needed to produce this level?

(c) Compute $L(1)$. How does this compare with the threshold of pain which is around 140 decibels?

61. The pH of a solution is a measure of its acidity or alkalinity. Specifically, $\text{pH} = -\log\left[\text{H}^+\right]$ where $\left[\text{H}^+\right]$ is the hydrogen ion concentration in moles per liter. A solution with a pH less than 7 is an acid, one with a pH greater than 7 is a base (alkaline) and a pH of 7 is regarded as neutral.

(a) The hygrogen ion concentration of pure water is $\left[\text{H}^+\right] = 10^{-7}$. Find its pH.

(b) Find the pH of a solution with $\left[\text{H}^+\right] = 6.3\times10^{-13}$.

(c) The pH of gastric acid (the acid in your stomach) is about 0.7. What is the corresponding hydrogen ion concentration?

4.2 Properties of Logarithms

Learning Objectives

- Use the definition of common and natural logarithms in solving equations and simplifying expressions.
- Use the change of base property to evaluate logarithms.
- Solve exponential equations using logarithmic properties.
- Combine and/or expand logarithmic expressions.
- Solve basic logarithmic equations using properties of logarithms and exponentials.

Common and Natural Logarithms

A commonly used base for logarithms is base 10. Logarithms with base 10 are referred to as **common logarithms** and we usually leave off the base when writing these logarithms, so that $\log_{10} x = \log x$. Below is an application of common logarithms.

Example 4.2.1. The state of Utah has one of the fastest growing populations in the country with an annual growth rate of about 2%. The number of years it takes for a population with the growth rate of r to double in size is $y = \frac{\log 2}{\log(1+r)}$. At the current rate, how many years will it take for the population of Utah to double in size? If the growth rate is reduced to 1.5%, how many years will it take for the population to double in size?

Solution. For an annual growth rate of 2%, we set $r = 0.02$ and use a calculator to determine *y*.

$$y = \frac{\log 2}{\log(1+0.02)} = \frac{\log 2}{\log 1.02} \approx 35.0028$$

We find that it will take about 35 years for the population of Utah to double in size. For the annual growth rate of 1.5%, we set $r = 0.015$.

$$y = \frac{\log 2}{\log(1+0.015)} = \frac{\log 2}{\log 1.015} \approx 46.5555$$

By reducing the growth rate to 1.5%, the time for the population of Utah to double in size becomes approximately 46 and one-half years.

□

Another important base is an irrational number designated by the letter 'e'. This number arises naturally in the study of Calculus and financial transactions. The choice of the letter e is by the prolific mathematician Leonhard Euler who popularized it. The value of e is approximately 2.718281828459.

The number e may be defined as the value of the exponential term $\left(1+\frac{1}{m}\right)^m$ where the value of m gets large, as seen from the following table.

m	$\left(1+\frac{1}{m}\right)^m$
1	$\left(1+\frac{1}{1}\right)^1 = 2$
100	$\left(1+\frac{1}{100}\right)^{100} = 2.70481\cdots$
10,000	$\left(1+\frac{1}{10000}\right)^{10000} = 2.71814\cdots$
1,000,000	$\left(1+\frac{1}{1000000}\right)^{1000000} = 2.71828\cdots$

A **natural exponential** is an exponential with base e. Since $e>1$, we have already studied the graph and properties of the exponential function $f(x)=e^x$. A logarithm with base e is called a **natural logarithm**. Natural logarithms are used even more often than common logarithms in mathematics, and so have their own notation. We use the notation 'ln' for 'natural logarithm', so that $\log_e x = \ln x$.

Properties of Logarithms

We next introduce some properties of logarithms that we will apply in calculating values of logarithms, combining or expanding logarithmic expressions, and solving both exponential and logarithmic equations. We begin with a property that will allow us to rewrite a logarithm as a quotient of logarithms having a different base that the original logarithm.

Example 4.2.2. Show that $\log_2 3 = \frac{\log 3}{\log 2}$.

Solution. By definition, the logarithm $\log_2 3$ is the power of 2 what gives 3, so we must show this power is $\frac{\log 3}{\log 2}$. We let $m=\log 3$ and $n=\log 2$ and proceed with converting each equation to exponential form.

$$m=\log 3 \Rightarrow 3=10^m$$
$$n=\log 2 \Rightarrow 2=10^n \Rightarrow 2^{\frac{1}{n}}=10$$

It follows that

$$\begin{aligned} 2^{\frac{\log 3}{\log 2}} &= 2^{\frac{m}{n}} \\ &= \left(2^{\frac{1}{n}}\right)^m \\ &= (10)^m \\ &= 3 \end{aligned}$$

Thus, $\frac{\log 3}{\log 2}$ is the power of 2 that gives 3, and we conclude that $\log_2 3=\frac{\log 3}{\log 2}$.

□

This property can be generalized as follows.

Change of Base Property

Let a and b be positive numbers, not equal to 1, and let x be a positive number.

$$\log_b x=\frac{\log_a x}{\log_a b}$$

The two logarithm buttons commonly found on calculators are the ‘LOG’ and ‘LN’ buttons which correspond to the common and natural logarithms, respectively. For calculation purposes, we choose the new base a to be 10 or e:

$$\log_b x=\frac{\log x}{\log b}=\frac{\ln x}{\ln b}$$

Example 4.2.3. Convert the expression $\log_4 5$ to base e. Verify your answer using a calculator.

Solution. Applying the change of base property with $b=4$, $x=5$ and $a=e$ leads us to write $\log_4 5=\frac{\ln 5}{\ln 4}$. Evaluating this with a calculator results in $\frac{\ln 5}{\ln 4}\approx 1.16$. How do we check that this really is the value of $\log_4 5$? By definition, $\log_4 5$ is the exponent we put on 4 to get 5. You can check this using a calculator.[5]

□

Our next properties are simply restatements of results from $f(x)=b^x$ and $g(x)=\log_b x$ being inverses of each other. In **Section 4.1**, we verified that these functions are inverses when $b=2$. Before moving on, we verify that $f(x)=b^x$ and $g(x)=\log_b x$ are inverses for any value of b.

$$\begin{aligned}(f\circ g)(x)&=f(g(x))\\&=f(\log_b x)\\&=b^{\log_b x}\\&=x\end{aligned}$$

since $\log_b x$ is the power of b that gives x

$$\begin{aligned}(g\circ f)(x)&=g(f(x))\\&=g(b^x)\\&=\log_b b^x\\&=x\end{aligned}$$

since x is the power of b that gives b^x

The last step in each part of the verification process gives us an inverse property, as stated below.

Inverse Properties

Let b be a positive number, not equal to 1.

$b^{\log_b x}=x$, for any positive number x

$\log_b b^x = x$, for any real number x

A result of these properties is that we can think of the exponential and logarithmic functions as 'undoing' each other.

Example 4.2.4. Show that $\log_b b=1$ and that $\log_b 1=0$.

Solution. We use the property $\log_b b^x = x$ to verify each of these equations. For $\log_b b$, we note that $\log_b b=\log_b b^1$, and it follows directly that this is equivalent to 1. We also note that $\log_b 1$ can be written as $\log_b b^0$, and confirm that this is equivalent to 0.

□

We continue with an example that leads into our next property of logarithms.

Example 4.2.5. Show that $\log 4^5 = 5\log 4$.

[5] Of course, if it is lying to us about the first answer it gave us, at least it is being consistent. ☺

Solution. By definition, $\log 4^5$ is the power of 10 that gives 4^5, so we must show this power is $5\log 4$. We proceed with a simplification of $10^{5\log 4}$.

$$\begin{aligned} 10^{5\log 4} &= \left(10^{\log 4}\right)^5 \\ &= (4)^5 \\ &= 4^5 \end{aligned}$$

We have shown that $10^{5\log 4} = 4^5$, and thus verified that $\log 4^5 = 5\log 4$.

□

This property can be generalized as follows.

Exponent Property of Logarithms

Let *b* be a positive number, not equal to 1, *m* any real number, and *x* a positive number.

$$\log_b x^m = m\log_b x$$

This frequently used property can be thought of as 'putting the exponent of an argument out front as the coefficient'. A simple, but useful, consequence is obtained by setting $x = b$:

$$\begin{aligned} \log_b b^m &= m\log_b b \\ &= m(1) \\ &= m \end{aligned}$$

That is, $\log_b b^m = m$.

Using Properties of Logarithms to Solve Exponential Equations

Example 4.2.6. Solve $2^x = 3$.

Solution. The *x* we are looking for is the power of 2 that gives 3. By definition, $x = \log_2 3$, and by the change of base property, we find $x = \log_2 3 = \frac{\log 3}{\log 2}$. Another way we can solve this equation is to take the common logarithm of both sides, after which we use the exponent property of logarithms to move the exponent *x* in front of the logarithm.

$$\begin{aligned} 2^x &= 3 \\ \log 2^x &= \log 3 \\ x\log 2 &= \log 3 \\ x &= \frac{\log 3}{\log 2} \end{aligned}$$

□

We often use this technique of taking the logarithm of both sides, in the same base, of an exponential equation, as in the following example.

Example 4.2.7. Solve $7^{x+1} = 3^{-2x}$.

Solution. We begin by taking the natural logarithm of both sides, and then apply the exponent property of logarithms to both sides of the equation.

$$\begin{aligned} 7^{x+1} &= 3^{-2x} \\ \ln 7^{x+1} &= \ln 3^{-2x} \\ (x+1)\ln 7 &= -2x\ln 3 \end{aligned}$$

Even though the resulting equation looks complicated, keep in mind that $\ln 7$ and $\ln 3$ are just constants. The equation $(x+1)\ln 7 = -2x\ln 3$ is actually a linear equation and as such we gather all of the terms with x on one side, and the constants on the other. We then divide both sides by the coefficient of x, which we obtain by factoring.

$$\begin{aligned} (x+1)\ln 7 &= -2x\ln 3 \\ x\ln 7 + \ln 7 &= -2x\ln 3 \\ x\ln 7 + 2x\ln 3 &= -\ln 7 \\ x(\ln 7 + 2\ln 3) &= -\ln 7 && \text{factor out } x \\ x &= \frac{-\ln 7}{\ln 7 + 2\ln 3} \end{aligned}$$

The answer we have found is an exact solution. An approximate solution, found using a calculator, is $x \approx -0.47$.

□

In the previous example, we could have used any logarithm base; natural logarithm is the most frequently used, but you may see the base 3 or 7 used for this example. The final anwers may look different, but they all represent the same value. The next two examples lead us to our last two properties of logarithms.

Example 4.2.8. Show that $\ln 15 = \ln 3 + \ln 5$.

Solution. We show this relationship using the definition of logarithms. Since $\ln 15$ is the power of e that gives 15, we want to show this power is $\ln 3 + \ln 5$. Let's start with $e^{\ln 3 + \ln 5}$ and apply properties of exponents, along with the inverse property of logarithms.

$$\begin{aligned} e^{\ln 3 + \ln 5} &= e^{\ln 3} e^{\ln 5} \\ &= (3)(5) \\ &= 15 \end{aligned}$$

Thus, $\ln(3) + \ln(5)$ is the power of e that gives 15 and we have shown that $\ln(15) = \ln(3) + \ln(5)$.

□

Example 4.2.9. Show that $\log\left(\frac{u}{v}\right) = \log u - \log v$, where u and v are positive numbers.

Solution. By the definition of logarithms, we need to show the power of 10 that gives $\frac{u}{v}$ is $\log u - \log v$. We begin with $10^{\log u - \log v}$, to which we apply properties of exponents and the inverse property of logarithms.

$$\begin{aligned} 10^{\log u - \log v} &= 10^{\log u} 10^{-\log v} \\ &= \frac{10^{\log u}}{10^{\log v}} \\ &= \frac{u}{v} \end{aligned}$$

So $\log u - \log v$ is the power of 10 that gives $\frac{u}{v}$, verifying that $\log\left(\frac{u}{v}\right) = \log u - \log v$.

□

These properties can be generalized as follows.

Sum and Difference Properties of Logarithms

Let b be a positive number, not equal to 1, and let u and v be positive numbers.

$$\log_b(uv) = \log_b u + \log_b v$$

$$\log_b\left(\frac{u}{v}\right) = \log_b u - \log_b v$$

These are not really two different properties, since the second can be derived from the first as follows.

$$\begin{aligned} \log_b\left(\frac{u}{v}\right) &= \log_b\left(uv^{-1}\right) \\ &= \log_b u + \log_b v^{-1} \\ &= \log_b u - \log_b v \end{aligned}$$

It may help to remember these properties in words:

Log of a product is the sum of logs of its factors: $\log_b(uv) = \log_b u + \log_b v$.

Log of a quotient is the log of its top minus the log of its bottom: $\log_b\left(\frac{u}{v}\right) = \log_b u - \log_b v$.

Combining and/or Expanding Logarithmic Expressions

Example 4.2.10. Expand the following using the properties of logarithms. Write exponents as coefficients and simplify. Assume that all variables represent positive real numbers.

1. $\log_{0.1}\left(10x^2\right)$

2. $\log\sqrt[3]{\dfrac{100x^2}{y\,z^5}}$

Solution.

1. We expand $\log_{0.1}\left(10x^2\right)$ as follows.

$$\begin{aligned}\log_{0.1}\left(10x^2\right) &= \log_{0.1}10+\log_{0.1}x^2 && \text{sum property}\\ &= \log_{0.1}10+2\log_{0.1}x && \text{exponent property}\\ &= -1+2\log_{0.1}x && \text{since } (0.1)^{-1}=10\end{aligned}$$

We find that $\log_{0.1}\left(10x^2\right)$, when expanded, is equivalent to $2\log_{0.1}x-1$.

2. In expanding $\log\sqrt[3]{\dfrac{100x^2}{y\,z^5}}$, we begin by writing the cube root as the exponent $\dfrac{1}{3}$.

$$\begin{aligned}\log\sqrt[3]{\frac{100x^2}{y\,z^5}} &= \log\left(\frac{100x^2}{y\,z^5}\right)^{\frac{1}{3}}\\ &= \frac{1}{3}\log\left(\frac{100x^2}{y\,z^5}\right) && \text{exponent property}\\ &= \frac{1}{3}\left[\log\left(100x^2\right)-\log\left(y\,z^5\right)\right] && \text{difference property}\\ &= \frac{1}{3}\log\left(100x^2\right)-\frac{1}{3}\log\left(y\,z^5\right)\\ &= \frac{1}{3}\left(\log 100+\log x^2\right)-\frac{1}{3}\left(\log y+\log z^5\right) && \text{sum property}\\ &= \frac{1}{3}\log 100+\frac{1}{3}\log x^2-\frac{1}{3}\log y-\frac{1}{3}\log z^5\\ &= \frac{1}{3}\log 100+\frac{2}{3}\log x-\frac{1}{3}\log y-\frac{5}{3}\log z && \text{exponent property}\\ &= \frac{2}{3}+\frac{2}{3}\log x-\frac{1}{3}\log y-\frac{5}{3}\log z && \text{since } 10^2=100\end{aligned}$$

Finally, we have the expanded expression $\dfrac{2}{3}\log x-\dfrac{1}{3}\log y-\dfrac{5}{3}\log z+\dfrac{2}{3}$.

□

Example 4.2.11. Use the properties of logarithms to write the following as a single logarithm.

1. $2\log x-3\log(x-1)$

2. $4\ln u-\ln(5u)+\ln\left(\dfrac{1}{u}\right)$

Solution.

1. In the expression $2\log x-3\log(x-1)$ we have a difference of logarithms. However, before we can use the difference property, we need to deal with the coefficients of 2 on $\log x$ and 3 on $\log(x-1)$. This can be handled using the exponent property.

$$\begin{aligned} 2\log x-3\log(x-1) &= \log x^2-\log(x-1)^3 && \text{exponent property} \\ &= \log\left(\frac{x^2}{(x-1)^3}\right) && \text{difference property} \end{aligned}$$

2. The expression $4\ln u-\ln(5u)+\ln\left(\frac{1}{u}\right)$ contains both a difference and a sum of logarithms. Again, before we can do either, we need to deal with the coefficient of 4 on $\ln u$.

$$\begin{aligned} 4\ln u-\ln(5u)+\ln\left(\frac{1}{u}\right) &= \ln u^4-\ln(5u)+\ln\left(\frac{1}{u}\right) && \text{exponent property} \\ &= \ln\left(\frac{u^4}{5u}\right)+\ln\left(\frac{1}{u}\right) && \text{difference property} \\ &= \ln\left(\frac{u^4}{5u}\cdot\frac{1}{u}\right) && \text{sum property} \\ &= \ln\left(\frac{u^2}{5}\right) \end{aligned}$$

□

Using Properties to Solve Basic Logarithmic Equations

Example 4.2.12. Solve $\log x-\log 2=1$.

Solution. We start by combining the logarithmic terms into one term.

$$\begin{aligned} \log x-\log 2 &= 1 \\ \log\left(\frac{x}{2}\right) &= 1 && \text{difference property} \\ \frac{x}{2} &= 10^1 && \text{change to exponential form} \\ x &= 20 \end{aligned}$$

To check this answer, we replace x with 20 in the original equation:

$$\begin{aligned} \log 20-\log 2 &= \log\left(\frac{20}{2}\right) \\ &= \log 10 \\ &= 1 \end{aligned}$$

□

Checking potential solutions is important in solving logarithmic equations, as we see in the next example.

Example 4.2.13. Solve $\log_2 x + \log_2(x-1) = 1$.

Solution. We first combine the logarithmic terms into one term.

$$
\begin{aligned}
\log_2 x + \log_2(x-1) &= 1 \\
\log_2\big(x(x-1)\big) &= 1 \quad \text{sum property} \\
x(x-1) &= 2^1 \quad \text{change to exponential form} \\
x^2 - x - 2 &= 0 \\
(x-2)(x+1) &= 0
\end{aligned}
$$

We have two potential solutions: $x = -1$ and $x = 2$. We recall that the domain of a logarithm consists only of positive numbers. The potential solution $x = -1$ is not acceptable since neither $\log_2 x$ nor $\log_2(x-1)$ is defined for this x value. We refer to $x = -1$ as an extraneous solution. It is easy to check that $x = 2$ does satisfy this equation. Thus, our answer is $x = 2$.

□

In general, we will have to check potential solutions of logarithmic equations, or at least be sure the logarithmic terms in the equation are defined for those values. We will discuss this later on. For now, we wrap up this section by summarizing the properties of logarithms.

Properties of Logarithms

Let a and b be positive numbers, not equal to 1; x, u and v positive numbers; m any real number.

- $\log_b x$ is the power of b that gives x: $\log_b x = y \Leftrightarrow b^y = x$
- $\log_b 1 = 0$ and $\log_b b = 1$
- $b^{\log_b x} = x$
- $\log_b b^x = x$
- $\log_b u = \log_b v \Rightarrow u = v$
- $\log_b x = \dfrac{\log_a x}{\log_a b}$
- $\log_b x^m = m\log_b x$
- $\log_b(uv) = \log_b u + \log_b v$
- $\log_b\left(\dfrac{u}{v}\right) = \log_b u - \log_b v$

4.2 Exercises

1. What is the purpose of the change of base formula? Why is it useful when using a calculator?

2. When does an extraneous solution occur? How can an extraneous solution be recognized?

In Exercises 3 – 6, use the change of base property to convert the given expression to an expression with the indicated base.

3. $\log_7 15$ to base e

4. $\log_{14} 55.875$ to base 10

5. $\log_3(x+2)$ to base 10

6. $\log(x^2+1)$ to base e

In Exercises 7 – 12, use the change of base property to approximate the logarithm to five decimal places.

7. $\log_3 12$

8. $\log_5 80$

9. $\log_6 72$

10. $\log_4\left(\frac{1}{10}\right)$

11. $\log_{\frac{3}{5}} 1000$

12. $\log_{\frac{2}{3}} 50$

In Exercises 13 – 30, solve the equation analytically.

13. $3^{2x}=5$

14. $5^{-x}=2$

15. $5^x=-2$

16. $3^{x-1}=29$

17. $9^{x-10}=1$

18. $1.005^{12x}=3$

19. $e^{-5730k}=\frac{1}{2}$

20. $3^{x-1}=2^x$

21. $2^{x+1}=5^{2x-1}$

22. $3^{2x+1}=7^{x-2}$

23. $3^{x-1}=\left(\frac{1}{2}\right)^{x+5}$

24. $7^{3+7x}=3^{4-2x}$

25. $\log_4(4x)-\log_4\left(\frac{x}{4}\right)=3$

26. $\log_4(x)-\log_4(3)=1$

27. $\log_5(x-4)+\log_5 x=1$

28. $\log_2(x-1)+\log_2(x-3)=3$

29. $\log_3(x-4)+\log_3(x+4)=2$

30. $\log_5(2x+1)+\log_5(x+2)=1$

In Exercises 31 – 45, expand the logarithm and simplify. Assume when necessary that all quantities represent positive real numbers.

31. $\ln(x^3y^2)$

32. $\log_2\left(\frac{128}{x^2+4}\right)$

33. $\log_5\left(\frac{z}{25}\right)^3$

34. $\log\left(1.23\times 10^{37}\right)$

35. $\ln\left(\dfrac{\sqrt{z}}{xy}\right)$

36. $\log_5\left(x^2-25\right)$

37. $\log_{\sqrt{2}}\left(4x^3\right)$

38. $\log_{\frac{1}{3}}\left(9x\left(y^3-8\right)\right)$

39. $\log\left(1000x^3y^5\right)$

40. $\log_3\left(\dfrac{x^2}{81y^4}\right)$

41. $\ln\sqrt[4]{\dfrac{xy}{ez}}$

42. $\log_6\left(\dfrac{216}{x^3y}\right)^4$

43. $\log\left(\dfrac{100x\sqrt{y}}{\sqrt[3]{10}}\right)$

44. $\log_{\frac{1}{2}}\left(\dfrac{4\sqrt[3]{x^2}}{y\sqrt{z}}\right)$

45. $\ln\left(\dfrac{\sqrt[3]{x}}{10\sqrt{yz}}\right)$

In Exercises 46 – 57, use the properties of logarithms to write the expression as a single logarithm.

46. $\log\left(2x^4\right)+\log\left(3x^5\right)$

47. $\ln\left(6x^9\right)-\ln\left(3x^2\right)$

48. $4\ln x+2\ln y$

49. $\log_2 x+\log_2 y+\log_2 z$

50. $\log_3 x-2\log_3 y$

51. $\dfrac{1}{2}\log_3 x-2\log_3 y-\log_3 z$

52. $2\ln x-3\ln y-4\ln z$

53. $\log x-\dfrac{1}{3}\log z+\dfrac{1}{2}\log y$

54. $-\dfrac{1}{3}\ln x-\dfrac{1}{3}\ln y+\dfrac{1}{3}\ln z$

55. $\log_5 x-3$

56. $3-\log x$

57. $\log_7 x+\log_7\left(x-3\right)-2$

58. Give numerical examples to show that, in general,

(a) $\log_b\left(x+y\right)\neq\log_b x+\log_b y$

(b) $\log_b\left(x-y\right)\neq\log_b x-\log_b y$

(c) $\log_b\left(\dfrac{x}{y}\right)\neq\dfrac{\log_b x}{\log_b y}$

59. Research the history of logarithms, including the origin of the word ‘logarithm’ itself. Why is the abbreviation of the natural logarithm ‘ln’ and not ‘nl’?

60. There is a scene in the movie ‘Apollo 13’ in which several people at Mission Control use slide rules to verify a computation. Was that scene accurate? Look for other pop culture references to logarithms and slide rules.

4.3 Exponential Equations and Functions

Learning Objectives

- Solve exponential equations.
- Determine *x*- and *y*-intercepts of exponential functions.
- Graph exponential functions.
- Solve applications of exponential functions.

An **exponential equation** is an equation that has an exponent containing a variable. Some examples of exponential equations are $7^{x+1}=3^{-2x}$, $5\left(10^{-2x}\right)-73=0$ and $e^{2x}+3e^{x}-10=0$, the first of which we solved in **Section 4.2**. In this section, we continue solving exponential equations and graphing exponential functions.

Solving Exponential Equations

We begin with a strategy for solving exponential equations that includes methods from the previous two sections.

Solving Exponential Equations

1. Rewrite the original equation in the form $b^m=a^n$ or $b^m=y$, if possible .[6]
2. For the case $b^m=a^n$:
   a) If $b=a$, use the one-to-one property to reduce it to the new equation $m=n$.
   b) If $b\neq a$, take the logarithm of both sides[7] and use the exponent property of logarithms to get a new equation.
3. For the case $b^m=y$, if $y\leq 0$ the equation has no solution.[8] If $y>0$, take the logarithm of both sides[9] and use the exponent property of logarithms to get a new equation.
4. Solve the new equation.

---

[6] This strategy fails if the equation cannot be written in one of these forms.
[7] Use the same base!
[8] If $y\leq 0$, this equation has no solution since the range of $y=b^x$ is the set of positive numbers.
[9] Again, same base!

In step 3, rather than taking the logarithm of both sides of $b^m = y$, we could convert the equation to its logarithmic form $m = \log_b y$.

Reducing the equation to one of the forms stated in our strategy may require additional operations, as we will demonstrate shortly. Note also that this strategy will not generate extraneous solutions so checking answers is not required, except to verify our work. In the following four examples, the reader should become familiar with the strategy for solving exponential equations by identifying the steps in each solution.

Example 4.3.1. Solve the equation $2^{x^2+5x} = \frac{1}{16}$.

Solution. Since $\frac{1}{16} = \frac{1}{2^4} = 2^{-4}$, we can rewrite $2^{x^2+5x} = \frac{1}{16}$ as $2^{x^2+5x} = 2^{-4}$. Applying the one-to-one property, we set the exponents equal to each other to arrive at the new equation $x^2 + 5x = -4$.

$$x^2 + 5x = -4$$
$$x^2 + 5x + 4 = 0$$
$$(x+4)(x+1) = 0$$

We set each factor equal to zero, and find a solution of $x = -4$ or $x = -1$.

□

While this first example is similar to equations we solved in **Section 4.1**, the next example requires additional steps to write the equation in the form $b^m = b^n$.

Example 4.3.2. Solve the equation $2\left(5^{3x-1}\right) = 10\left(25^x\right)$.

Solution. To write both sides using the same base, we first divide through by 2 to get $5^{3x-1} = 5\left(25^x\right)$.

$$5^{3x-1} = 5\left(25^x\right)$$
$$5^{3x-1} = 5\left(5^{2x}\right) \quad \text{since } 25 = 5^2$$
$$5^{3x-1} = 5^{2x+1} \quad \text{since } 5 = 5^1$$

We find $3x - 1 = 2x + 1$, from which $x = 2$.

□

The next example requires additional steps to write the equation in the form $b^m = y$.

Example 4.3.3. Solve the equation $5\left(10^{-2x}\right) - 73 = 0$.

Solution. We begin by isolating the exponential term.

$$5\left(10^{-2x}\right)-73=0$$
$$5\left(10^{-2x}\right)=73$$
$$10^{-2x}=\frac{73}{5}$$

Noting that we have a base of 10, we continue by taking the common logarithm of both sides.

$$\log 10^{-2x}=\log\left(\frac{73}{5}\right)$$
$$-2x\log 10=\log\left(\frac{73}{5}\right) \quad \text{exponent property}$$
$$-2x=\log\left(\frac{73}{5}\right) \quad \text{since } \log 10=1$$
$$x=-\frac{1}{2}\log\left(\frac{73}{5}\right)$$

The solution is $x=-\frac{1}{2}\log\left(\frac{73}{5}\right)$. We may find an approximate solution, by calculator, to be $x\approx-0.582$.

□

We note that, in the preceeding example, converting $10^{-2x}=\frac{73}{5}$ to the logarithmic form $-2x=\log\left(\frac{73}{5}\right)$ would have resulted in a quicker solution.

Example 4.3.4. Solve the equation $e^{2x}+3e^{x}-10=0$.

Solution. This equation cannot be written in one of the desired forms, but it is of the quadratic form since $e^{2x}=\left(e^{x}\right)^{2}$. We may choose to use a substitution by letting $u=e^{x}$. Then $e^{2x}=\left(e^{x}\right)^{2}=u^{2}$.

$$e^{2x}+3e^{x}-10=0$$
$$u^{2}+3u-10=0$$
$$(u-2)(u+5)=0$$

We find $u=2$ or $u=-5$. Substituting these results back into the exponential equation, we have $e^{x}=2$ or $e^{x}=-5$. The second equation has no solution since $-5\le 0$. Taking the natural logarithm of both sides of the first equation, we get

$$\ln e^{x}=\ln 2$$
$$x=\ln 2 \quad \text{inverse property}$$

Our final, and only, solution is $x=\ln 2$.

□

Graphing Exponential Functions

Exponential functions were first introduced and defined in **Section 4.1**. Less formally, we can think of exponential functions as functions which have an exponent that contains a variable. Some examples of exponential functions are $f(x) = 2^{x+1} - 3$, $f(x) = 3^{2x+1} - 1$ and $f(x) = 10^{1-x^2}$, the first of which was graphed in **Section 4.1**. Before we continue with graphs of exponential functions, we note that finding intercepts is often important in graphing and we apply our equation solving skills in finding *x*-intercepts in the following example.

Example 4.3.5. Find the *x*-intercepts of the graph of $f(x) = 2^{x^2-5} - 16$.

Solution. To find the *x*-intercepts, we set $y = f(x) = 0$ and solve for *x*.

$$\begin{aligned} 2^{x^2-5} - 16 &= 0 \\ 2^{x^2-5} &= 16 \\ 2^{x^2-5} &= 2^4 \\ x^2 - 5 &= 4 \quad \text{one-to-one property} \\ x^2 &= 9 \end{aligned}$$

We find $x = \pm 3$, for *x*-intercepts of $(-3,0)$ and $(3,0)$.

□

We return to graphing exponential functions, making use of the following strategy.

Graphing Exponential Functions

1. Find the domain. Recall that $y = b^x$ is defined for all real *x* values.
2. Find the *x*- and *y*-intercepts, if any exist.
3. Use transformations if the graph can be obtained through shifts, reflections, and/or scalings of the graph of a function $y = b^x$.
4. Plot additional points, as needed, to identify or confirm the general shape of the graph.
5. Find the horizontal asymptote(s), if any exist.
   Recall that for $b > 0$, $y = b^x \to 0$ as $x \to -\infty$ and for $0 < b < 1$, $y = b^x \to 0$ as $x \to \infty$.
6. Sketch a smooth curve that passes through intercepts and points, and approaches asymptote(s).

Generally, we expect the graph to have one horizontal asymptote. However, it is possible to have a function, for example a piecewise defined function, whose graph has no horizontal asymptote or more than one horizontal asymptote.

Example 4.3.6. Sketch the graph of $f(x)=3^{2x+1}-1$.

Solution.

1. The domain of this function is $(-\infty,\infty)$.

2. To find the x-intercepts, we set $y=f(x)=0$.

$$\begin{aligned} 3^{2x+1}-1&=0 \\ 3^{2x+1}&=1 \\ 3^{2x+1}&=3^0 \\ 2x+1&=0 \quad \text{one-to-one property} \end{aligned}$$

We get $x=-\frac{1}{2}$ for an x-intercept of $\left(-\frac{1}{2},0\right)$. For the y-intercept, setting $x=0$, we have $f(0)=3^1-1=2$. The y-intercept is the point $(0,2)$.

3. We may use transformations of the graph of $y=3^x$ to graph $y=f(x)=3^{2x+1}-1$. The change of input from x to $2x+1$ tells us the graph of $y=3^x$ will shift to the left by one unit, and be horizontally scaled by $\frac{1}{2}$. The change of -1 to this output tells us to shift the graph down one unit.

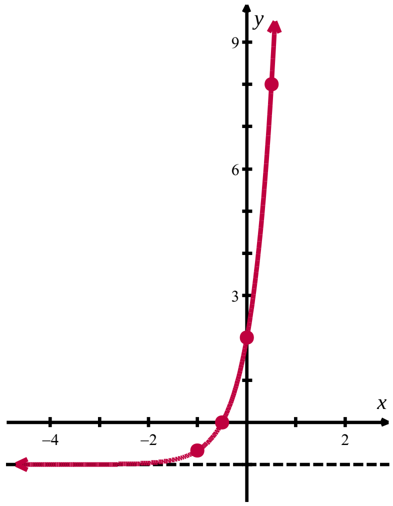
4. We may use the additional points in the table to help with the shape of the curve.

x	$y=f(x)=3^{2x+1}-1$	(x,y)
-1	$3^{2(-1)+1}-1=3^{-1}-1=-\frac{2}{3}$	$\left(-1,-\frac{2}{3}\right)$
$\frac{1}{2}$	$3^{2\left(\frac{1}{2}\right)+1}-1=3^2-1=8$	$\left(\frac{1}{2},8\right)$

5. Since we have a shift of one unit down from the graph of $y=3^x$, whose horizontal asymptote is the line $y=0$, we can see that the horizontal asymptote is the line $y=-1$. We can also verify this by evaluating f for large negative values of x.

6. We plot the intercepts, additional points and horizontal asymptote, then draw a smooth curve through the points that approaches the asymptote and has shape similar to $y = 3^x$.

Figure 4.3. 1



$$y = f(x) = 3^{2x+1} - 1$$

□

Example 4.3.7. Sketch the graph of $f(x) = 10^{1-x^2}$.

Solution.

1. The domain of this function is $(-\infty, \infty)$ since $1 - x^2$ is defined for any *x* value.

2. To find the *x*-intercepts, we set $y = f(x) = 10^{1-x^2} = 0$. There is no solution since $10^{1-x^2} > 0$ for all *x* values; thus no *x*-intercept. To determine the *y*-intercept, we set $x = 0$ to get $f(0) = 10^{1-(0)^2} = 10$. The *y*-intercept is $(0, 10)$.

3. We cannot use transformations since our function is not a transformation of $f(x) = 10^x$ that we have seen before.

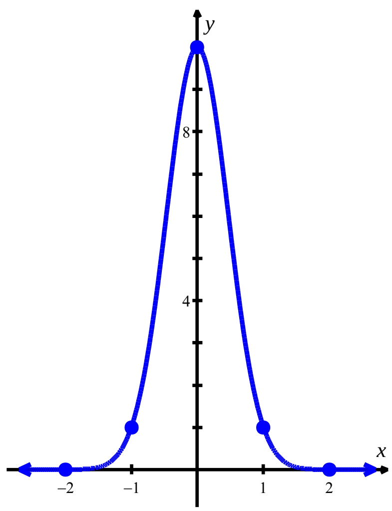
4. We identify a few more points in the following table.

x	$y=f(x)=10^{1-x^2}$	(x,y)
-2	$10^{1-(-2)^2}=10^{-3}=\frac{1}{1000}$	$\left(-2,\frac{1}{1000}\right)$
-1	$10^{1-(-1)^2}=10^0=1$	$(-1,1)$
1	$10^{1-(1)^2}=10^0=1$	$(1,1)$
2	$10^{1-(2)^2}=10^{-3}=\frac{1}{1000}$	$\left(2,\frac{1}{1000}\right)$

5. By considering the values of $f(x)=10^{1-x^2}$ as $x\to\pm\infty$, like $f(-2)=\frac{1}{1000}$ and $f(2)=\frac{1}{1000}$, we see that the horizontal asymptote is the *x*-axis, $y=0$, and that the graph approaches the horizontal asymptote both to the left and to the right.

6. We draw a smooth curve through the points we have identified, approaching the *x* axis, our horizontal asymptote, in both directions.

Figure 4.3. 2



$y=f(x)=10^{1-x^2}$

□

Applications of Exponential Functions

As mentioned earlier, exponential functions and equations occur frequently in everyday life. Now that we can solve exponential equations, we can also solve real life applications. Below is a more general form of an example we have already seen.

Example 4.3.8. Suppose a new car loses a fixed proportion of its value every year. Its value after t years is $V(t) = V_0(1-r)^t$ where V_0 is its initial value, and r is the proportion of yearly loss. If the car was bought initially for \$25,000 and loses 20% of its value every year, after how many years will it be worth \$10,000?

Solution. Plugging in the initial value $V_0 = 25000$ and proportion $r = 0.2$, we have

$$\begin{aligned} V(t) &= 25000(1-0.2)^t \\ &= 25000(0.8)^t \end{aligned}$$

We want to find t when $V(t) = 10000$.

$$\begin{aligned} 25000(0.8)^t &= 10000 \\ (0.8)^t &= \frac{10000}{25000} \\ 0.8^t &= 0.4 \\ \ln 0.8^t &= \ln 0.4 \\ t \ln 0.8 &= \ln 0.4 \\ t &= \frac{\ln 0.4}{\ln 0.8} \approx 4.106 \end{aligned}$$

After about 4.106 years, or about 4 years and 1.3 months, this car will be worth \$10,000.

□

We will discuss applications more thoroughly in **Section 4.5**, but for now return to logarithmic functions in **Section 4.4**.

4.3 Exercises

1. Give an example of a type of exponential equation that cannot be solved using the strategy at the beginning of this section.

2. How many intercepts will the graph of an exponential function of the form $f(x)=a^{bx+c}$ have?

In Exercises 3 – 35, solve the equation analytically.

3. $64\cdot 4^{3x}=16$

4. $3^{2x+1}\cdot 3^{x}=243$

5. $2^{-3n}\cdot\frac{1}{4}=2^{n+2}$

6. $625\cdot 5^{3x+3}=125$

7. $2e^{6x}=13$

8. $e^{r+10}-10=-42$

9. $2000e^{0.1t}=4000$

10. $-8\cdot 10^{p+7}=-24$

11. $7e^{3n-5}+5=-89$

12. $e^{-3k}+6=44$

13. $-5e^{9x-8}-8=-62$

14. $-6e^{9x+8}+2=-74$

15. $7e^{8x+8}-5=-95$

16. $4e^{3x+3}-7=53$

17. $8e^{-5x-2}-4=-90$

18. $10e^{8x+3}+2=8$

19. $3e^{3-3x}+6=-31$

20. $500\left(1-e^{2x}\right)=250$

21. $30-6e^{-0.1x}=20$

22. $\dfrac{100e^{x}}{e^{x}+2}=50$

23. $\dfrac{5000}{1+2e^{-3t}}=2500$

24. $\dfrac{150}{1+29e^{-0.8t}}=75$

25. $25\left(\frac{4}{5}\right)^{x}=10$

26. $e^{2x}=2e^{x}$

27. $7e^{2x}=28e^{-6x}$

28. $e^{2x}-e^{x}-132=0$

29. $e^{2x}-e^{x}-6=0$

30. $e^{2x}-3e^{x}-10=0$

31. $e^{2x}=e^{x}+6$

32. $4^{x}+2^{x}=12$

33. $e^{x}-3e^{-x}=2$

34. $e^{x}+15e^{-x}=8$

35. $3^{x}+25\cdot 3^{-x}=10$

In Exercises 36 – 47, sketch the graph of $y=f(x)$.

36. $f(x)=3^{\frac{x}{2}}-2$

37. $f(x)=\left(\frac{1}{3}\right)^{-x}-1$

38. $f(x)=4^{-x+1}$

39. $f(x)=e^{-x}+2$

40. $f(x)=2-e^{x}$

41. $f(x)=8-e^{-x}$

42. $f(x)=3\left(2^{x}\right)+1$

43. $f(x)=5\left(3^{-x}\right)$

44. $f(x)=2\left(\frac{1}{3}\right)^{-x}$

45. $f(x) = 2^{x^2}$

46. $f(x) = 2^{1-x^2}$

47. $f(x) = 8 - 2^{x^2}$

48. The population of a small town is modeled by the equation $P = 1650e^{0.5t}$ where t is measured in years. In approximately how many years will the town's population reach 20,000?

49. Atmospheric pressure P in pounds per square inch is represented by the formula $P = 14.7e^{-0.21x}$, where x is the number of miles above sea level. To the nearest foot, how high is the peak of a mountain with an atmospheric pressure of 8.369 pounds per square inch?

4.4 Logarithmic Equations and Functions

Learning Objectives

- Determine the domain of a logarithmic function.
- Determine *x*- and *y*-intercepts of logarithmic functions.
- Graph logarithmic functions.
- Solve logarithmic equations.
- Solve applications of logarithmic functions.

Logarithmic functions were first introduced and defined in **Section 4.1**. We identify a logarithmic function as a function involving a logarithm that has a variable in its argument. Some examples of logarithmic functions are $f(x)=\log_3\left(\frac{1}{2}x+1\right)+1$, $f(x)=\log|x-1|$ and $f(x)=\ln\left(\frac{x}{x-1}\right)$. We begin this section with determining domains of logarithmic functions.

Finding Domains of Logarithmic Functions

We recall that logarithms are only defined for positive numbers. It follows that the domain of a logarithm consists of those values for which its argument is a positive number.

Example 4.4.1. Find the domain of the following functions.

1. $f(x)=\log_2\left(x^2+1\right)$
2. $f(x)=\log_3\left(\frac{1}{2}x+1\right)+1$
3. $f(x)=\ln\left(\frac{x}{x+1}\right)$

Solution.

1. The logarithm $\log_2\left(x^2+1\right)$ is defined when $x^2+1>0$. This holds for all real numbers since $x^2+1\geq 1>0$ for all *x* values. Thus, the domain of this function is all real numbers, or $(-\infty,\infty)$.

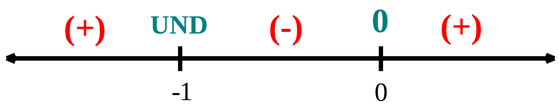
2. For the function $f(x)=\log_3\left(\frac{1}{2}x+1\right)+1$, we need $\frac{1}{2}x+1>0$. This occurs when

$$\begin{aligned}\frac{1}{2}x&>-1\\ x&>-2\end{aligned}$$

The domain is the interval $(-2,\infty)$.

3. For the function $f(x)=\ln\left(\frac{x}{x+1}\right)$, the argument of the natural logarithm must be greater than zero, so we need to solve $\frac{x}{x+1}>0$. We let $r(x)=\frac{x}{x+1}$ represent the left side of this inequality, and note that r is undefined at $x=-1$ and is zero at $x=0$. Referring to **Section 3.4**, and evaluating the sign of r in each interval, we construct a sign diagram as follows.

Figure 4.4. 1



The domain is the set of all *x* values less than −1 and all *x* values greater than zero, which is the set $(-\infty,-1)\cup(0,\infty)$.

□

Graphing Logarithmic Functions

In **Section 4.1**, we graphed basic logarithmic function and discussed their properties. We also graphed some transformations of basic logarithmic functions. Here, we state a general strategy for graphing logarithmic functions.

Graphing Logarithmic Functions

1. Find the domain. Recall that the argument of a logarithmic function must be greater than zero.
2. Find the *x*- and *y*-intercepts, if any exist.
3. Use transformations if the graph can be derived through shifts, reflections, and/or scalings of the graph of a function $y=\log_b x$.
4. Plot additional points, as needed, to identify or confirm the general shape of the graph.[10]
5. Find the vertical asymptote(s), if any exist. Recall that $y=\log_b x\to-\infty$ as $x\to 0$.
6. Sketch a smooth curve that passes through intercepts and points, and approaches asymptote(s).

Generally, if the graph has a vertical asymptote, it occurs at the end of intervals that make up the domain or at *x* values that make the argument of the logarithm zero.

[10] To simplify calculations, use powers of the base as the argument.

Example 4.4.2. Sketch the graph of $f(x)=\log_3\left(\frac{1}{2}x+1\right)+1$.

Solution.

1. In **Example 4.4.1**, we found that the domain of this function is $(-2,\infty)$.

2. To find the x-intercepts, we set $y=f(x)=0$.

$$\log_3\left(\frac{1}{2}x+1\right)+1=0$$
$$\log_3\left(\frac{1}{2}x+1\right)=-1$$
$$\frac{1}{2}x+1=3^{-1}$$
$$\frac{1}{2}x+1=\frac{1}{3}$$
$$\frac{1}{2}x=-\frac{2}{3}$$

We get $x=-\frac{4}{3}$ for an x-intercept of $\left(-\frac{4}{3},0\right)$. For the y-intercept, setting $x=0$, we have $f(0)=\log_3(1)+1=0+1=1$. The y-intercept is the point $(0,1)$.

3. We may use transformations of the graph of $y=\log_3 x$ to graph $y=f(x)=\log_3\left(\frac{1}{2}x+1\right)+1$. The change of input from x to $\frac{1}{2}x+1$ tells us the graph of $y=\log_3 x$ will shift to the left by one unit and be horizontally scaled by a factor of 2. The change to this output by +1 tells us to shift the graph up one unit.

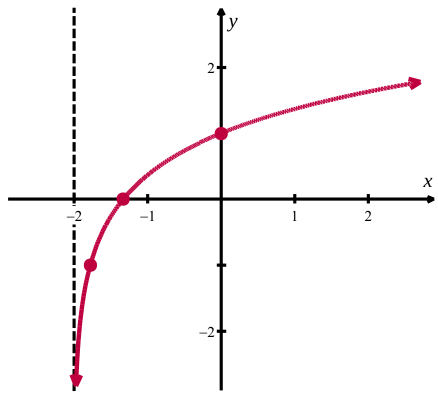
4. For additional points, we try powers of the base 3 for the value of the argument, $\frac{1}{2}x+1$, and look for points different than the x- and y-intercepts.

$\frac{1}{2}x+1$	x	$y=f(x)=\log_3\left(\frac{1}{2}x+1\right)+1$	(x,y)
$3^{-2}=\frac{1}{9}$	$\frac{1}{2}x+1=\frac{1}{9}\Rightarrow x=-\frac{16}{9}$	$y=-2+1=-1$	$\left(-\frac{16}{9},-1\right)$
$3^{1}=3$	$\frac{1}{2}x+1=3\Rightarrow x=4$	$y=1+1=2$	$(4,2)$

5. After applying transformations to the input of $y=\log_3 x$, we see that its vertical asymptote of $x=0$ is shifted left one unit, to $x=-1$, and then multiplied by two, for a resulting vertical asymptote of $x=-2$. We can also verify this by evaluating f for values of *x* close to −2.

6. We plot the intercepts, additional points and vertical asymptote, then draw a smooth curve through the points that approaches the asymptote and has shape similar to $y=\log_3 x$.

Figure 4.4. 2



$$y=f(x)=\log_3\left(\frac{1}{2}x+1\right)+1$$

□

Example 4.4.3. Sketch the graph of $f(x)=\log|x-1|$.

Solution.

1. The domain of this function inludes *x* values for which $|x-1|>0$. Since $|x-1|\geq 0$ for all *x* values and $|x-1|=0$ only for $x=1$, the domain is the set of all real numbers except one: $\{x\,|\,x\neq 1\}$ or $(-\infty,1)\cup(1,\infty)$.

2. To find the *x*-intercepts, we set $y=f(x)=\log|x-1|=0$.

$$\begin{aligned}\log|x-1|&=0\\|x-1|&=10^0\\|x-1|&=1\\x-1&=\pm 1\end{aligned}$$

We get $x=0$ or $x=2$ for *x*-intercepts of $(0,0)$ and $(2,0)$. The *x*-intercept at $(0,0)$ is also our *y*-intercept.

3. We cannot use transformations since our function is not a transformation of $y=\log x$ that we have seen before.

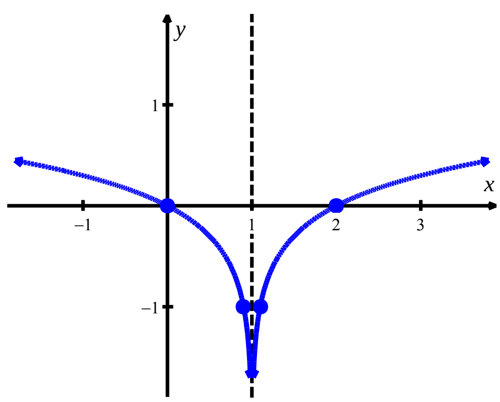
4. To add a few more points, we try powers of the base 10 for the value of the argument, $|x-1|$, looking for points that are different than the intercepts.

$\|x+1\|$	x	$y=f(x)=\log\|x-1\|$	(x,y)
$10^{-2}=\frac{1}{100}$	$\|x-1\|=\frac{1}{100}\Rightarrow x=\frac{99}{100},\frac{101}{100}$	-2	$\left(\frac{99}{100},-2\right),\left(\frac{101}{100},-2\right)$
$10^{-1}=\frac{1}{10}$	$\|x-1\|=\frac{1}{10}\Rightarrow x=\frac{9}{10},\frac{11}{10}$	-1	$\left(\frac{9}{10},-1\right),\left(\frac{11}{10},-1\right)$
$10^{1}=10$	$\|x-1\|=10\Rightarrow x=-9,11$	1	$(-9,1),(11,1)$

5. By checking *x* values near 1, the end value of the domain, we have $f\left(1\pm\frac{1}{10}\right)=-1$ and $f\left(1\pm\frac{1}{100}\right)=-2$. Continuing to get closer to $x=1$ from both sides, we would find that $f(x)=\log|x-1|\to-\infty$, so the vertical asymptote is the line $x=1$ and the graph approaches this vertical asymptote from both sides.

6. We draw a smooth curve through the points we have identified, approaching the vertical asymptote, $x=1$, from both sides of the line. Note that some points are not included in the following graph.

Figure 4.4. 3



$y=f(x)=\log|x-1|$

Notice that although we did not include the points $(-9,1)$ and $(11,1)$ in the graph, we used them as a guide on how slowly the graph is rising on both left and right sides.

□

Solving Logarithmic Equations

A **logarithmic equation** is an equation that contains a logarithm with a variable in its argument. One example of a logarithmic equation is $\log_2 x + \log_2(x-1) = 1$, which we solved in **Section 4.2**. In this section, we continue using properties of logarithms to solve logarithmic equations. In our next example, we apply the one-to-one property in solving $\ln(x^2 - 3x - 1) = \ln(2 - x)$. As we discovered in **Section 4.2**, once we have potential solutions, we must check to verify they are not extraneous.

Example 4.4.4. Solve the equation $\ln(x^2 - 3x - 1) = \ln(2 - x)$.

Solution. We can apply the one-to-one property, $\log_b u = \log_b v \Rightarrow u = v$.

$$\begin{aligned} \ln(x^2 - 3x - 1) &= \ln(2 - x) \\ x^2 - 3x - 1 &= 2 - x \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \end{aligned}$$

We have two potential solutions: $x = -1$ and $x = 3$.

Check for $x = -1$: Left Side $= \ln(x^2 - 3x - 1) = \ln(1 + 3 - 1) = \ln 3$

Right Side $= \ln(2 - x) = \ln(2 + 1) = \ln(3) =$ Left Side

Check for $x = 3$: Left Side $= \ln(x^2 - 3x - 1) = \ln(9 - 9 - 1) = \ln(-1)$ is not defined

Right Side $= \ln(2 - x) = \ln(2 - 3) = \ln(-1)$ which is also not defined

So $x = 3$ is not a solution. The only solution to this equation is $x = -1$.

□

We continue with a general strategy for solving logarithmic equations.

Solving Logarithmic Equations

1. Rewrite the original equation in the form $\log_b u = y$ or $\log_b u = \log_b v$, if possible.[11]
2. For the case $\log_b u = y$, convert the equation to its equivalent form $u = b^y$.
3. For the case $\log_b u = \log_b v$, use the one-to-one property to reduce it to the equation $u = v$.
4. Solve the new equation to find all potential solutions of the original equation.
5. Check each potential solution in the original equation. Those that satisfy the original equation are its solutions.

[11] This strategy fails if the equation cannot be written in one of these forms.

The reason we must check the potential solutions is that logarithms are only defined for positive numbers, not all real numbers as in exponentials. Another way of ensuring that a potential solution is acceptable is to check that it is in the domain of logarithm(s) in the original equation. Of course, this doesn't verify that we haven't made a mistake along the way. In general, checking of the potential solutions is a required part of the solution process and is not optional.

In the following two examples, the reader should become familiar with the strategy for solving logarithmic equations by identifying the steps in each solution.

Example 4.4.5. Solve the equation $\log_3(4-5x)=1+\log_3(x-4)$.

Solution. We move both logarithms to the same side, then use the difference property to combine them into one logarithm.

$$\begin{aligned}\log_3(4-5x)&=1+\log_3(x-4)\\ \log_3(4-5x)-\log_3(x-4)&=1\\ \log_3\left(\frac{4-5x}{x-4}\right)&=1\end{aligned}$$

The equation $\log_3\left(\frac{4-5x}{x-4}\right)=1$ can be written in exponential form.

$$\frac{4-5x}{x-4}=3^1=3$$

We solve the new equation by, first of all, multiplying through by $x-4$.

$$\begin{aligned}4-5x&=3(x-4)\\ 4-5x&=3x-12\\ -8x&=-16\end{aligned}$$

The potential solution is $x=2$. In checking the potential solution of $x=2$, we find that $\log_3(4-5x)=\log_3(-6)$, which is not defined. Therefore, $x=2$ is not acceptable and this equation does not have a solution, or we say its solution set is the empty set.

□

Example 4.4.6. Solve the equation $\log(x+2)+\log(5-x)=1$.

Solution. We use the sum property to write the logarithms as one term, and proceed to write the equation in the form $\log_b u = y$.

$$\log(x+2)+\log(5-x)=1$$
$$\log[(x+2)(5-x)]=1$$
$$\log(-x^2+3x+10)=1$$

Converting the new equation into exponential form, we have $-x^2+3x+10=10^1=10$. We solve the new equation by first subtracting 10 from each side.

$$-x^2+3x=0$$
$$x(-x+3)=0$$

The potential solutions are $x=0$ and $x=3$. Since the potential solution values of $x=0$ and $x=3$ result in positive arguments for $\log(x+2)$ and $\log(5-x)$, both of these values are acceptable. Of course, to be sure of our work, we could plug them in to check that they satisfy the equation. Our solution includes both $x=0$ and $x=3$.

□

Applications of Logarithmic Functions

Logarithmic functions and equations also occur frequently in everyday life. Now that we can solve logarithmic equations, we can also solve real life applications. Below is a problem similar to one we saw in **Section 4.2**.

Example 4.4.7. Let $P(t)$ be the population of the state of Utah, in millions, t years after 1970. In 1970, Utah had a population of about 1.06 million. Assuming a constant growth rate of 2%, the population of Utah satisfies the equation $\ln P(t)=\ln 1.06+0.02t$. Find the population of Utah as a function of t. The population of Utah in 2018 is about 3.16 million. Compare this actual population with your calculation.

Solution. We can convert our equation to exponential form and simplify.

$$\ln P(t)=\ln 1.06+0.02t$$
$$P(t)=e^{\ln 1.06+0.02t}$$
$$P(t)=e^{\ln 1.06}e^{0.02t}$$
$$P(t)=1.06e^{0.02t}$$

For the year 2018, $t=2018-1970=48$. According to this model, the population in the year 2018 is

$$P(48)=1.06e^{0.02(48)}$$
$$\approx 2.768$$

This is about 2.77 million, so the model has underestimated the true population of 3.16 million.

□

We note that, in reality, the population growth rate is not constant. Over the years, the population growth rate of the state of Utah has decreased from over 2% to less than 2%. We end this section with a note that many more applications, of both logarithmic and exponential functions, are coming up shortly in **Section 4.5**.

4.4 Exercises

1. What type(s) of transformation(s), if any, affect the domain of a logarithmic function?

2. What type(s) of transformation(s), if any, affect the range of a logarithmic function?

In Exercises 3 – 14, find the domain of the function.

3. $f(x)=\ln(2-x)$

4. $f(x)=\log\left(x-\frac{3}{7}\right)$

5. $h(x)=-\log(3x-4)+3$

6. $g(x)=\ln(2x+6)-5$

7. $f(x)=\log_3(15-5x)+6$

8. $f(x)=\ln(x^2+4)$

9. $f(x)=\log_7(4x+8)$

10. $f(x)=\ln(4x-20)$

11. $f(x)=\log(x^2+9x+18)$

12. $f(x)=\log\left(\frac{x+2}{x^2-1}\right)$

13. $f(x)=\log\left(\frac{x^2+9x+18}{4x-20}\right)$

14. $f(x)=\ln(7-x)+\ln(x-4)$

In Exercises 15 – 28, solve the equation analytically.

15. $\log(3x-1)=\log(4-x)$

16. $\log_2 x^3=\log_2 x$

17. $\ln(8-x^2)=\ln(2-x)$

18. $\log_5(18-x^2)=\log_5(6-x)$

19. $\ln(x^2-99)=0$

20. $\log\left(\frac{x}{10^{-3}}\right)=4.7$

21. $-\log x=5.4$

22. $10\log\left(\frac{x}{10^{-12}}\right)=150$

23. $6-3\log_5(2x)=0$

24. $3\ln x-2=1-\ln x$

25. $\log_{169}(3x+7)-\log_{169}(5x-9)=\frac{1}{2}$

26. $\ln(x+1)-\ln x=3$

27. $2\log_7 x=\log_7 2+\log_7(x+12)$

28. $\log x-\log 2=\log(x+8)-\log(x+2)$

In Exercises 29 – 40, sketch the graph of $y = f(x)$.

29. $f(x) = 2\log(x+20) - 1$

30. $f(x) = -\ln(8-x)$

31. $f(x) = -10\ln\left(\frac{x}{10}\right)$

32. $f(x) = \log_2(x+1)$

33. $f(x) = \log_3(-x)$

34. $f(x) = \log_2(-x+3)$

35. $f(x) = \log_3(2x) - 4$

36. $f(x) = \ln(x-1)$

37. $f(x) = 2\log x + 1$

38. $f(x) = \log\left(\frac{1}{2}x - 1\right)$

39. $f(x) = \log_2|x|$

40. $f(x) = \log_2|x+1|$

41. Let $P(t)$ be the population of Arizona, in millions, t years after 1970. In 1970 Arizona had a population of 1.8 million. Assuming a constant growth rate of 3%, the population of Arizona satisfies the equation $\ln P(t) = \ln 1.8 + 0.03t$. Use this equation to estimate the population of Arizona in 2018. The population of Arizona in 2018 is about 7.34 million. Compare this with your estimate.

42. Let $P(t)$ be the population of the United States of America, in millions, t years after 1970. In 1970, the USA had a population of about 210 million. Assuming a constant growth rate of 1%, the population of the USA satisfies the equation $\ln P(t) = \ln 210 + 0.01t$. Use this equation to estimate the population of the USA in 2018. The population of the USA in 2018 is about 327 million. Compare this with your estimate.

43. Let $P(t)$ be the population of Canada, in millions, t years after 1970. In 1970, Canada had a population of about 21.5 million. Assuming a constant growth rate of 1.2%, the population of Canada satisfies the equation $\ln P(t) = \ln 21.5 + 0.012t$. Use this equation to estimate the population of Canada in 2018. The population of Canada in 2018 is about 36.9 million. Compare this with your estimate.

44. Let $P(t)$ be the population of China, in millions, t years after 1970. In 1970, China had a population of about 825 million people. Assuming a constant growth rate of 0.5%, the population of China satisfies the equation $\ln P(t) = \ln 825 + 0.005t$. Use this equation to estimate the population of China in 2018. The population of China in 2018 is about 1415 million. Compare this with your estimate.

4.5 Applications of Exponentials and Logarithms

Learning Objectives

- Use compound interest formulas to solve financial applications.
- Solve applications of uninhibited growth and decay.
- Solve additional applications represented by exponential and logarithmic models.

Exponential and logarithmic functions are used to model a wide variety of behaviors in the real world. In the examples that follow, note that while the applications are drawn from many different disciplines, the mathematics remains essentially the same. Due to the applied nature of the problems we will examine in this section, a calculator is often used to express our answers as decimal approximations.

Compound Interest

Let's start with an example. Suppose we invest $10,000 in a savings account. This initial amount is called the **principal**. Assume the savings account pays 3% annual interest compounded semiannually; that is, twice a year or every six months, which is referred to as the **compounding period**. The interest is calculated at the end of each compounding period and added to the principal. After the first 6 months, the interest earned is one-half [12] of 3% of $10,000:

$$\left(\frac{1}{2}\right)(0.03)(\$10{,}000)=\left(\frac{0.03}{2}\right)(\$10{,}000)=\$150$$

That is, the amount in our account at the end of the first 6 months is $\$10{,}000+\$150=\$10{,}150$. At the end of the second 6 months, the interest on the new amount of $10,150 is

$$\left(\frac{0.03}{2}\right)(\$10{,}150)=\$152.25$$

This results in an amount of $\$10{,}150+\$152.25=\$10{,}302.25$ at the end of the second 6 months. At the end of the third 6 months, interest on $10,302.25 is

$$\left(\frac{0.03}{2}\right)(\$10{,}302.25)=\$154.53$$

[12] Since interest is compounded twice a year, we earn half of the annual interest rate after 6 months. Had interest been compounded quarterly, or four times per year, we would have earned one-fourth of the annual interest after 3 months.

The new amount, at the end of the third 6 months, is $\$10,302.25+\$154.53=\$10,456.78$.

The pattern found in these calculations results in a compound interest formula. We let P represent the principal, r the interest rate[13], n the number of compounding periods per year, and A_k the amount in the account after the *k*th compounding. We find that A_1, the amount in our account after one compounding period, is

$$A_1=P+\left(\frac{r}{n}\right)P=P\left(1+\frac{r}{n}\right)$$

After the second compounding period, we have

$$\begin{aligned}A_2&=A_1+\left(\frac{r}{n}\right)A_1\\&=A_1\left(1+\frac{r}{n}\right)\\&=\left[P\left(1+\frac{r}{n}\right)\right]\left(1+\frac{r}{n}\right)\\&=P\left(1+\frac{r}{n}\right)\left(1+\frac{r}{n}\right)\\&=P\left(1+\frac{r}{n}\right)^2\end{aligned}$$

After three compounding periods, the amount in our account is

$$\begin{aligned}A_3&=A_2+\left(\frac{r}{n}\right)A_2\\&=A_2\left(1+\frac{r}{n}\right)\\&=\left[P\left(1+\frac{r}{n}\right)^2\right]\left(1+\frac{r}{n}\right)\\&=P\left(1+\frac{r}{n}\right)^2\left(1+\frac{r}{n}\right)\\&=P\left(1+\frac{r}{n}\right)^3\end{aligned}$$

It follows that, after *k* compounding periods, we have $A_k=P\left(1+\frac{r}{n}\right)^k$. Since we compound the interest *n* times per year, after *t* years, we have $k=n\cdot t$ compoundings. We have just derived the following general formula for compound interest.

[13] When used in calculations, interest is written in decimal notation. For example, as we have shown, 3% must be written as 0.03.

Compounded Interest Formula

If a principal amount of P dollars is invested in an account earning interest at an annual rate r, compounded n times per year, the amount A in the account after t years is $A = P\left(1+\frac{r}{n}\right)^{nt}$ dollars.

Remember to input r in decimal notation!

Example 4.5.1. Suppose \$2000 is invested in an account earning 3% annual interest compounded monthly.

1. How much will be in this account after 6 years?
2. How long will it take for the value of this account to reach \$3500?
3. How long will it take for the principal of \$2000 to double in value?

Solution. The principal P is \$2000 and annual interest rate is 3%, or $r = 0.03$, compounded $n = 12$ times per year. The total accumulation in this account after t years is $A = 2000\left(1+\frac{0.03}{12}\right)^{12t}$.

1. After 6 years, setting $t = 6$, we calculate that the total accumulation is

$$\begin{aligned} A &= 2000\left(1+\frac{0.03}{12}\right)^{(12)(6)} \\ &= 2000(1.0025)^{72} \\ &\approx 2393.8969 \end{aligned}$$

There will be about \$2393.90 in the account after 6 years.[14]

2. To find the time required for the amount to reach \$3500, we set $A = 3500$ and solve for t.

$$\begin{aligned} 3500 &= 2000\left(1+\frac{0.03}{12}\right)^{12t} \\ \frac{3500}{2000} &= \left(1+\frac{0.03}{12}\right)^{12t} && \text{divide by 2000} \\ 1.75 &= (1.0025)^{12t} && \text{simplify} \\ \ln 1.75 &= \ln 1.0025^{12t} && \text{take the natural logarithm of both sides} \\ \ln 1.75 &= (12t)\ln 1.0025 && \text{exponent property} \\ \ln 1.75 &= (12\ln 1.0025)t \\ t &= \frac{\ln 1.75}{12\ln 1.0025} \approx 18.6772 \end{aligned}$$

[14] You may want to check out how banks do rounding. This could affect your \$2393.90.

It will take about 18.68 years, or a few days more than 18 years and 8 months, for the value of the account to reach \$3500.

3. To determine the time required for the original principal of \$2000 to double, we look for the value of t for which $A = 2(2000) = 4000$. We solve the following exponential equation.

$$4000 = 2000\left(1+\frac{0.03}{12}\right)^{12t}$$

$$\frac{4000}{2000} = \left(1+\frac{0.03}{12}\right)^{12t} \qquad \text{divide by 2000}$$

$$2 = (1.0025)^{12t} \qquad \text{simplify}$$

$$\ln 2 = \ln 1.0025^{12t} \qquad \text{take the natural logarithm of both sides}$$

$$\ln 2 = (12t)\ln 1.0025 \qquad \text{exponent property}$$

$$\ln 2 = (12\ln 1.0025)t$$

$$t = \frac{\ln 2}{12\ln 1.0025} \approx 23.1338$$

It will take about 23.13 years, or close to 23 years and 2 months, for the accumulation in this account to double in value.

□

In the next example, we know everything except the interest rate. This example may send us searching for investment opportunities but, as a side note, higher interest rates are often linked to riskier investments.

Example 4.5.2. Suppose we want to invest \$10,000, and hope to have \$15,000 in 8 years. What interest rate, compounded annually, must we seek to achieve this desired result?

Solution. We have the a principal of \$10,000 with annual compounding (once a year), so $P = 10000$ and $n = 1$. For the time period of 8 years, we use $t = 8$, and we set $A = 15000$ to give us the following equation, which we proceed to solve.

$$15000 = 10000\left(1+\frac{r}{1}\right)^{(1)(8)}$$

$$1.5 = (1+r)^{8} \qquad \text{divide by 10000 and simplify}$$

$$\sqrt[8]{1.5} = 1+r \qquad \text{since } 1+r>0$$

$$r = \sqrt[8]{1.5} - 1 \approx 0.05199$$

So, we will look for an interest rate of approximately 5.2%, compounded annually.

□

The more times an investment is compounded per year, the higher the total accumulation will be. However, there is a limit to the growth as the number of compounding periods per year increases. Let's take another look at the compound interest formula $A = P\left(1+\frac{r}{n}\right)^{nt}$, replacing $\frac{r}{n}$ with $\frac{1}{\left(\frac{n}{r}\right)}$.

$$\begin{aligned} A &= P\left(1+\frac{1}{\left(\frac{n}{r}\right)}\right)^{nt} \\ &= P\left(1+\frac{1}{\left(\frac{n}{r}\right)}\right)^{\left(\frac{n}{r}\right)rt} \qquad \text{since } n = \left(\frac{n}{r}\right)\cdot r \\ &= P\left[\left(1+\frac{1}{\left(\frac{n}{r}\right)}\right)^{\left(\frac{n}{r}\right)}\right]^{rt} \end{aligned}$$

Now recall from **Section 4.2** that as m gets large, $\left(1+\frac{1}{m}\right)^{m} \to e$. In the above formula, if we think of $\frac{n}{r}$ as a variable, then as $\frac{n}{r}$ becomes large, $\left(1+\frac{1}{\left(\frac{n}{r}\right)}\right)^{\left(\frac{n}{r}\right)} \to e$. In fact, since the interest rate r is a fixed value, $\frac{n}{r}$ is indeed becoming larger as n, the number of compounding periods, increases. In fact, as $n \to \infty$,

$$A = P\left(1+\frac{r}{n}\right)^{nt} \to Pe^{rt}$$

We see that the largest amount we can achieve through increasing the number of compounding periods is $A = Pe^{rt}$. Here, we say that interest is compounded continuously, and use the following formula.

Continuously Compounded Interest Formula

If a principal amount of P dollars is invested in an account earning annual interest at a rate r, compounded continuously, the total accumulation after t years is $A = Pe^{rt}$ dollars.

Example 4.5.3. Suppose you invest your first annual bonus of $4000 in a retirement account which promises a minimum of 5% interest compounded continuously. How long will it take for this investment to triple in value?

Solution. For an initial amount of $4000 and interest rate of at least 5%, we set $P = 4000$ and $r = 0.05$. The investment has tripled when the retirement account reaches three times $4000, so we set $A = 12000$. Plugging these values into $A = Pe^{rt}$, we get

$$12000 = 4000e^{0.05t}$$
$$3 = e^{0.05t} \quad \text{divide by 4000 and simplify}$$
$$\ln 3 = \ln e^{0.05t} \quad \text{take the natural logarithm of both sides}$$
$$\ln 3 = 0.05t \quad \text{exponent property}$$
$$t = \frac{\ln 3}{0.05} \approx 21.9722$$

It will take about 22 years for this investment to triple in value.

□

Exponential Growth and Decay

It turns out that many natural phenomena also experience exponential growth or decay like, for example, uninhibited population growth, or decay of radioactive material.

Exponential Growth or Decay

Suppose a substance grows or decays exponentially as a function of a variable t that represents time, and its amount at time t is $A(t)$. Then $A(t) = A_0e^{kt}$ where A_0 is the initial amount and k is the relative or exponential growth/decay rate.

- If $k > 0$, A grows exponentially.
- If $k < 0$, A decays exponentially.

A few notes are in order.

- A_0 is called the initial amount since it is the value of A at time zero: $A(0) = A_0e^0 = A_0$. We often read 'A sub zero' as 'A naught'.
- In this formula, we can use any positive number b, not equal to one, as the base in place of e, but doing so will change the value of k and its physical meaning.
- This formula and the formula for continuously compounded interest, $A = Pe^{rt}$, are the same. The only differences are the use of A_0 in place of P and k in place of r. So, there is no need to memorize two different formulas.

Example 4.5.4. In order to perform anthrosclerosis research, epithelial cells are harvested from discarded umbilical tissue and grown in the laboratory. A technician observes that a culture of twelve thousand cells grows to five million cells in one week. Assuming that the cells grow exponentially, find a formula for the number of cells, $A(t)$, in thousands, after t days.

Solution. We start with the formula $A(t)=A_0e^{kt}$. Then, since A is to give the number of cells in thousands, we know $A_0=12$, from which $A(t)=12e^{kt}$. To complete the formula, we must determine the growth rate k. We know that after one week the number of cells has grown to five million. Since t measures days and the unit of A is thousands, this translates to $A(7)=5000$. Noting that, additionally, $A(7)=12e^{7k}$, we can put these two equations together and solve to find k.

$$\begin{aligned}
12e^{7k} &= 5000 \\
e^{7k} &= \frac{5000}{12} && \text{divide by 12} \\
\ln e^{7k} &= \ln\left(\frac{5000}{12}\right) && \text{take the natural logarithm of both sides} \\
7k &= \ln\left(\frac{5000}{12}\right) && \text{inverse property} \\
k &= \frac{1}{7}\ln\left(\frac{5000}{12}\right) \approx 0.8618
\end{aligned}$$

The number of cells, in thousands, after t days is $A(t)\approx 12e^{0.8618t}$.

□

Example 4.5.5. Iodine-131 is a commonly used radioactive isotope that helps detect how well the thyroid is functioning. Iodine-131 decays exponentially and after approximately 8 days only one-half of the original amount remains. If 5 grams of Iodine-131 is present initially, find a function that gives the amount of Iodine-131, in grams, t days later.

Solution. We begin with the formula $A(t)=A_0e^{kt}$, with $A_0=5$, so that $A(t)=5e^{kt}$. To complete the formula, we need to determine the growth rate k. Since one-half of the original amount remains after approximately 8 days, we use $A(8)=2.5$ and note that, additionally, $A(8)=5e^{8k}$.

$$\begin{aligned}
5e^{8k} &= 2.5 \\
e^{8k} &= 0.5 && \text{divide by 5} \\
\ln e^{8k} &= \ln(0.5) && \text{take the natural logarithm of both sides} \\
8k &= \ln(0.5) && \text{inverse property}
\end{aligned}$$

So $k = \frac{\ln(0.5)}{8} \approx -0.08664$ and the amount of Iodine-131 after t days is approximately $A(t) = 5e^{-0.08664t}$.

□

You may be wondering when the exponential model is an appropriate model. It turns out that when the growth or decay rate in each time period is proportional to the original amount, we have exponential growth or decay. Examples are a 2% annual interest rate, or population growth rate, and the daily 8.664% decay rate for Iodine-131. The general derivation of this fact will be done in Calculus. We also note that the exponential growth/decay rate k is the ratio of increase or decrease in a time period to the original size so we refer to k as the relative growth rate.

Another property of exponential growth or decay is that the amount of a substance will double or halve in a fixed period, known as doubling time or half-life, respectively, regardless of the initial amount present. In an example from an earlier section, we saw that the doubling time for the population of Utah, assuming 2% annual growth rate, is 35 years. According to **Example 4.5.5**, the half-life of Iodine-131 is 8 days.

Example 4.5.6. The half-life of Iodine-131 is approximately 8 days. If 5 grams of Iodine-131 is present initially, how many days will it take until only 0.625 grams will remain? Do not solve using the function model in the last example.

Solution. Since, after every 8 days, one-half of the starting amount remains, we can form the following table.

t days	Grams of Iodine-131
0	5
8	$\frac{5}{2} = 2.5$
16	$\frac{2.5}{2} = 1.25$
24	$\frac{1.25}{2} = 0.625$

It will take 24 days for the Iodine-131 to decay to only 0.625 grams.

□

Had we been asked to find the time until 1 gram remained, a more prudent approach would be to use the formula $A(t) = 5e^{-0.08664t}$ from **Example 4.5.5**, as follows.

$1 = 5e^{-0.08664t}$

$0.2 = e^{-0.08664t}$ divide by 5

$\ln 0.2 = \ln e^{-0.08664t}$ take the natural logarithm of both sides

$\ln 0.2 = -0.08664t$ inverse property

$t = \dfrac{\ln 0.2}{-0.08664} \approx 18.576$

We see it would take slightly over 18 and a half days for the original 5 grams to decay to 1 gram.

Other Exponential and Logarithmic Models

Example 4.5.7. In 1619, Kepler discovered the relationship $\log T = 1.5\log d - 2.95$ between the period[15] T, in Earth years, of a planet and its average distance d, in millions of miles, from the Sun. Solve this equation for the period T. Although at that time only the six inner planets were known, this formula correctly predicted the periods of other planets unknown at that time. Find the period of the planet Uranus with the average distance of $d = 1783$ million miles from the Sun.

Solution. To solve for T, we convert this equation to exponential form.

$\log T = 1.5\log d - 2.95$

$T = 10^{1.5\log d - 2.95}$ change to exponential form

$T = 10^{1.5\log d}10^{-2.95}$

$T = \left(10^{\log d}\right)^{1.5}10^{-2.95}$

$T = \left(d\right)^{1.5}10^{-2.95}$ inverse property

$T = 10^{-2.95}d^{1.5}$

To find the period of Uranus, we plug in $d = 1783$ and solve for T.

$$T = 10^{-2.95}1783^{1.5} \approx 84.4748$$

The period of Uranus is about 84.5 Earth years.

□

The formula in the prior example was also used to predict the position of the asteroid belt in our solar system, giving reason to think that the asteroid belt is made of material that failed to form a planet.

Example 4.5.8. A defrosted turkey at temperature of 32° F is placed in an oven of constant temperature 350° F. The temperature of the turkey after t hours is $T(t) = 350 - 318e^{kt}$, where k is a

[15] The period of a planet is the time it takes for a planet to complete one revolution of the Sun.

constant. If after 2 hours the temperature of the turkey reaches 125° F, find the constant k. Determine how long it will take for the turkey to reach the safe consumption temperature of 172° F.

Solution. To find k, we set $t = 2$ to get $T(2) = 350 - 318e^{2k}$ and note that $T(2) = 125$.

$$\begin{aligned} 125 &= 350 - 318e^{2k} \\ -225 &= -318e^{2k} \\ \frac{225}{318} &= e^{2k} \\ \ln\left(\frac{225}{318}\right) &= 2k \\ k &= \frac{1}{2}\ln\left(\frac{225}{318}\right) \approx -0.173 \end{aligned}$$

So $T(t) \approx 350 - 318e^{-0.173t}$, and we solve $T(t) = 172$ to find the time it takes the turkey to reach 172° F.

$$\begin{aligned} 350 - 318e^{-0.173t} &= 172 \\ -318e^{-0.173t} &= -178 \\ e^{-0.173t} &= \frac{178}{318} \\ \ln e^{-0.173t} &= \ln\left(\frac{178}{318}\right) \\ -0.173t &= \ln\left(\frac{178}{318}\right) \\ t &= \frac{1}{-0.173}\ln\left(\frac{178}{318}\right) \approx 3.354 \end{aligned}$$

It will take about 3 hours and 21 minutes for the turkey to reach 172° F.

□

The model used in the above example is called Newton's Law of Heating and Cooling. We move on to one last example before ending this section, and chapter.

Example 4.5.9. The number of students, N, in hundreds, at Salt Lake Community College who have acted on the rumor 'Free popsicles at the library!', and received a free popsicle, can be modeled using the equation $N(t) = \dfrac{84}{1 + 2799e^{-t}}$, where $t \geq 0$ is the number of days after June 4, 2018. How many students got a free popsiscle on June 4, 2018? After how many days, will 4200 students have picked up their free popsicle?

Solution. On June 4, 2018, $t = 0$ and

$$\begin{aligned} N(0) &= \frac{84}{1+2799e^{0}} \\ &= \frac{84}{2800} \\ &= \frac{3}{100} \end{aligned}$$

So the number of students who got a free popsicle on June 4, 2018, was $\frac{3}{100}$ hundreds, or $\frac{3}{100} \times 100 = 3$.

To find how many days it takes for 4200 students to get a free popsicle, we need to solve $N(t) = 42$.

$$\begin{aligned} \frac{84}{1+2799e^{-t}} &= 42 \\ 84 &= 42\left(1+2799e^{-t}\right) \\ \frac{84}{42} &= 1+2799e^{-t} \\ 2-1 &= 2799e^{-t} \\ \frac{1}{2799} &= e^{-t} \\ \ln\left(\frac{1}{2799}\right) &= \ln e^{-t} \\ -t &= \ln\left(\frac{1}{2799}\right) \end{aligned}$$

We find $t = -\ln\left(\frac{1}{2799}\right) \approx 7.937$. It takes about 8 days for 4200 students to get a free popsicle. That's a lot of popsicles!

□

The model used in **Example 4.5.9** is called the logistic model.

4.5 Exercises

1. What is the effect of interest on a savings account being compounded monthly versus quarterly?

2. How is continuously compounded interest related to exponential growth and decay?

In Exercises 3 – 8, find each of the following.

(a) the amount *A* in the account as a function of the term of the investment *t* in years;

(b) how much is in the account after 5 years, 10 years, 30 years and 35 years, rounding your answers to the nearest cent;

(c) how long it will take the initial investment to double, rounding your answer to the nearest year.

3. $500 is invested in an account that offers 0.75%, compounded monthly.

4. $500 is invested in an account that offers 0.75%, compounded continuously.

5. $1000 is invested in an account that offers 1.25%, compounded monthly.

6. $1000 is invested in an account that offers 1.25%, compounded continuously.

7. $5000 is invested in an account that offers 2.125%, compounded monthly.

8. $5000 is invested in an account that offers 2.125%, compounded continuously.

9. Look back at your answers to Exercises 3 – 8. What can be said about the difference between monthly compounding and continuously compounding the interest in those situations? With the help of your classmates, discuss scenarios where the difference between monthly and continuously compounded interest would be more dramatic. Try varying the interest rate, the term of the investment and the principal. Use computations to support your answer.

10. How much money needs to be invested now to obtain $2000 in 3 years if the interest rate in a savings account is 0.25%, compounded continuously? Round your answer to the nearest cent.

11. How much money needs to be invested now to obtain $5000 in 10 years if the interest rate in a CD is 2.25%, compounded monthly? Round your interest to the nearest cent.

12. If the Annual Percentage Rate (APR) for a savings account is 0.25% compounded monthly, use the equation $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$ to answer the following.

    (a) For a principal of $2000, how much is in the account after 8 years?

(b) If the original principal was \$2000 and the account now contains \$4000, how many years have passed since the original investment, assuming no other additions or withdrawals have been made?

(c) What principal should be invested so that the account balance is \$2000 in three years?

13. If the Annual Percentage Rate (APR) for a 36-month Certificate of Deposit (CD) is 2.25%, compounded monthly, use the equation $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$ to answer the following.

(a) For a principal of \$2000, how much is in the account after 8 years?

(b) If the original principal was \$2000 and the account now contains \$4000, how many years have passed since the original investment, assuming no other additions or withdrawals have been made?

(c) What principal should be invested so that the account balance is \$2000 in three years?

(d) The Annual Percentage Yield is the simple[16] interest rate that returns the same amount of interest after one year as the compound interest does. Compute the APY for this investment.

14. A finance company offers a promotion on \$5000 loans. The borrower does not have to make any payments for the first three years, however interest will continue to be charged to the loan at 29.9% compounded continuously. What amount will be due at the end of the three year period, assuming no payments are made? If the promotion is extended an additional three years, and no payments are made, what amount will be due?

15. Use the equation $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$ to show that the time it takes for an investment to double in value does not depend on the principal *P*, but rather depends on the APR and the number of compoundings per year. Let $n = 12$ and with the help of your classmates compute the doubling time for a variety of rates *r*. Then look up the Rule of 72 and compare your answers to what that rule says. If you're really interested[17] in Financial Mathematics, you could also compare and contrast the Rule of 72 with the Rule of 70 and the Rule of 69.

[16] There is no compounding with simple interest.
[17] Awesome pun!

In Exercises 16 – 20, we list some radioactive isotopes and their associated half-lives. Assume that each decays according to the formula $A(t) = A_0 e^{kt}$ where A_0 is the initial amount of the material and *k* is a constant representing the rate of decay. For each isotope:

(a) Find the decay constant *k*. Round your answer to four decimal places.

(b) Find a function that gives the amount of isotope *A* that remains after time *t*. (Keep the units of *A* and *t* the same as the given data.

(c) Determine how long it takes for 90% of the material to decay. Round your answer to two decimal places. (HINT: If 90% of the material decays, how much is left?)

16. Cobalt 60, used in food irradiation, initial amount 50 grams, half-life of 5.27 years.

17. Phosphorus 32, used in agriculture, initial amount 2 milligrams, half-life 14 days.

18. Chromium 51, used to track red blood cells, initial amount 75 milligrams, half-life 27.7 days.

19. Americium 241, used in smoke detectors, initial amount 0.29 micrograms, half-life 432.7 years.

20. Uranium 235, used for nuclear power, initial amount 1 kg, half-life 704 million years.

21. With the help of your classmates, show that the time it takes for 90% of each isotope listed in Exercises 16 – 20 to decay does not depend on the initial amount of the substance, but rather on only the decay constant *k*. Find a formula, in terms of *k* only, to determine how long it takes for 90% of a radioactive isotope to decay.

22. The Gross Domestic Product (GDP) of the US (in billions of dollars) *t* years after the year 2000 can be modeled by

$$G(t) = 9743.77e^{0.0514t}$$

(a) Find and interpret $G(0)$.

(b) According to the model, what should have been the GDP in 2007? In 2010? (According to the US Department of Commerce, the 2007 GDP was $14,369.1 billion and the 2010 GDP was $14,657.8 billion.)

23. The diameter *D* of a tumor, in millimeters, *t* days after it is detected is given by

$$D(t) = 15e^{0.0277t}$$

(a) What was the diameter of the tumor when it was originally detected?

(b) How long until the diameter of the tumor doubles?

24. Under optimal conditions, the growth of a certain strain of *E. Coli* is modeled by the Law of Uninhibited Growth $A(t) = A_0 e^{kt}$ where A_0 is the initial number of bacteria and t is the elapsed time, measured in minutes. From numerous experiments, it has been determined that the doubling time of this organism is 20 minutes. Suppose 1000 bacteria are present initially.

(a) Find the growth constant k. Round your answer to four decimal places.

(b) Find a function that gives the number of bacteria $A(t)$ after t minutes.

(c) How long until there are 9000 bacteria? Round your answer to the nearest minute.

25. Yeast is often used in biological experiments. A research technician estimates that a sample of yeast suspension contains 2.5 million organisms per cubic centimeter (cc). Two hours later, she estimates the population density to be 6 million organisms per cc. Let t be the time elapsed since the first observation, measured in hours. Assume that the yeast growth follows the Law of Uninhibited Growth $A(t) = A_0 e^{kt}$.

(a) Find the growth constant k. Round your answer to four decimal places.

(b) Find a function that gives the number of yeast (in millions) per cc $A(t)$ after t hours.

(c) What is the doubling time for this strain of yeast?

26. The Law of Uninhibited Growth also applies to situations where an animal is re-introduced into a suitable environment. Such a case is the reintroduction of wolves to Yellowstone National Park. According to the National Park Service, the wolf population in Yellowstone National Park was 52 in 1996 and 118 in 1999. Using these data, find a function of the form $A(t) = A_0 e^{kt}$ that models the number of wolves t years after 1996. (Use $t = 0$ to represent the year 1996. Also, round your value of k to four decimal places.) According to the model, how many wolves were in Yellowstone in 2002? (The recorded number is 272.)

27. During the early years of a community, it is not uncommon for the population to grow according to the Law of Uninhibited Growth. According to the Painesville Wikipedia entry, in 1860, the village of Painesville had a population of 2649. In 1920, the population was 7272. Use these two data points to fit a model of the form $A(t) = A_0 e^{kt}$ where $A(t)$ is the number of Painesville Residents t years after 1860. (Use $t = 0$ to represent the year 1860. Also, round the value of k to four decimal places.) According to this model, what was the population of Painesville in 2010? (The 2010 census gave the population as 19,563.) What could be some causes for such a vast discrepancy?

28. The population of Sasquatch in Salt Lake County is modeled by

$$P(t) = \frac{120}{1 + 3.167e^{-0.05t}}$$

where $P(t)$ is the population of Sasquatch t years after 2010.

(a) Find and interpret $P(0)$.

(b) Find the population of Sasquatch in Salt Lake County in 2013. Round your answer to the nearest Sasquatch.

(c) When will the population of Sasquatch in Salt Lake County reach 60? Round your answer to the nearest year.

29. The half-life of the radioactive isotope Carbon-14 is about 5730 years.

(a) Use the equation $A(t) = A_0 e^{kt}$ to express the amount of Carbon-14 left from an initial N milligrams as a function of time t in years.

(b) What percentage of the original amount of Carbon-14 is left after 20,000 years?

(c) If an old wooden tool is found in a cave and the amount of Carbon-14 present in it is estimated to be only 42% of the original amount, approximately how old is the tool?

(d) Radiocarbon dating is not as easy as these exercises might lead you to believe. With the help of your classmates, research radiocarbon dating and discuss why our model is somewhat over-simplified.

30. Carbon-14 cannot be used to date inorganic material such as rocks, but there are many other methods of radiometric dating which estimate the age of rocks. One of them, Rubidium-Strontium dating, uses Rubidium-87 which decys to Strontium-87 with a half-life of 50 billion years. Use the equation $A(t) = A_0 e^{kt}$ to express the amount of Rubidium-87 left from an initial 2.3 micrograms as a function of time t in billions of years. Research this and other radiometric techniques and discuss the margins or error for various methods with your classmates.

31. Use the equation $A(t) = A_0 e^{kt}$ to show that $k = -\frac{\ln 2}{h}$ where h is the half-life of the radioactive isotope.

Key Equations

Change of Base Formula: $\log_b x = \dfrac{\log_a x}{\log_a b}$

Compound Interest Formula: $A = P\left(1+\dfrac{r}{n}\right)^{nt}$ where *A* is the amount after *t* years, *P* is the principal, *r* is the interest rate, *n* is the number of compounding periods per year

Continuous Compound Interest Formula: $A = Pe^{rt}$ where *A* is the amount after *t* years, *P* is the principal, *r* is the interest rate

Difference Property of Logarithms: $\log_b\left(\dfrac{u}{v}\right) = \log_b u - \log_b v$

Equivalence between Exponentials and Logarithms: $\log_b y = x \Leftrightarrow b^x = y$

Exponent Property of Logarithms: $\log_b x^m = m\log_b x$

Exponential Function: $f(x) = b^x$ where *b* is a positive real number and $b \neq 1$

Exponential Growth/Decay: $A(t) = A_0 e^{kt}$ where A_0 is the initial amount, *k* is the growth/decay rate

Logarithmic Function: $f(x) = \log_b y$ where $y > 0$ and *b* is a positive constant other than 1; Inverse of an exponential function

Sum Property of Logarithms: $\log_b(uv) = \log_b u + \log_b v$

Key Terms

Common Logarithm: Logarithm with base 10

Compounding Period: number of times per year interest is compounded

Doubling Time: amount of time for a population to double

Exponential Equation: an equation that has an exponent containing a variable

Exponential Function: A function of the form $f(x) = b^x$ where *b* is a positive real number and $b \neq 1$

Half-life: amount of time for a substance to decrease by half

Logarithmic Equation: an equation that contains a logarithm with a variable in its argument

Logarithmic Function: The inverse of an exponential function; $\log_b y$ is the power of *b* that gives *y*

Natural Logarithm: Logarithm with base *e*

Principal: initial amount invested

CHAPTER 5
HOOKED ON CONICS

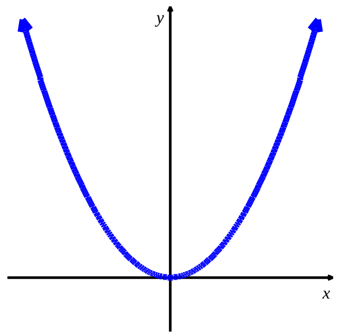


Figure 5.0. 1

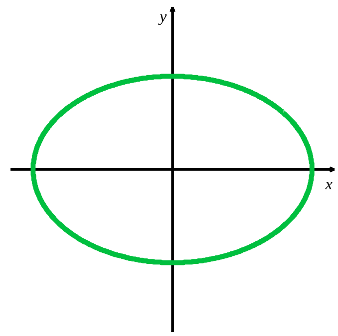


Figure 5.0. 2

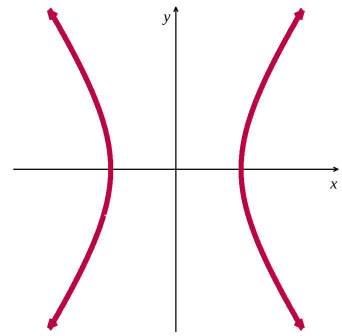


Figure 5.0. 3

Chapter Outline

Introduction

In Chapter 5, we look at circles, parabolas, ellipses and hyperbolas, both from their geometric interpretation as conic sections and as curves defined by a set of points in the coordinate plane. By the end of this chapter, you should be able to look at an equation representing a conic and a) determine which conic is represented: circle, parabola, ellipse, or hyperbola, b) find distinguishing values and points (e.g. distances to and/or locations of vertex/vertices, directrix, focus/foci, major axis, the slopes of asymptotes, all where applicable), c) sketch graphs of conics, and d) use knowledge of conics in application problems.

All of the conic sections are introduced in Section 5.1 through the slicing of a double-napped right circular cone by a plane to show the resulting conic sections. In Section 5.1 we are careful to distinguish between the non-degenerate cases that will be studied in this chapter and the degenerate cases that may also occur; it is important that you understand that degenerate cases may occur even though they will not be emphasized in the chapter. Armed with a general geometric idea of how we might understand conic sections, you will revisit each conic, one at a time, analytically in the next four sections.

In Section 5.2, circles are explored. The section begins by revisiting the definition of a circle and then introduces students to the standard form of a circle; an equation that allows you to think of a circle as a set of ordered pairs that satisfy the equation. In standard form, you can readily find the center and radius of the circle. Two ideas should be taken from this section; first transformations. In general, the standard form of the equation of a circle is a transformation by (h,k) from the origin of the Pythagorean theorem (the basis of the equation of a circle.) Second, in this form, *x* and *y* are variables (the ordered pairs that can be plotted on a plane that satisfy the equation, thus they will remain *x* and *y* in the equation), while *h* and *k* are constants (they must be found and will be 'numbers' that replace *h* and *k* in the final form of the equation). Notice, *x* and *y*, and *h* and *k*, are used in the equation of a circle in the same way *x* and *y*, and *m* and *b* are used in the slope intercept equation/form of a line. Once you are familiar with the standard form of a circle, you learn how to manipulate equations of circles not in standard form, using completing the square, so that it is in standard form and you can readily find the center and radius of the circle.

In Section 5.3, parabolas are explored. You have worked with parabolas before in this course and possibly Intermediate Algebra. This exploration is different however in two ways; first, you will be introduced to the focus and directrix of a parabola, and second, you will work with parabolas that have horizontal lines of symmetry. As with the previous section, you start by being given the standard form for a parabola, how to identify key features of the parabola from that equation, how to graph the parabola and key features, and then finally how to manipulate equations that are not in standard form (using complete the square techniques) so that you can find key features and construct a graph.

In Section 5.4 ellipses are explored. This is likely the first time you have explored this conic section, but you should be able to build from the understandings you developed with circles and parabolas in the previous sections, as we move on to ellipses. Again, the section begins by introducing the standard form for the ellipse and how to use it to find the vertices, major axis, foci and to graph the relation. Then you do the same for equations not in standard form, using completing the square techniques to manipulate the equations.

The last section, 5.5, follows the same structure as the previous sections, but for the hyperbola. By the end of the section, you should be able to identify the vertices, foci, and slopes of asymptotes for a hyperbola, graph the hyperbola, and manipulate an equation in non-standard form for a hyperbola to accomplish all of the above. Additionally, you should be able to look at non-standard forms of all conic section equations (without rotations) and determine which conic is represented. You should be able to convert a non-standard form to an equation in standard form, identify key features, and sketch a graph.

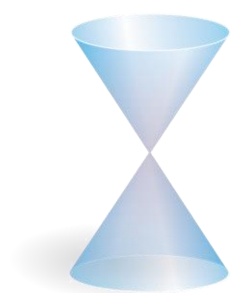
5.1 The Conic Sections

Learning Objective

- Recognize the conic sections that result from slicing a cone with a plane.

In this chapter, we study the **conic sections** – literally sections of a cone. Consider a double-napped right circular cone as seen below.[1] The point of intersection of the upper nappe with the lower nappe is called the vertex.

Figure 5.1. 1



Circles, Ellipses, Parabolas and Hyperbolas

If we slice the cone with a horizontal plane, not containing the vertex, the resulting curve is a **circle**.

Figure 5.1. 2

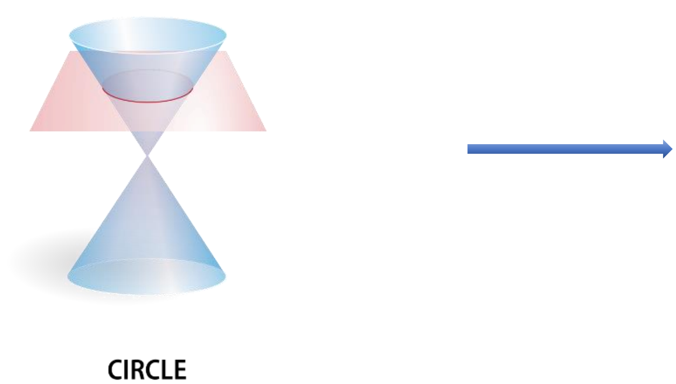
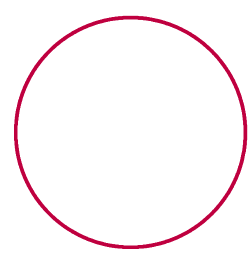


Figure 5.1. 3



[1] Graphics in this section are courtesy of Scott Nicholson. Thank you Scott!

Tilting the plane ever so slightly produces a closed curve called an **ellipse**. This will be the case for any nonhorizontal plane slicing through the two sides of the upper-half or lower-half cone.

Figure 5.1. 4

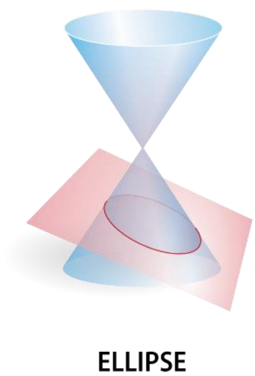
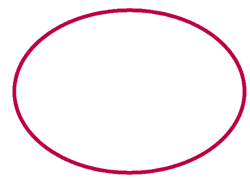


Figure 5.1. 5



If the slicing plane is parallel to the left or the right side of the cone, and does not go through the vertex, the resulting curve is a **parabola**.

Figure 5.1. 6

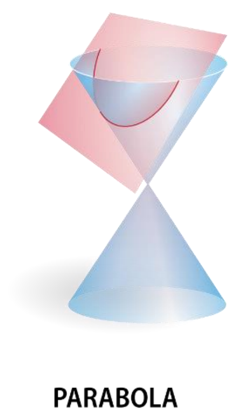
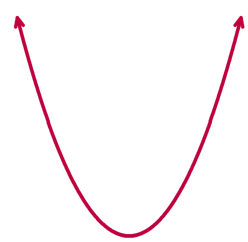


Figure 5.1. 7



If we slice the cone with a vertical plane, not through the vertex, the cross section consists of two disjoint curves, called a **hyperbola**.

Figure 5.1. 8

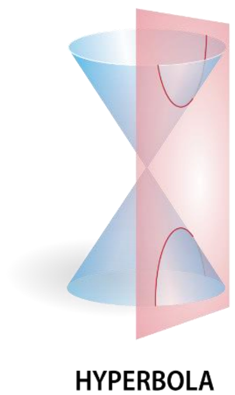
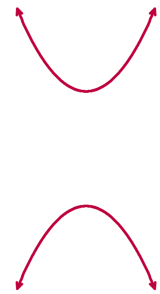


Figure 5.1. 9



The Degenerate Conic Sections

If the slicing plane contains the vertex of the cone, we get the so-called **degenerate conics**: a point, a line or two intersecting lines.

Figure 5.1. 10

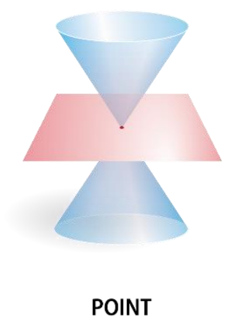


Figure 5.1. 11

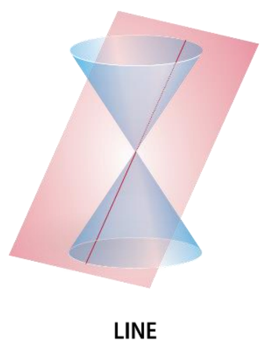
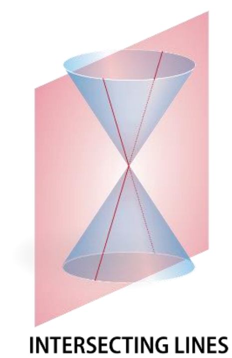


Figure 5.1. 12



In the remainder of this chapter, we will focus our discussion on the non-degenerate cases: circles, parabolas, ellipses and hyperbolas. As you know, points on a circle are equidistant from a point called the center. These other non-degenerate curves also have distance properties. We will use the distance properties to derive their equations.

5.2 Circles

Learning Objectives

- Define a circle in a plane.
- Determine whether an equation represents a circle.
- Graph a circle from a given equation.
- Determine the center and radius of a circle.
- Find the equation of a circle from stated properties.
- Identify the Unit Circle.

The Definition of a Circle

Recall from Geometry that a circle can be determined by fixing a point (called the center) and a positive number (called the radius) as follows.

Definition 5.1. A **circle** with center (h,k) and radius $r>0$ is the set of all points (x,y) in the plane whose distance to (h,k) is r.

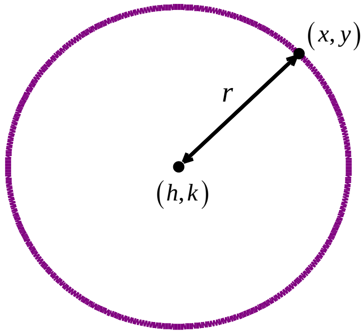


Figure 5.2. 1

By the definition, a point (x,y) is on the circle if its distance to (h,k) is r. We express this relationship algebraically, using the distance formula:[2]

$$r=\sqrt{(x-h)^2+(y-k)^2}$$

[2] Distance Formula: The distance d between the points $P(x_0,y_0)$ and $Q(x_1,y_1)$ is $d=\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}$.

By squaring both sides of this equation, and noting that $r>0$, we get an equivalent equation which gives us the standard equation of a circle.

The Standard Equation of a Circle

Equation 5.1. The Standard Equation of a Circle: The equation of a circle with center (h,k) and radius $r>0$ is $(x-h)^2+(y-k)^2=r^2$.

Example 5.2.1. Write the standard equation of the circle with center $(-2,3)$ and radius 5.

Solution. Here, $(h,k)=(-2,3)$ and $r=5$ so we get

$$(x-h)^2+(y-k)^2=r^2$$
$$(x-(-2))^2+(y-3)^2=(5)^2$$
$$(x+2)^2+(y-3)^2=25$$

□

Example 5.2.2. Find the center and radius of the circle given by the equation $(x+2)^2+(y-1)^2=4$. Graph the circle.

Solution. Matching our equation with the standard form of a circle, **Equation 5.1**, we see that $x+2$ is $x-h$, so $h=-2$, and $y-1$ is $y-k$, so $k=1$. This tells us that our center is $(-2,1)$. We also see that $r^2=4$, and since $r>0$, we get $r=2$. Thus we have a circle centered at $(-2,1)$ with a radius of 2.

To graph the circle, we first plot the center $(-2,1)$. Since the radius is 2, we can locate four points on the circle by plotting points 2 units to the left, right, up and down from the center. These four points can then be connected by a smooth curve to complete the graph.

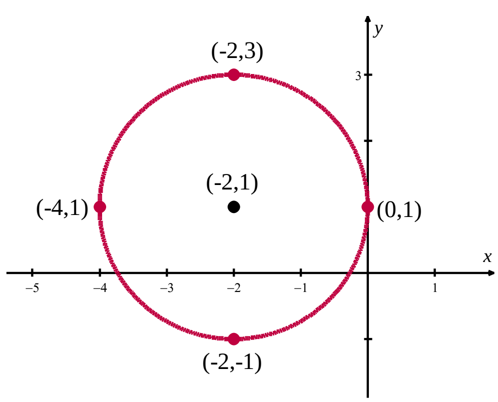


Figure 5.2. 2

□

If we were to expand the equation in the previous example and gather like terms, instead of the easily recognizable $(x+2)^2+(y-1)^2=4$, we'd be contending with $x^2+4x+y^2-2y+1=0$. If we're given such an equation, we can complete the square in each of the variables to see if it fits the form given in **Equation 5.1** by following the steps given below.

To Write the Equation of a Circle in Standard Form

1. Position all terms containing variables on one side of the equation, grouping terms with 'like' variables together. Position the constant on the other side.
2. Complete the square on both variables as needed.
3. Divide both sides by the coefficient of the squares. (For circles, they will be the same.)

Example 5.2.3. Complete the square to find the center and radius of $3x^2-6x+3y^2+4y-4=0$.

Solution.

$$3x^2-6x+3y^2+4y-4=0$$

$$3x^2-6x+3y^2+4y=4 \qquad \text{add 4 to both sides}$$

$$3\left(x^2-2x\right)+3\left(y^2+\frac{4}{3}y\right)=4 \qquad \text{factor out leading coefficients before competing squares}$$

$$3\left(x^2-2x+1\right)+3\left(y^2+\frac{4}{3}y+\frac{4}{9}\right)=4+3(1)+3\left(\frac{4}{9}\right) \qquad \text{multiply by leading coefficients when adding values to right hand side}$$

$$3(x-1)^2+3\left(y+\frac{2}{3}\right)^2=\frac{25}{3} \qquad \text{factor and simplify}$$

$$(x-1)^2+\left(y+\frac{2}{3}\right)^2=\frac{25}{9} \qquad \text{divide both sides by 3}$$

From **Equation 5.1**, we identify $x-1$ as $x-h$, so $h=1$, and $y+\frac{2}{3}$ as $y-k$, so $k=-\frac{2}{3}$. Hence, the center is $(h,k)=\left(1,-\frac{2}{3}\right)$. Furthermore, we see that $r^2=\frac{25}{9}$ so the radius is $r=\frac{5}{3}$.

□

It is possible to obtain equations like $(x-3)^2+(y+1)^2=0$ or $(x-3)^2+(y+1)^2=-1$, neither of which describes a circle. Do you see why not? The reader is encouraged to think about what, if any, points lie on the graphs of these two equations. The next example uses the midpoint formula in conjunction with the ideas presented so far in this section.

Example 5.2.4. Write the standard equation of the circle which has $(-1,3)$ and $(2,4)$ as the endpoints of a diameter.

Solution. We recall that a diameter of a circle is a line segment containing the center and two points on the circle. Plotting the given data yields

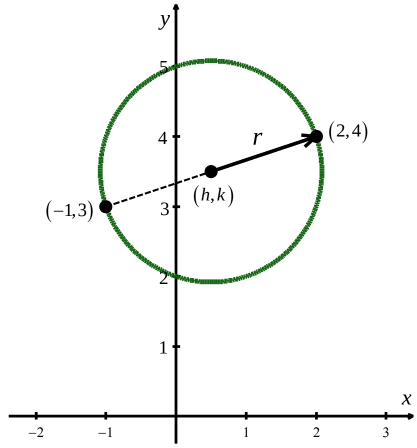


Figure 5.2. 3

Since the given points are endpoints of a diameter, we use the midpoint formula[3] to find the center (h,k).

$$\begin{aligned}(h,k) &= \left(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2}\right)\\ &= \left(\frac{-1+2}{2},\frac{3+4}{2}\right)\\ &= \left(\frac{1}{2},\frac{7}{2}\right)\end{aligned}$$

The diameter of the circle is the distance between the given points, so we know that half of the distance is the radius. Thus,

$$\begin{aligned}r &= \frac{1}{2}\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}\\ &= \frac{1}{2}\sqrt{(2-(-1))^2+(4-3)^2}\\ &= \frac{1}{2}\sqrt{3^2+1^2}\\ &= \frac{\sqrt{10}}{2}\end{aligned}$$

Finally, since $\left(\frac{\sqrt{10}}{2}\right)^2=\frac{10}{4}$, the equation of the circle is $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{7}{2}\right)^2=\frac{10}{4}$.

□

We close this section with a circle that is often referenced in Trigonometry.

[3] Midpoint Formula: The midpoint M of the line segment connecting the points $P(x_0,y_0)$ and $Q(x_1,y_1)$ is $M=\left(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2}\right)$.

The Unit Circle

Definition 5.2. The **Unit Circle** is the circle centered at $(0,0)$ with a radius of 1. The standard equation of the Unit Circle is $x^2 + y^2 = 1$.

Example 5.2.5. Find the points on the Unit Circle that have a *y*-coordinate of $\frac{\sqrt{3}}{2}$.

Solution. We replace *y* with $\frac{\sqrt{3}}{2}$ in the equation $x^2 + y^2 = 1$ to determine the *x*-coordinate(s).

$$x^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$$
$$x^2 + \frac{3}{4} = 1$$
$$x^2 = \frac{1}{4}$$
$$x = \pm\sqrt{\frac{1}{4}}$$
$$x = \pm\frac{1}{2}$$

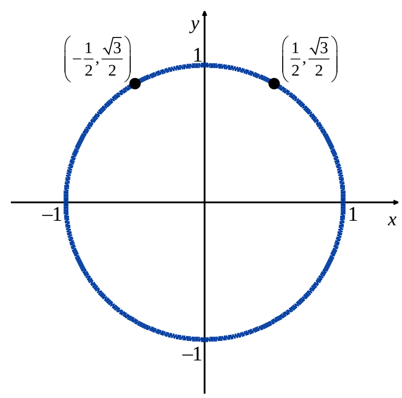


Figure 5.2. 4

We conclude that the points on the Unit Circle with a y-coordinate of $\frac{\sqrt{3}}{2}$ are $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$.

□

5.2 Exercises

1. Define a circle in terms of its center.
2. In the standard equation of a circle, explain why r must be greater than 0.

In Exercises 3 – 8, complete the square in order to put the equation into standard form. Identify the center and the radius or explain why the equation does not represent a circle.

3. $x^2 - 4x + y^2 + 10y = -25$
4. $-2x^2 - 36x - 2y^2 - 112 = 0$
5. $x^2 + y^2 + 8x - 10y - 1 = 0$
6. $x^2 + y^2 + 5x - y - 1 = 0$
7. $4x^2 + 4y^2 - 24y + 36 = 0$
8. $x^2 + x + y^2 - \frac{6}{5}y = 1$

In Exercises 9 – 26, find the center and radius. Graph the circle.

9. $(x+5)^2 + (y+3)^2 = 1$
10. $(x-2)^2 + (y-3)^2 = 9$
11. $(x-4)^2 + (y+2)^2 = 16$
12. $(x+2)^2 + (y-5)^2 = 4$
13. $x^2 + (y+2)^2 = 25$
14. $(x-1)^2 + y^2 = 36$
15. $(x-1)^2 + (y-3)^2 = \frac{9}{4}$
16. $x^2 + y^2 = 64$
17. $x^2 + y^2 = 49$
18. $2x^2 + 2y^2 = 8$
19. $x^2 + y^2 + 2x + 6y + 9 = 0$
20. $x^2 + y^2 - 6x - 8y = 0$
21. $x^2 + y^2 - 4x + 10y - 7 = 0$
22. $x^2 + y^2 + 12x - 14y + 21 = 0$
23. $x^2 + y^2 + 6y + 5 = 0$
24. $x^2 + y^2 - 10y = 0$
25. $x^2 + y^2 + 4x = 0$
26. $x^2 + y^2 - 14x + 13 = 0$

In Exercises 27 – 40, find the standard form of the equation of the circle which has the given properties.

27. Center $(-1,-5)$, Radius 10
28. Center $(4,-2)$, Radius 3
29. Center $\left(-3,\frac{7}{13}\right)$, Radius $\frac{1}{2}$
30. Center $(5,-9)$, Radius $\ln(8)$
31. Center $(-e,\sqrt{2})$, Radius π
32. Center (π,e^2), Radius $\sqrt[3]{91}$
33. Center $(3,5)$, containing the point $(-1,-2)$
34. Center $(3,6)$, containing the point $(-1,4)$
35. Center $(3,-2)$, containing the point $(3,6)$
36. Center $(6,-6)$, containing the point $(2,-3)$
37. Center $(4,4)$, containing the point $(2,2)$
38. Center $(-5,6)$, containing the point $(-2,3)$

39. Endpoints of a diameter are $(3,6)$ and $(-1,4)$

40. Endpoints of a diameter are $\left(\frac{1}{2},4\right)$ and $\left(\frac{3}{2},-1\right)$

41. The Giant Wheel at Cedar Point is a circle with diameter 128 feet which sits on an 8 foot tall platform, resulting in an overall height of 136 feet. Find an equation for the wheel assuming that its center lies on the *y*-axis and that the ground is the *x*-axis.

42. Verify that the following points lie on the Unit Circle: $(\pm 1,0)$, $(0,\pm 1)$, $\left(\pm\frac{\sqrt{2}}{2},\pm\frac{\sqrt{2}}{2}\right)$, $\left(\pm\frac{1}{2},\pm\frac{\sqrt{3}}{2}\right)$ and $\left(\pm\frac{\sqrt{3}}{2},\pm\frac{1}{2}\right)$.

5.3 Parabolas

Learning Objectives

- Define a parabola in a plane.
- Determine whether an equation represents a parabola.
- Graph a parabola from a given equation.
- Determine the vertex, focus and directrix of a parabola.
- Find the equation of a parabola from a graph or from stated properties.
- Solve applications of parabolas.

We have already learned that the graph of a quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, is called a **parabola**. We may also define parabolas in terms of distance.

The Definition of a Parabola

Definition 5.3. Let *F* be a point in the plane and let *D* be a line not containing *F*. A **parabola** is the set of all points equidistant from *F* and *D*. The point *F* is called the **focus** of the parabola and the line *D* is called the **directrix** of the parabola.

Schematically, we have the following.

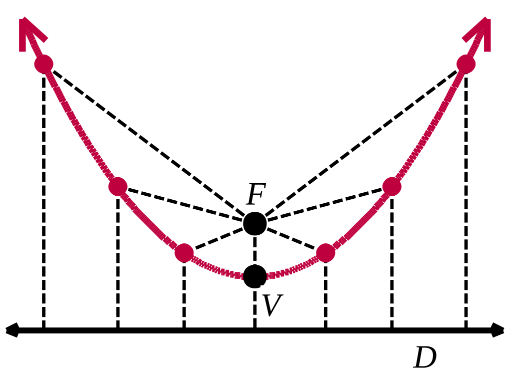


Figure 5.3. 1

Each dashed line from the point *F* to a point on the curve has the same length as the dashed line from the point on the curve to the line *D*. The point suggestively labeled *V* is, as you should expect, the **vertex**. The vertex is the point on the parabola closest to the focus.

The Equation of a Vertical Parabola with Vertex $(0,0)$

We want to use only the distance definition of parabola to derive the equation of a vertical[4] parabola. Let p denote the directed[5] distance from the vertex to the focus, which by definition is the same as the distance from the vertex to the directrix. For simplicity, assume that the vertex is $(0,0)$ and that the parabola opens upward. Hence, the focus is $(0, p)$ and the directrix is the line $y = -p$. Our picture becomes

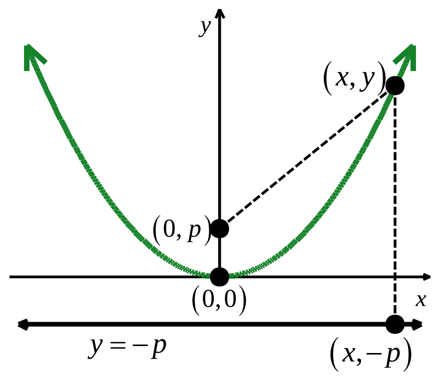


Figure 5.3. 2

From the definition of parabola, we know the distance from $(0, p)$ to (x, y) is the same as the distance from $(x,-p)$ to (x, y). Using the distance formula[6], we get

$$
\begin{aligned}
\sqrt{(x-0)^2+(y-p)^2} &= \sqrt{(x-x)^2+(y-(-p))^2} \\
\sqrt{x^2+(y-p)^2} &= \sqrt{(y+p)^2} \\
x^2+(y-p)^2 &= (y+p)^2 && \textbf{square both sides} \\
x^2+y^2-2py+p^2 &= y^2+2py+p^2 && \textbf{expand quantities} \\
x^2 &= 4py && \textbf{gather like terms}
\end{aligned}
$$

Solving for y yields $y = \dfrac{x^2}{4p}$, which is a quadratic function of the form $y = ax^2$, with $a = \dfrac{1}{4p}$ and vertex $(0,0)$.

[4] A 'vertical' parabola opens either upward or downward.
[5] We'll talk more about what 'directed' means later.
[6] Distance Formula: The distance d between the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is $d = \sqrt{(x_1-x_0)^2+(y_1-y_0)^2}$.

We know from previous experience that if the coefficient of x^2 is negative, the parabola opens downward. In the equation $y=\frac{x^2}{4p}$, this happens when $p<0$. In our formulation, we say that *p* is a 'directed distance' from the vertex to the focus: if $p>0$, the focus is above the vertex; if $p<0$, the focus is below the vertex. The **focal length** of a parabola is $|p|$.

Equation 5.2. The Standard Equation of a Vertical Parabola with Vertex (0,0): The equation of a vertical parabola with vertex $(0,0)$ and focal length $|p|$ is

$$x^2=4py$$

If $p>0$, the parabola opens upward; if $p<0$, the parabola opens downward.

Notice that in the standard equation of the parabola, above, only one of the variables, *x*, is squared. This is a quick way to distinguish an equation of a parabola from that of a circle since both variables are squared in the equation of a circle.

Example 5.3.1. Graph the parabola given by the equation $x^2=-6y$. Find the focus and directrix.

Solution. We recognize this as the form given in **Equation 5.2**. We see that $4p=-6$ so $p=-\frac{3}{2}$. Since $p<0$, the focus will be below the vertex and the parabola will open downward.

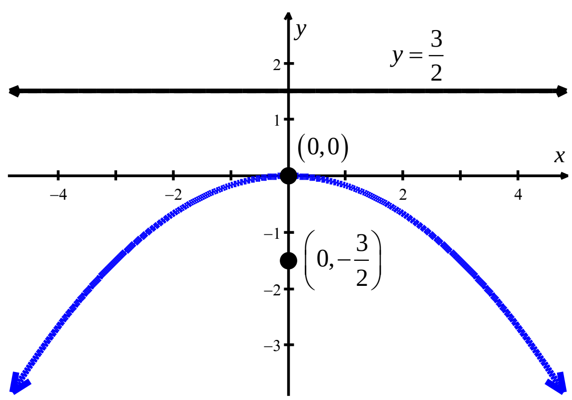


Figure 5.3. 3

The distance from the vertex to the focus is $|p|=\frac{3}{2}$, which means the focus is $\frac{3}{2}$ units below the vertex. Thus, we find the focus at $\left(0,-\frac{3}{2}\right)$. The directrix, then, is $\frac{3}{2}$ units above the vertex, so it is the line $y=\frac{3}{2}$.

□

Of all of the information requested in the previous example, only the vertex is part of the graph of the parabola. So in order to get a sense of the actual shape of the graph, we need some more information.

While we could plot a few points randomly, a more useful measure of how wide a parabola opens is the length of the parabola's latus rectum. The **latus rectum** of a parabola is the line segment parallel to the directrix which contains the focus. The endpoints of the latus rectum are, then, two points on opposite sides of the parabola. Graphically, we have the following.

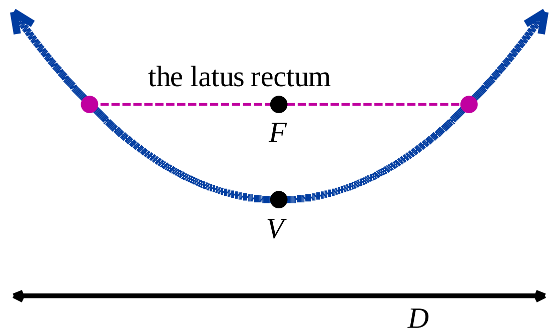


Figure 5.3. 4

It turns out[7] that the length of the latus rectum, called the **focal diameter** of the parabola, is $|4p|$, which, in light of **Equation 5.2**, is easy to find. In our last example, for instance, when graphing $x^2 = -6y$, we can use the fact that the focal diameter is $|-6| = 6$, which means the parabola is 6 units wide at the focus. To generate a more accurate graph, we can plot points 3 units to the left and right of the focus.

The following diagram summarizes the key features of the parabola.

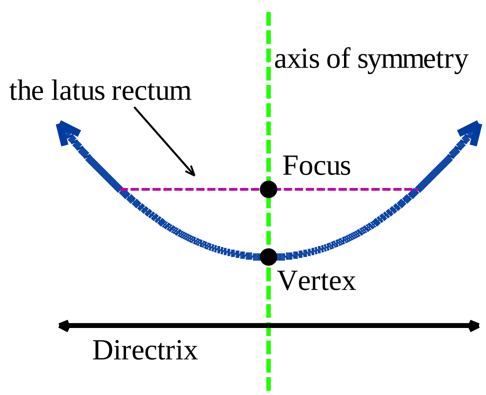


Figure 5.3. 5

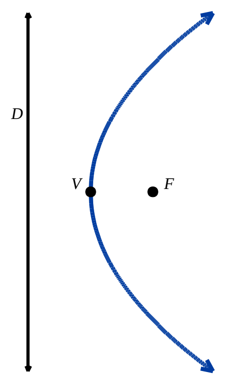
We learned about the parabola's vertex and axis of symmetry while studying quadratic functions. Notice that the **axis of symmetry** passes through both the focus and the vertex and is perpendicular to the directrix.

[7] Consider this an exercise.

The Equation of a Horizontal Parabola with Vertex $(0,0)$

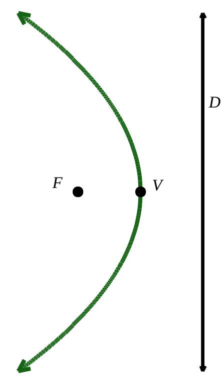
If we interchange the roles of *x* and *y*, we can produce horizontal parabolas: parabolas which open to the right or to the left. The directrices[8] of such animals would be vertical lines and the focus would either lie to the right or to the left of the vertex, as seen below.

Figure 5.3. 6



Parabola opening to the right

Figure 5.3. 7



Parabola opening to the left

Equation 5.3. The Standard Equation of a Horizontal Parabola with Vertex (0,0): The equation of a horizontal parabola with vertex $(0,0)$ and focal length $|p|$ is

$$y^2 = 4px$$

If $p > 0$, the parabola opens to the right; if $p < 0$, it opens to the left.

Example 5.3.2. Graph the parabola given by the equation $y^2 = 12x$. Find the focus and directix and the endpoints of the latus rectum.

Solution. We recognize this as the form given in **Equation 5.3**. We see that $4p = 12$ so $p = 3$. Since $p > 0$, the focus will be to the right of the vertex and the parabola will open to the right. The distance from the vertex to the focus is $|p| = 3$, which means the focus is 3 units to the right. If we start at the vertex, $(0,0)$, and move 3 units to the right, we arrive at the focus $(3,0)$. The directix, then, is 3 units to the left of the vertex, and is the vertical line $x = -3$. Since the focal diameter is $|4p| = 12$, the parabola is 12 units wide at the focus, and thus there are points 6 units above and below the focus on the parabola. The resulting endpoints of the latus rectum are $(3,6)$ and $(3,-6)$.

[8] plural of directrix

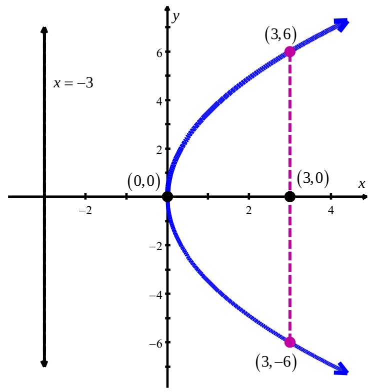


Figure 5.3. 8

□

The Equation of a Parabola with Vertex (h,k)

If we choose to place the vertex at an arbitrary point (h,k), we arrive at the following formulas using either transformations or re-deriving the formula from **Definition 5.3**.

> **Equation 5.4. The Standard Equation of a Parabola with Vertex (*h*, *k*):**
>
> - The equation of a vertical parabola with vertex (h,k) and focal length $|p|$ is
> $$(x-h)^2 = 4p(y-k)$$
>   If $p>0$, the parabola opens upward; if $p<0$, it opens downward.
> - The equation of a horizontal parabola with vertex (h,k) and focal length $|p|$ is
> $$(y-k)^2 = 4p(x-h)$$
>   If $p>0$, the parabola opens to the right; if $p<0$, it opens to the left.

Example 5.3.3. Graph the parabola given by the equation $(x+1)^2 = 8(y-3)$. Find the vertex, focus, directrix and the length of the latus rectum.

Solution. This is a vertical parabola of the form $(x-h)^2 = 4p(y-k)$. Here, $x-h$ is $x+1$ so that $h=-1$, and $y-k$ is $y-3$, from which $k=3$. Hence, the vertex is $(-1,3)$. We also see that $4p=8$, so $p=2$. Since $p>0$, the focus will be above the vertex and the parabola will open upward.

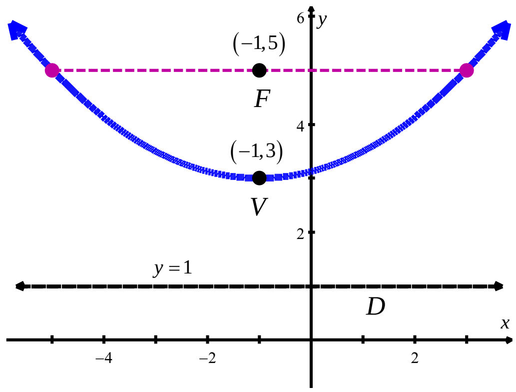


Figure 5.3. 9

The distance from the vertex to the focus is $|p|=2$, which means the focus is 2 units above the vertex. From $(-1,3)$, we move up 2 units and find the focus at $(-1,5)$. The directrix, then, is 2 units below the vertex, so it is the line $y=1$. We see that the length of the latus rectum, also the focal diameter, is $|4p|=8$. Thus, there are points on the parabola that are 4 units to the left and 4 units to the right of the focus. These points, $(-5,5)$ and $(3,5)$, are the endpoints of the latus rectum.

□

Example 5.3.4. Find the standard form of the parabola with focus $(2,1)$ and directrix $y=-4$.

Solution. Sketching the data yields

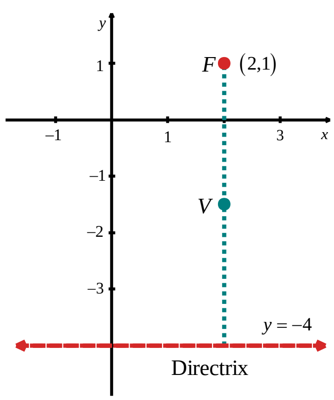


Figure 5.3. 10

From the diagram, we see the parabola opens upward. (Take a moment to think about it if you don't see that immediately.) Hence, the vertex lies below the focus and has an *x*-coordinate of 2.

To find the *y*-coordinate, we note that the vertex lies halfway between the focus and the directrix, and that the distance from the focus to the directrix is $1-(-4)=5$. Thus, the the vertex lies $\frac{5}{2}$ units below the focus. Starting at $(2,1)$ and moving down $\frac{5}{2}$ units leaves us at $\left(2,-\frac{3}{2}\right)$, which is our vertex.

Since the parabola opens upward, we know *p* is positive. Thus, $p=\frac{5}{2}$. Plugging all of this data into $(x-h)^2=4p(y-k)$ gives us

$$(x-2)^2=4\left(\frac{5}{2}\right)\left(y-\left(-\frac{3}{2}\right)\right)$$
$$(x-2)^2=10\left(y+\frac{3}{2}\right)$$

□

As with circles, not all parabolas will come to us in the forms of **Equation 5.4**. If we encounter an equation with two variables in which exactly one variable is squared, we can attempt to put the equation into a standard form using the following steps.

To Write the Equation of a Parabola in Standard Form

1. Position all terms containing the variable which is squared on one side of the equation and position terms containing the non-squared variable and the constant on the other side.

2. Complete the square if necessary, and then divide both sides of the equation by the coefficient of the square.

3. On the non-squared side of the equation, factor out the coefficient from the variable and the constant.

Example 5.3.5. Consider the equation $y^2+4y+8x=4$. Put this equation into standard form and graph the parabola. Find the vertex, focus and directrix.

Solution. We need a perfect square (in this case, using *y*) on the left-hand side of the equation and we factor out the coefficient of the non-squared variable (in this case, *x*) on the other side.

$$y^2+4y+8x=4$$
$$y^2+4y=-8x+4$$
$$y^2+4y+4=-8x+4+4 \quad \text{complete the square in } y \text{ only}$$
$$(y+2)^2=-8x+8 \quad \text{factor}$$
$$(y+2)^2=-8(x-1)$$

Now that the equation is in the form given in **Equation 5.4**, we see that $x-h$ is $x-1$ so that $h=1$, and $y-k$ is $y+2$, from which $k=-2$. Hence, the vertex is $(1,-2)$. We also see that $4p=-8$ so that $p=-2$.

Since $p<0$, the focus will be left of the vertex and the parabola will open to the left. The distance from the vertex to the focus is $|p|=2$, which means the focus is 2 units to the left of the vertex. If we start at $(1,-2)$ and move left 2 units, we arrive at the focus $(-1,-2)$. The directrix, then, is 2 units to the right of the vertex. If we move right 2 units from $(1,-2)$, we'd be on the vertical line $x=3$. Since the focal diameter is $|4p|=8$, the parabola is 8 units wide at the focus and there are points on the parabola that are 4 units above and 4 units below the focus.

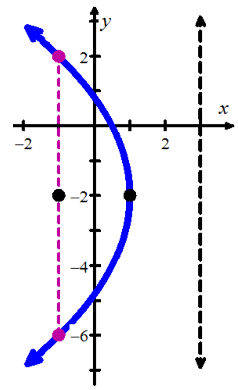


Figure 5.3. 11

□

Applications of Parabolas

In studying quadratic functions, we have seen parabolas used to model physical phenomena such as the trajectories of projectiles. Other applications of the parabola concern its reflective property which necessitates knowing about the focus of a parabola. For example, many satellite dishes are formed in the shape of a **paraboloid**, a parabola revolved about its axis of symmetry, as depicted in the following illustration.

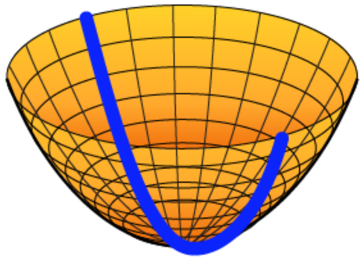


Figure 5.3. 12

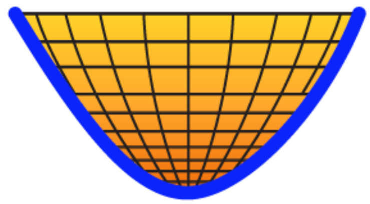


Figure 5.3. 13

Every cross section through the vertex of a paraboloid is a parabola with the same focus. To see why this is important, imagine the dotted lines below as electromagnetic waves heading toward a parabolic dish. It turns out that the waves reflect off the parabola and concentrate at the focus which then becomes the optimal place for the receiver.

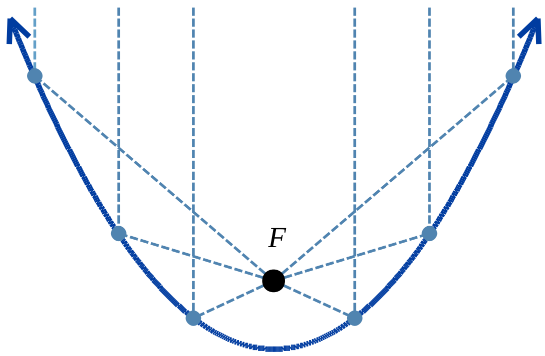


Figure 5.3. 14

If, on the other hand, we imagine the dotted lines as emanating from the focus, we see that the waves are reflected off the parabola in a coherent fashion as in the case of a flashlight. Here, the bulb is placed at the focus and the light rays are reflected off a parabolic mirror to give directional light.

Example 5.3.6. A satellite dish is to be constructed in the shape of a paraboloid. If the receiver placed at the focus is located 2 feet above the vertex of the dish, and the dish is to be 12 feet wide, how deep will the dish be?

Solution. One way to approach this problem is to determine the equation of the parabola suggested to us by this data. For simplicity, we'll assume the vertex is $(0,0)$ and the parabola opens upward. Our standard form for such a parabola is $x^2 = 4py$. Since the focus is 2 feet above the vertex, we know $p = 2$ so that $x^2 = 8y$.

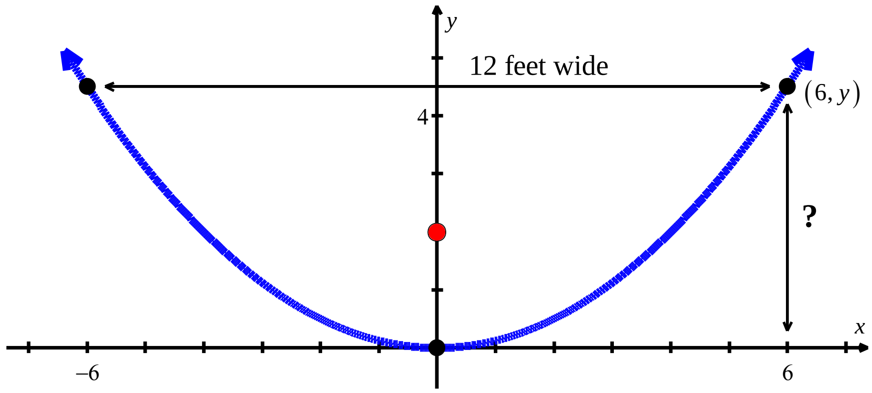


Figure 5.3. 15

With the dish having a width of 12 feet, we know the edge is 6 feet from the vertex. To find the depth, we are looking for the y value when $x = 6$. Substituting $x = 6$ into the equation of the parabola yields

$$6^2 = 8y$$
$$y = \frac{36}{8} = 4.5$$

Hence, the dish will be 4.5 feet deep.

□

Parabolas are used to design many objects we use every day, such as telescopes, suspension bridges, microphones and radar equipment. Parabolic mirrors have a unique reflecting property. When rays of light parallel to the parbola's axis of symmetry are directed toward any surface of the mirror, the light is reflected directly to the focus. (See the diagram preceding **Example 5.3.6**.) Parabolic mirrors have the ability to focus the sun's energy to a single point, raising the temperature hundreds of degrees in a matter of seconds. Thus, parabolic mirrors are featured in many low-cost, energy efficient, solar products such as solar cookers, solar heaters, and even travel-sized fire starters.

Example 5.3.7. A cross-section of a design for a travel-sized solar fire starter is shown below.

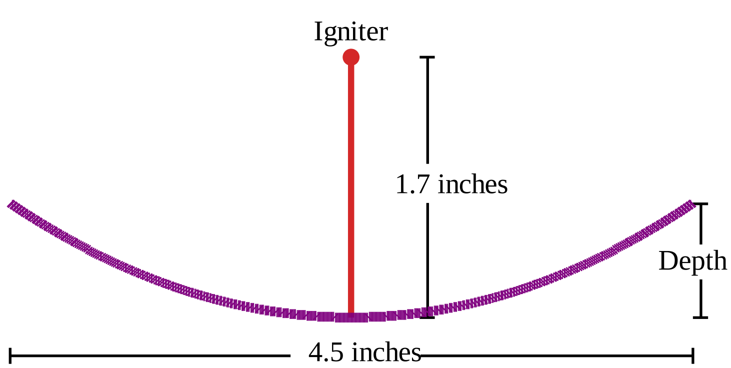


Figure 5.3. 16

The sun's rays reflect off the parabolic mirror toward an object attached to the igniter. Because the igniter is located at the focus of the parabola, the reflected rays cause the object to burn in just seconds. Use the dimensions provided in the figure to determine the depth of the fire starter.

Solution. We begin by finding an equation for the parabola that models the fire starter. We assume that the vertex of the parabolic mirror is the origin of the coordinate plane and that the parabola has the standard form $x^2 = 4py$, where $p > 0$. The igniter, which is the focus, is 1.7 inches above the vertex of the dish. Thus we have $p = 1.7$, which we substitute into $x^2 = 4py$ to find an equation for the parabola.

$$x^2 = 4(1.7)y$$
$$x^2 = 6.8y$$

The dish extends $\frac{4.5}{2} = 2.25$ inches on either side of the origin. We can substitute 2.25 for x in the equation to find the depth of the dish.

$$x^2 = 6.8y$$
$$(2.25)^2 = 6.8y$$
$$y \approx 0.74$$

The dish is approximately 0.74 inches deep.

□

5.3 Exercises

1. Define a parabola in terms of its focus and directrix.
2. If the equation of a parabola is written in standard form and p is positive and the directrix is a vertical line, then what can we conclude about its graph?
3. If the equation of a parabola is written in standard form and p is negative and the directrix is a horizontal line, then what can we conclude about its graph?
4. As the graph of a parabola becomes wider, what happens to the distance between the focus and the directrix?

In Exercises 5 – 8, determine whether the given equation represents a parabola. If it does, rewrite the equation in standard form.

5. $y^2 = 4 - x^2$
6. $y = 4x^2$
7. $y^2 + 12x - 6y - 51 = 0$
8. $3x^2 - 6y^2 = 12$

In Exercises 9 – 16, find the vertex, the focus and the directrix. Graph the parabola. Include the endpoints of the latus rectum in your sketch.

9. $(x-3)^2 = -16y$
10. $\left(x+\frac{7}{3}\right)^2 = 2\left(y+\frac{5}{2}\right)$
11. $(y-2)^2 = -12(x+3)$
12. $(y+4)^2 = 4x$
13. $(x-1)^2 = 4(y+3)$
14. $(x+2)^2 = -20(y-5)$
15. $(y-4)^2 = 18(x-2)$
16. $\left(y+\frac{3}{2}\right)^2 = -7\left(x+\frac{9}{2}\right)$

In Exercises 17 – 22, put the equation into standard form. Find the vertex, the focus and the directrix. Graph the parabola.

17. $y^2 - 10y - 27x + 133 = 0$
18. $25x^2 + 20x + 5y - 1 = 0$
19. $x^2 + 2x - 8y + 49 = 0$
20. $2y^2 + 4y + x - 8 = 0$
21. $x^2 - 10x + 12y + 1 = 0$
22. $3y^2 - 27y + 4x + \frac{211}{4} = 0$

In Exercises 23 – 31, find the standard form of the equation of the parabola which has the given properties.

23. Vertex $(0,0)$, Directrix $y = 4$, Focus $(0,-4)$

24. Vertex $(0,0)$, Directrix $x=4$, Focus $(-4,0)$

25. Vertex $(-2,3)$, Directrix $x=-\frac{7}{2}$, Focus $\left(-\frac{1}{2},3\right)$

26. Vertex $(1,2)$, Directrix $y=\frac{11}{3}$, Focus $\left(1,\frac{1}{3}\right)$

27. Vertex $(0,0)$ with endpoints of the latus rectum $(2,1)$ and $(-2,1)$

28. Vertex $(0,0)$ with endpoints of the latus rectum $(-2,4)$ and $(-2,-4)$

29. Vertex $(1,2)$ with endpoints of the latus rectum $(-5,5)$ and $(7,5)$

30. Vertex $(-3,-1)$ with endpoints of the latus rectum $(0,5)$ and $(0,-7)$

31. Vertex $(-8,-9)$, containing the points $(0,0)$ and $(-16,0)$

In Exercises 32 – 36, given the graph of the parabola, determine its equation.

32.

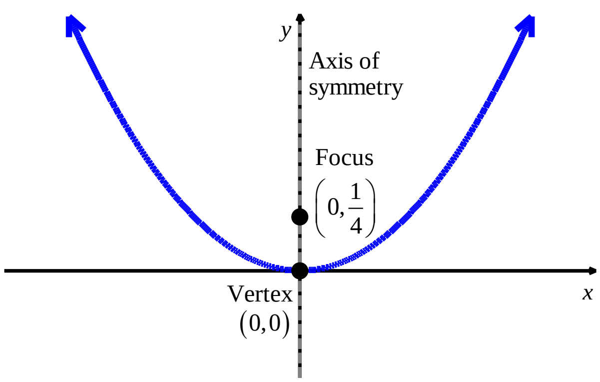


Figure 5.3.17

33.

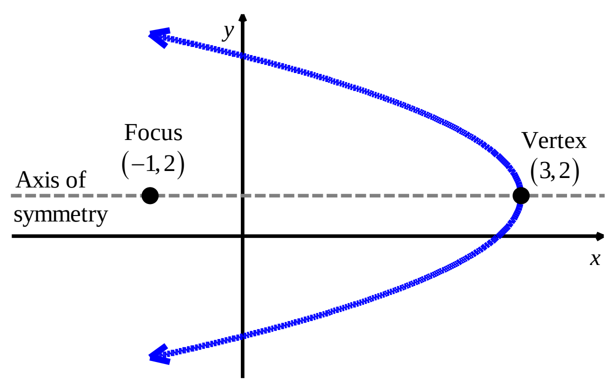


Figure 5.3.18

34.

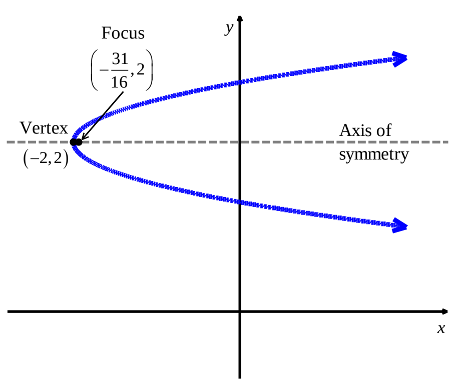


Figure 5.3.19

35.

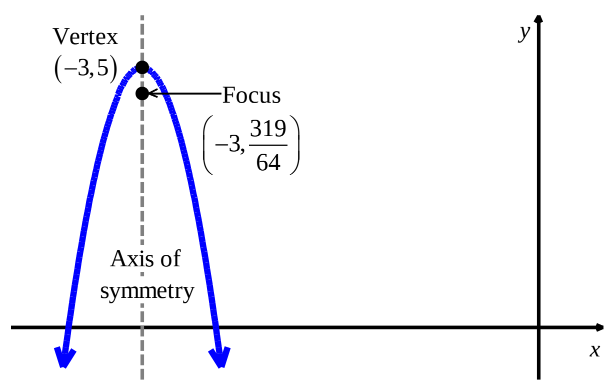


Figure 5.3.20

36.

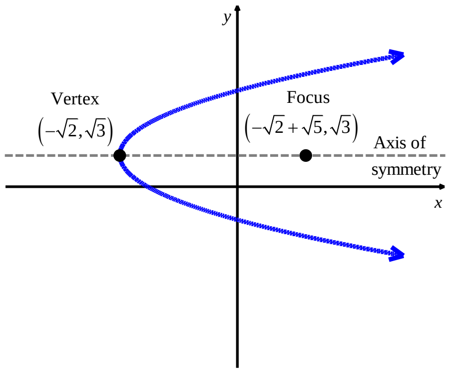


Figure 5.3.21

37. The mirror in Carl's flashlight is a paraboloid. If the mirror is 5 centimeters in diameter and 2.5 centimeters deep, where should the light bulb be placed so it is at the focus of the mirror?

38. A parabolic Wi-Fi antenna is constructed by taking a flat sheet of metal and bending it into a parabolic shape. If the cross section of the antenna is a parabola which is 45 centimeters wide and 25 centimeters deep, where should the receiver be placed to maximize reception?

39. A parabolic arch is constructed which is 6 feet wide at the base and 9 feet tall in the middle. Find the height of the arch exactly 1 foot in from the base of the arch.

40. A satellite dish is shaped like a paraboloid. The receiver is to be located at the focus. If the dish is 12 feet across at its opening and 4 feet deep at its center, where should the receiver be placed?

41. A searchlight is shaped like a paraboloid. A light source is located 1 foot from the base along the axis of symmetry. If the opening of the searchlight is 3 feet across, find the depth.

42. An arch is in the shape of a parabola. It has a span of 100 feet and a maximum height of 20 feet. Find the equation of the parabola, and determine the height of the arch 40 feet from the center.

43. Balcony-sized solar cookers have been designed for families living in India. The top of a dish has a diameter of 1,600 mm. The sun's rays reflect off the parabolic mirror toward the "cooker", which is placed 320 mm from the base. Find the depth of the cooker.

5.4 Ellipses

Learning Objectives

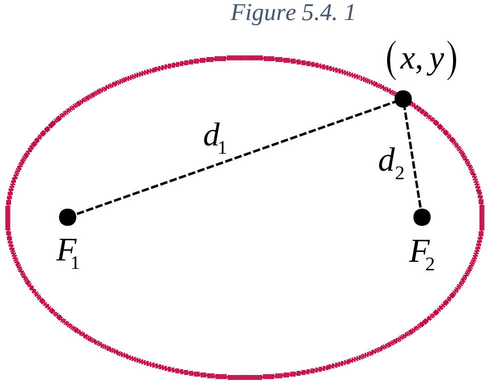
- Define an ellipse in a plane.
- Determine whether an equation represents an ellipse.
- Graph an ellipse from a given equation.
- Determine the center, vertices, foci and eccentricity of an ellipse.
- Find the equation of an ellipse from a graph or from stated properties.
- Solve applications of ellipses.

In the definition of a circle, **Definition 5.1**, we fixed a point called the **center** and considered all of the points which were a fixed distance r from that one point. For our next conic section, the ellipse, we fix two distinct points and use a distance d in our definition.

The Definition of an Ellipse

Definition 5.4. Given two distinct points F_1 and F_2 in the plane and a fixed distance d, an **ellipse** is the set of all points (x, y) in the plane such that the sum of the distance from F_1 to (x, y) and the distance from F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci**[9] of the ellipse.

Figure 5.4. 1



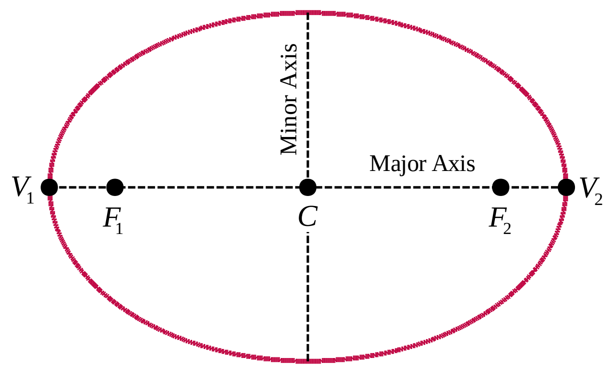
$d_1 + d_2 = d$ for all (x, y) on the ellipse

[9] 'Foci' is the plural of 'focus'.

We may imagine taking a length of string and anchoring it to two points on a piece of paper. The curve traced out by taking a pencil and moving it so the string is always taut is an ellipse.

The **center** of the ellipse is the midpoint of the line segment connecting the two foci. The **major axis** of the ellipse is the line segment connecting two opposite ends of the ellipse which also contains the center and foci. The **minor axis** of the ellipse is the line segment connecting two opposite ends of the ellipse which contains the center but is perpendicular to the major axis. The **vertices** of an ellipse are the points of the ellipse which lie on the major axis. Notice that the center is also the midpoint of the major axis, hence it it the midpoint of the vertices.

Figure 5.4. 2



An ellipse with center C, foci F_1 and F_2, vertices V_1 and V_2

The Equation of an Ellipse with Center at $(0,0)$

Note that the major axis is the longer of the two axes through the center, and likewise, the minor axis is the shorter of the two. In order to derive the standard equation of an ellipse, we assume that the ellipse has its center at $(0,0)$, its major axis along the x-axis, foci $(c,0)$ and $(-c,0)$, and vertices $(a,0)$ and $(-a,0)$. We will label the y-intercepts of the ellipse as $(0,b)$ and $(0,-b)$. We assume a, b, and c are all positive numbers.

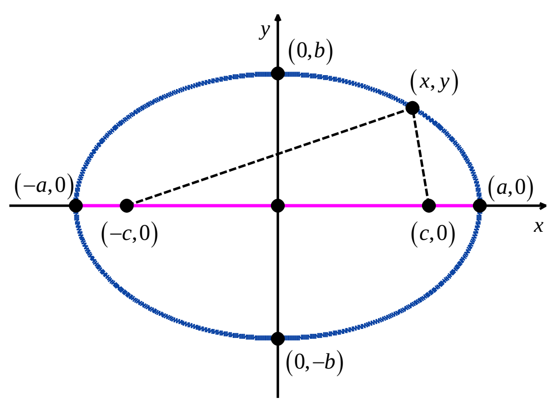


Figure 5.4. 3

Note that since $(a,0)$ is on the ellipse, it must satisfy the conditions of **Definition 5.4**. That is, the distance from $(-c,0)$ to $(a,0)$ plus the distance from $(c,0)$ to $(a,0)$ must equal the fixed distance d. Since all of these points lie on the x-axis, we get the following.

$$\begin{aligned}[\text{distance from } (-c,0) \text{ to } (a,0)] + [\text{distance from } (c,0) \text{ to } (a,0)] &= d \\ (a+c)+(a-c) &= d \\ 2a &= d\end{aligned}$$

In other words, the fixed distance *d* mentioned in the definition of the ellipse is none other than the length of the major axis. We now use the fact that $(0,b)$ is on the ellipse, along with the fact that $d = 2a$, to get

$$\begin{aligned}[\text{distance from } (-c,0) \text{ to } (0,b)] + [\text{distance from } (c,0) \text{ to } (0,b)] &= 2a \\ \sqrt{(0-(-c))^2+(b-0)^2}+\sqrt{(0-c)^2+(b-0)^2} &= 2a \quad \textbf{apply distance formula} \\ \sqrt{b^2+c^2}+\sqrt{b^2+c^2} &= 2a \\ 2\sqrt{b^2+c^2} &= 2a \\ \sqrt{b^2+c^2} &= a\end{aligned}$$

From this, we get $a^2 = b^2 + c^2$, or $b^2 = a^2 - c^2$, which will prove useful later. Now consider a point (x,y) on the ellipse. Applying **Definition 5.4**, we get

$$\begin{aligned}[\text{distance from } (-c,0) \text{ to } (x,y)] + [\text{distance from } (c,0) \text{ to } (x,y)] &= 2a \\ \sqrt{(x-(-c))^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2} &= 2a \quad \textbf{apply distance formula} \\ \sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2} &= 2a\end{aligned}$$

In order to make sense of this situation, we need to make good use of Intermediate Algebra.

$$\begin{aligned}\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2} &= 2a \\ \sqrt{(x+c)^2+y^2} &= 2a-\sqrt{(x-c)^2+y^2} \\ \left(\sqrt{(x+c)^2+y^2}\right)^2 &= \left(2a-\sqrt{(x-c)^2+y^2}\right)^2 && \textbf{square both sides} \\ (x+c)^2+y^2 &= 4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2 && \textbf{expand} \\ 4a\sqrt{(x-c)^2+y^2} &= 4a^2+(x-c)^2-(x+c)^2 && \textbf{simplify} \\ 4a\sqrt{(x-c)^2+y^2} &= 4a^2-4cx && \textbf{expand and simplify} \\ a\sqrt{(x-c)^2+y^2} &= a^2-cx && \textbf{simplify} \\ \left(a\sqrt{(x-c)^2+y^2}\right)^2 &= \left(a^2-cx\right)^2 && \textbf{square both sides} \\ a^2\left((x-c)^2+y^2\right) &= a^4-2a^2cx+c^2x^2 && \textbf{expand} \\ a^2x^2-2a^2cx+a^2c^2+a^2y^2 &= a^4-2a^2cx+c^2x^2 && \textbf{expand and distribute}\end{aligned}$$

$$a^2x^2 - c^2x^2 + a^2y^2 = a^4 - a^2c^2 \qquad \textbf{simplify}$$
$$\left(a^2 - c^2\right)x^2 + a^2y^2 = a^2\left(a^2 - c^2\right) \qquad \textbf{factor}$$

We are nearly finished. Recall that $b^2 = a^2 - c^2$ so that

$$\left(a^2 - c^2\right)x^2 + a^2y^2 = a^2\left(a^2 - c^2\right)$$
$$b^2x^2 + a^2y^2 = a^2b^2$$
$$\frac{b^2x^2}{a^2b^2} + \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2}$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

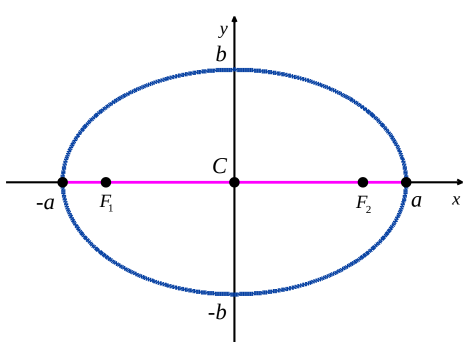
Equation 5.5. The Standard Equation of an Ellipse with Center (0,0): For positive unequal numbers *a* and *b*, the equation of an ellipse with center $(0,0)$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Some remarks about **Equation 5.5** are in order.

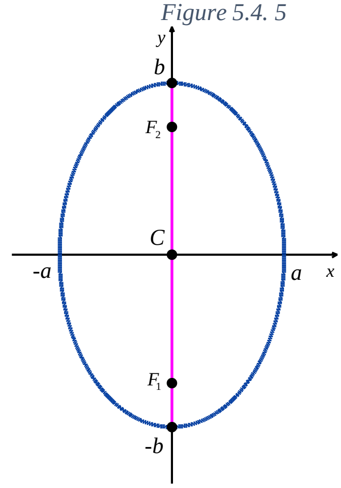
- The values *a* and *b* determine how far in the *x* and *y* directions, respectively, one counts from the center to arrive at points on the ellipse.
- If $a > b$, then we have an ellipse whose major axis is horizontal and, hence, the foci lie to the left and right of the center. In this case, as we've seen in the derivation, the distance from the center to the focus, *c*, can be found by $c = \sqrt{a^2 - b^2}$.

Figure 5.4. 4



$a > b$

Figure 5.4. 5



$b > a$

- If $b > a$, the roles of the major and minor axes are reversed and the foci lie above and below the center. In this case, $c = \sqrt{b^2 - a^2}$.
- In either case, c is the distance from the center to each focus, and $c = \sqrt{\text{larger denominator} - \text{smaller denominator}}$

Example 5.4.1. Identify the center, the vertices and the foci of the ellipse given by the equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$. Graph the ellipse.

Solution. The equation is in the standard form given by **Equation 5.5**. Thus, the ellipse has its center at $(0,0)$. We see that $a^2 = 16$ and $b^2 = 9$, from which $a = 4$ and $b = 3$. With $a > b$, this means that we move 4 units left and right from the center and 3 units up and down from the center to arrive at points on the ellipse, as seen below on the left.

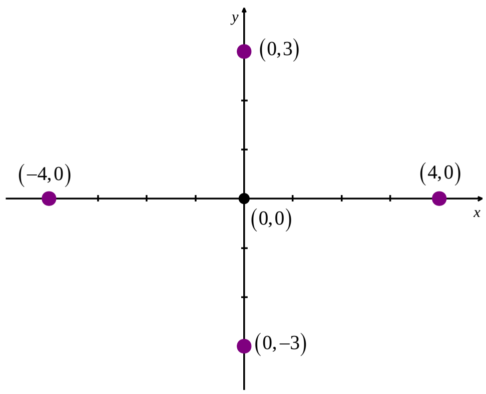


Figure 5.4. 6

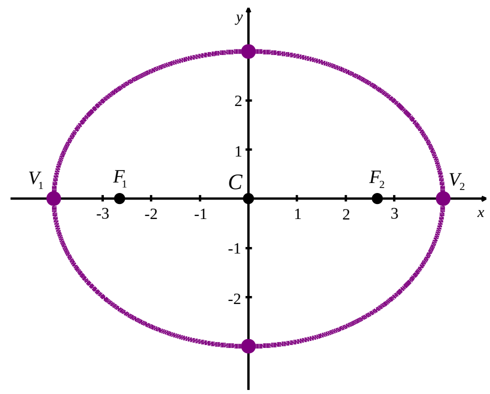


Figure 5.4. 7

Since we moved farther in the *x* direction than in the *y* direction, the major axis will lie along the *x*-axis and the minor axis will lie along the *y*-axis. The vertices are the points on the ellipse which lie along the major axis, so in this case they are the points $(-4,0)$ and $(4,0)$. To find the foci, we first evaluate *c* using $c = \sqrt{a^2 - b^2}$.

$$\begin{aligned} c &= \sqrt{16-9} \\ &= \sqrt{7} \end{aligned}$$

Thus, the foci lie $\sqrt{7}$ units left and right of the center, along the major axis, at $(-\sqrt{7},0)$ and $(\sqrt{7},0)$. Plotting all of this information, and connecting the four points $(4,0)$, $(0,3)$, $(-4,0)$ and $(0,-3)$ with a smooth curve to form the ellipse, gives the graph seen above on the right.

□

Example 5.4.2. Find the standard form of the equation of the ellipse that has vertices $(0,\pm 8)$ and foci $(0,\pm 5)$.

Solution. Plotting the data given to us, we have the scenario that is shown to the right. From this sketch, we know that the major axis lies along the *y*-axis. Since the center is the midpoint of the foci, we find it is $(0,0)$. Thus, we can use **Equation 5.5**, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

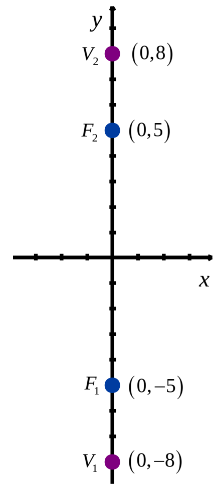


Figure 5.4. 8

Since one vertex is $(0,8)$ we have that $b=8$, so $b^2=64$. All that remains is to find a^2. From the focus at $(0,5)$, it follows that $c=5$. Noting that the major axis lies along the *y*-axis, and thus $b>a$, we have

$$c=\sqrt{b^2-a^2}$$
$$5=\sqrt{64-a^2}$$
$$25=64-a^2 \quad \text{after squaring both sides}$$
$$a^2=39$$

Substituting $a^2=39$ and $b^2=64$ into the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get a final answer of $\frac{x^2}{39}+\frac{y^2}{64}=1$.

□

The Equation of an Ellipse with Center (h,k)

To get the formula for the ellipse centered at (h,k), we could use transformations, or re-derive the equation using **Definition 5.4** and the distance formula, to obtain the formula below.

Equation 5.6. The Standard Equation of an Ellipse with Center (*h*,*k*): For positive unequal numbers *a* and *b*, the equation of an ellipse with center (h,k) is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

We note that a center of $(h,k)=(0,0)$ results in **Equation 5.5**.

Example 5.4.3. Identify the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, and the foci of the ellipse given by the equation $\frac{(x+1)^2}{9}+\frac{(y-2)^2}{25}=1$. Graph the ellipse.

Solution. We see that this formula is in the standard form of **Equation 5.6**. Here $x-h$ is $x+1$ so $h=-1$, and $y-k$ is $y-2$ so $k=2$. Hence, our ellipse is centered at $(-1,2)$. We see that $a^2=9$ so $a=3$, and $b^2=25$ so $b=5$. This means that we move 3 units left and right from the center and 5 units up and down from the center to arrive at points on the ellipse, as seen below on the left.

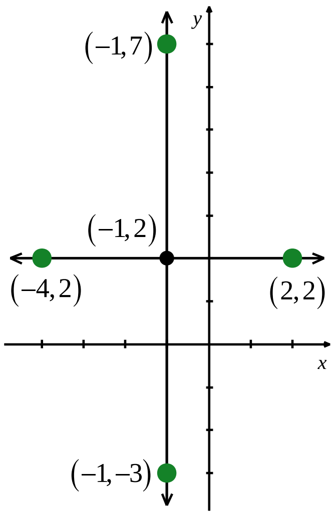


Figure 5.4. 9

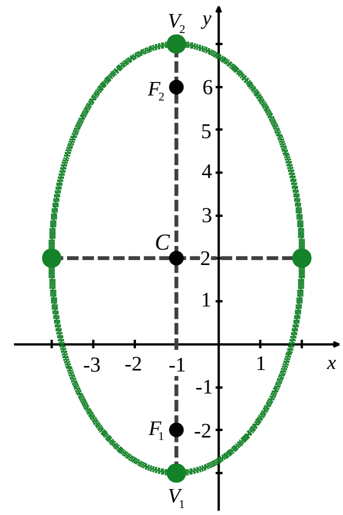


Figure 5.4. 10

With $b > a$, since we moved farther in the *y* direction than in the *x* direction, the major axis will lie along the vertical line $x = -1$, which means the minor axis lies along the horizontal line, $y = 2$. The vertices are the points on the ellipse which lie along the major axis so, in this case, they are the points $(-1,7)$ and $(-1,-3)$. The endpoints of the minor axis are $(-4,2)$ and $(2,2)$. To find the foci, we find

$$\begin{aligned} c &= \sqrt{25-9} \\ &= 4 \end{aligned}$$

Since the major axis is vertical, the foci lie 4 units above and below the center, at $(-1,6)$ and $(-1,-2)$. Plotting all this information and connecting the points $(2,2)$, $(-1,7)$, $(-4,2)$ and $(-1,-3)$ with a smooth curve to form the ellipse, gives the graph seen above on the right.

□

Example 5.4.4. Find the equation of the ellipse with foci $(2,1)$ and $(4,1)$, and vertex $(0,1)$.

Solution. Plotting the data given to us, we have the following.

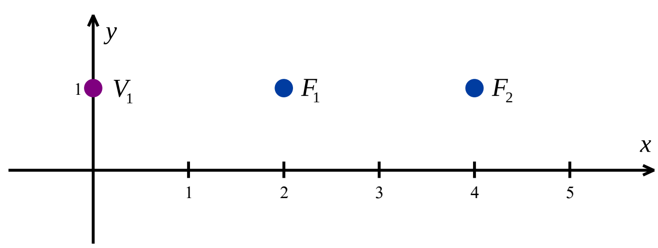


Figure 5.4. 11

From this sketch, we know that the major axis is horizontal, meaning $a > b$. Since the center is the midpoint of the foci, we know it is $(3,1)$. Since one vertex is $(0,1)$ we have that $a = 3$, so $a^2 = 9$. All that remains is to find b^2. The foci are 1 unit away from the center, and so we know $c = 1$. Since $a > b$, we have

$$\begin{aligned} c &= \sqrt{a^2 - b^2} \\ 1 &= \sqrt{9 - b^2} \\ 1 &= 9 - b^2 \quad \text{after squaring both sides} \\ b^2 &= 8 \end{aligned}$$

Substituting all of our findings into the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get a final answer of $\frac{(x-3)^2}{9} + \frac{(y-1)^2}{8} = 1$.

□

As with circles and parabolas, an equation may be given which is an ellipse, but isn't in the standard form of **Equation 5.6**. In those cases, as with circles and parabolas before, we will need to massage the given equation into the standard form.

To Write the Equation of an Ellipse in Standard Form

1. Position all terms containing variables on one side of the equation, grouping terms with 'like' variables together. Position the constant on the other side.
2. Complete the square in both variables as needed.
3. Divide both sides of the equation by the lone constant term value, provided the constant is not 0,[10] resulting in a lone constant term equal to 1.

Example 5.4.5. Put the equation $x^2 + 4y^2 - 2x + 24y + 33 = 0$ into standard form. Find the center, the vertices and the foci and use these in graphing the equation.

Solution. Since we have a sum of squares and the squared terms have unequal coefficients, it's a good bet we have an ellipse on our hands.[11] We need to complete both squares and then divide, if necessary, to get the right-hand side equal to 1.

$$\begin{aligned} x^2 + 4y^2 - 2x + 24y + 33 &= 0 \\ x^2 - 2x + 4y^2 + 24y &= -33 \quad \text{group variables and subtract 33 from both sides} \\ x^2 - 2x + 4\left(y^2 + 6y\right) &= -33 \quad \text{factor out leading coefficient from } y \text{ terms} \\ \left(x^2 - 2x + 1\right) + 4\left(y^2 + 6y + 9\right) &= -33 + 1 + 4(9) \quad \text{complete the square in both variables} \\ (x-1)^2 + 4(y+3)^2 &= 4 \quad \text{factor and simplify} \end{aligned}$$

---

[10] If the lone constant term is 0, the equation cannot be written in standard form. In this case, the equation will contain only a single solution and is graphically represented by a single point. If, after converting the equation to standard form, the constant term is 1 but the variable side of the equation now contains two negative square terms, the equation has no real solutions.

[11] The equation of a parabola has only one squared variable and the equation of a circle has two squared variables with identical coefficients.

$$\frac{(x-1)^2}{4}+\frac{4(y+3)^2}{4}=\frac{4}{4}$$ **divide both sides by 4**

$$\frac{(x-1)^2}{4}+\frac{(y+3)^2}{1}=1$$ **simplify**

Now that this equation is in the standard form of **Equation 5.6**, we see that $x-h$ is $x-1$ so $h=1$, and $y-k$ is $y+3$ so $k=-3$. Hence, our ellipse is centered at $(1,-3)$. We see that $a^2=4$ and $b^2=1$, from which $a=2$ and $b=1$. This means we move 2 units left and right from the center and 1 unit up and down from the center to arrive at points on the ellipse. Since we moved farther in the *x* direction than in the *y* direction, the major axis will lie along the horizontal line $y=-3$.

The vertices are the points on the ellipse which lie along the major axis so they are the points $(-1,-3)$ and $(3,-3)$. To find the foci, we find $c=\sqrt{4-1}=\sqrt{3}$, which means the foci lie $\sqrt{3}$ units from the center. Since the major axis is horizontal, the foci lie $\sqrt{3}$ units to the left and right of the center, at $(1-\sqrt{3},-3)$ and $(1+\sqrt{3},-3)$. Plotting all of this information gives the following.

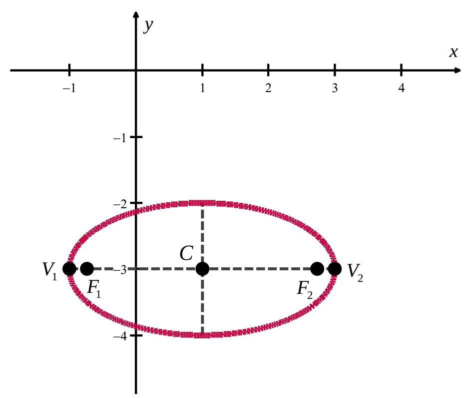


Figure 5.4. 12

□

The Eccentricity of an Ellipse

As you come across ellipses in the homework exercises, you'll notice they come in all shapes and sizes. Compare the two ellipses below.

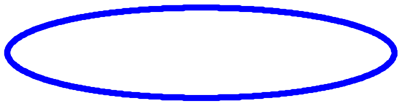


Figure 5.4. 13

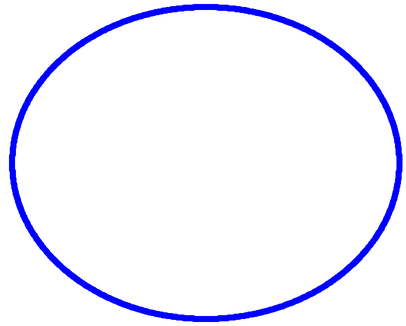


Figure 5.4. 14

Certainly, one ellipse is more round than the other. This notion of roundness is quantified below.

Definition 5.5. The **eccentricity** of an ellipse, denoted *e*, is the following ratio:

$$e = \frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}}$$

In an ellipse, the foci are closer to the center than the vertices, so $0 < e < 1$. The ellipse above on the left has eccentricity $e \approx 0.97$; for the ellipse on the right $e \approx 0.58$. In general, the closer the eccentricity is to 0, the more 'circular' the ellipse will appear; the closer the eccentricity is to 1, the more 'flat' the ellipse will appear.

Example 5.4.6. Find the equation of the ellipse whose vertices are $(\pm 5, 0)$ with eccentricity $e = \frac{1}{4}$.

Solution. As before, we plot the data given to us.

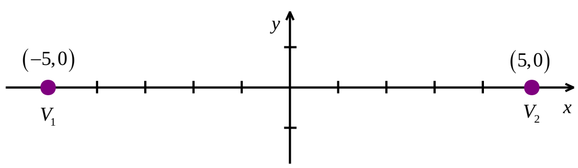


Figure 5.4. 15

From this sketch, we know that the major axis is horizontal, meaning $a > b$. With the vertices located at $(-5, 0)$ and $(5, 0)$ we see that the center is $(0, 0)$, since the center is the midpoint of the vertices. We also get $a = 5$ so $a^2 = 25$. All that remains is to find b^2. To that end, we use the eccentricity, $e = \frac{1}{4}$, to find *c*.

$$e = \frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}} = \frac{c}{a}$$

$$\frac{1}{4} = \frac{c}{5}$$

Then $c = \frac{5}{4}$ and we proceed to find b^2.

$$c = \sqrt{a^2 - b^2}$$
$$\frac{5}{4} = \sqrt{25 - b^2}$$
$$\frac{25}{16} = 25 - b^2 \quad \text{after squaring both sides}$$

It follows that $b^2 = \frac{375}{16}$. Substituting all of our findings into **Equation 5.5**, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, yields our final answer $\frac{x^2}{25} + \frac{16y^2}{375} = 1$.

□

Applications of Ellipses

As with parabolas, ellipses have a reflective property. If we imagine the dashed lines below representing sound waves, then the waves emanating from one focus reflect off the top of the ellipse and head toward the other focus.

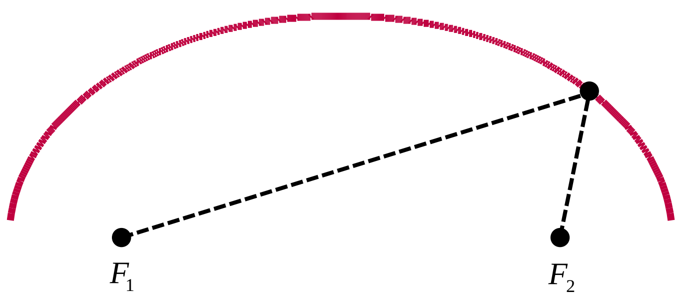


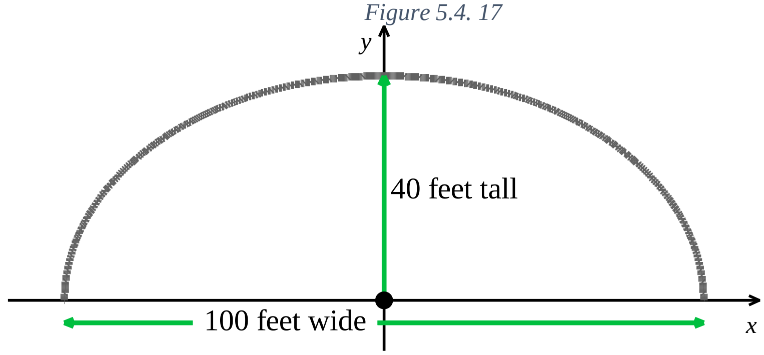
Figure 5.4. 16

Such geometry is exploited in the construction of so-called 'whispering galleries'. If a person whispers at one focus, a person standing at the other focus will hear the first person as if they were standing right next to them. We explore whispering galleries in our last example.

Example 5.4.7. Shawna and Spencer want to exchange secrets from across a crowded whispering gallery, which is a room having a cross section that is half of an ellipse. If the room is 40 feet high at the center and 100 feet wide at the floor, how far from the outer wall should each of them stand so that they will be positioned at the foci of the ellipse?

Solution. Graphing the data yields the following.

Figure 5.4. 17



It is most convenient to imagine this ellipse centered at $(0,0)$. Since the ellipse is 100 feet wide and 40 feet tall, we get $a = 50$ and $b = 40$. Hence, our ellipse has the equation $\frac{x^2}{50^2} + \frac{y^2}{40^2} = 1$. We are looking for the foci and we begin by finding $c = \sqrt{a^2 - b^2}$.

$$\begin{aligned} c &= \sqrt{50^2 - 40^2} \\ &= \sqrt{900} \\ &= 30 \end{aligned}$$

So the foci are 30 feet from the center. That means they are $50 - 30 = 20$ feet from the vertices. Hence, Shawna and Spencer should stand 20 feet from opposite ends of the gallery.

□

5.4 Exercises

1. Define an ellipse in terms of its foci.
2. What can be said about the symmetry of the graph of an ellipse with center at the origin and foci along the *y*-axis?

In Exercises 3 – 8, determine whether the given equation represents an ellipse. If it does, rewrite the equation in standard form.

3. $2x^2 + y = 4$
4. $4x^2 + 9y^2 = 36$
5. $4x^2 - y^2 = 4$
6. $4x^2 + 9y^2 = 1$
7. $4x^2 - 8x + 9y^2 - 72y + 112 = 0$
8. $4x^2 + 4y^2 = 1$

In Exercises 9 – 20, find the center, the vertices, the foci and the eccentricity. Graph the ellipse.

9. $\frac{x^2}{169} + \frac{y^2}{25} = 1$
10. $\frac{x^2}{9} + \frac{y^2}{25} = 1$
11. $\frac{x^2}{25} + \frac{y^2}{36} = 1$
12. $\frac{x^2}{16} + \frac{y^2}{9} = 1$
13. $\frac{(x-2)^2}{64} + \frac{(y-4)^2}{16} = 1$
14. $\frac{x^2}{2} + \frac{(y+1)^2}{5} = 1$
15. $\frac{(x-2)^2}{4} + \frac{(y+3)^2}{9} = 1$
16. $\frac{(x+5)^2}{16} + \frac{(y-4)^2}{1} = 1$
17. $\frac{(x-1)^2}{10} + \frac{(y-3)^2}{11} = 1$
18. $\frac{(x-1)^2}{9} + \frac{(y+3)^2}{4} = 1$
19. $\frac{(x+2)^2}{16} + \frac{(y-5)^2}{20} = 1$
20. $\frac{(x-4)^2}{8} + \frac{(y-2)^2}{18} = 1$

In Exercises 21 – 30, put the equation into standard form. Find the center, the vertices and the foci. Graph the ellipse.

21. $9x^2 + 25y^2 - 54x - 50y - 119 = 0$
22. $12x^2 + 3y^2 - 30y + 39 = 0$
23. $5x^2 + 18y^2 - 30x + 72y + 27 = 0$
24. $x^2 - 2x + 2y^2 - 12y + 3 = 0$
25. $9x^2 + 4y^2 - 4y - 8 = 0$
26. $6x^2 + 5y^2 - 24x + 20y + 14 = 0$
27. $4x^2 - 24x + 36y^2 - 360y + 864 = 0$
28. $4x^2 + 24x + 16y^2 - 128y + 228 = 0$
29. $4x^2 + 40x + 25y^2 - 100y + 100 = 0$
30. $9x^2 + 72x + 16y^2 + 16y + 4 = 0$

In Exercises 31 – 42, find the standard form of the equation of the ellipse which has the given properties.

31. Center $(3,7)$, Vertex $(3,2)$, Focus $(3,3)$

32. Center $(4,2)$, Vertex $(9,2)$, Focus $(4+2\sqrt{6},2)$

33. Center $(3,5)$, Vertex $(3,11)$, Focus $(3,5+4\sqrt{2})$

34. Center $(-3,4)$, Vertex $(1,4)$, Focus $(-3+2\sqrt{3},4)$

35. Foci $(0,\pm 5)$, Vertices $(0,\pm 8)$

36. Foci $(\pm 3,0)$ with minor axis length of 10

37. Vertices $(3,2)$ and $(13,2)$ with endpoints of the minor axis $(8,4)$ and $(8,0)$

38. Center $(5,2)$, Vertex $(0,2)$, eccentricity $\frac{1}{2}$

39. All points on the ellipse are in Quadrant IV except $(0,-9)$ and $(8,0)$. (One might also say that the ellipse is tangent to the axes at those two points.)

40. Center $(0,0)$, Focus $(4,0)$, containing the point $(0,3)$

41. Center $(0,0)$, Focus $(0,-2)$, containing the point $(5,0)$

42. Center $(0,0)$, Focus $(3,0)$, with the major axis twice as long as the minor axis

In Exercises 43 – 47, given the graph of the ellipse, determine its equation.

43.

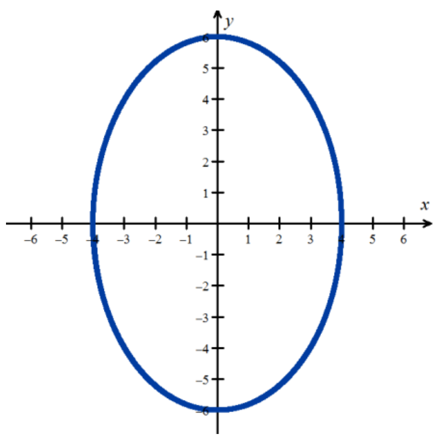


Figure 5.4.18

44.

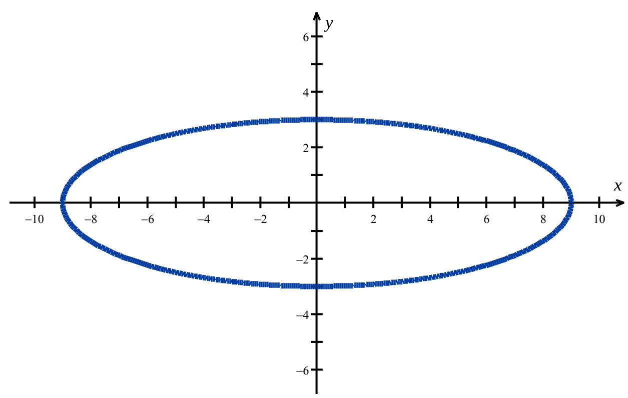


Figure 5.4.19

45.

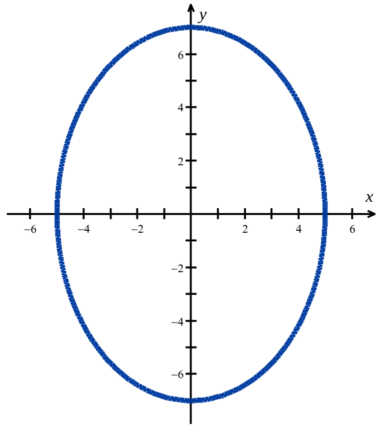


Figure 5.4. 20

46.

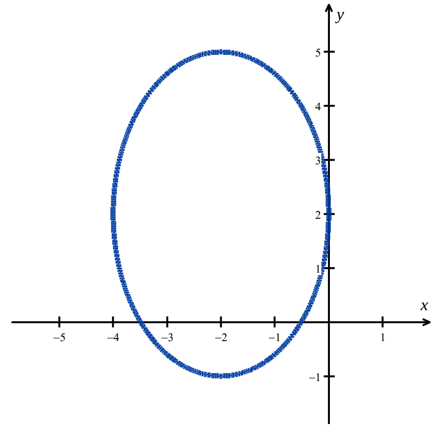


Figure 5.4.21

47.

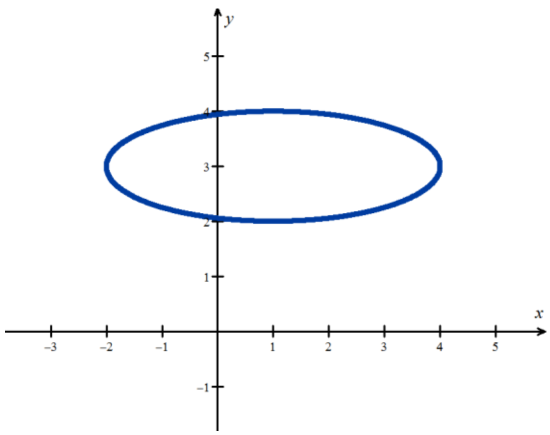


Figure 5.4.22

48. An elliptical arch[12] has a height of 8 feet and a span of 20 feet. Find an equation for the ellipse, and use it to find the height of the arch at a distance of 4 feet from the center.

49. An elliptical arch has a height of 12 feet and a span of 40 feet. Find an equation for the ellipse, and use it to find the distance from the center to a point at which the height is 6 feet.

50. A bridge is to be built in the shape of an elliptical arch and is to have a span of 120 feet. The height of the arch at a distance of 40 feet from the center is to be 8 feet. Find the height of the arch at its center.

51. An elliptical arch is constructed which is 6 feet wide at the base and 9 feet tall in the middle. Find the height of the arch exactly 1 foot in from the base of the arch. Compare your result with your answer to Exercise 39 in **Section 5.3**.

52. A person in a whispering gallery standing at one focus of the ellipse can whisper and be heard by a person standing at the other focus because all the sound waves that reach the ceiling are reflected to

[12] An elliptical arch is an arch in the shape of a semi-ellipse, or the top half of an ellipse.

the other person. If a whispering gallery has a length of 120 feet, and the foci are located 30 feet from the center, find the height of the ceiling at the center.

53. A person is standing 8 feet from the nearest wall in a whispering gallery. If that person is at one focus, and the other focus is 80 feet away, what is the length of the gallery and what is its height at the center?

54. The Earth's orbit around the sun is an ellipse with the sun at one focus and eccentricity $e \approx 0.0167$. The length of the semimajor axis (that is, half of the major axis) is defined to be 1 astronomical unit (AU). The vertices of the elliptical orbit are given special names: 'aphelion' is the vertex farthest from the sun, and 'perihelion' is the vertex closest to the sun. Find the distance in AU between the sun and aphelion and the distance in AU between the sun and perihelion.

55. Some famous examples of whispering galleries include St. Paul's Cathedral in London, England, and National Statuary Hall in Washington D.C. With the help of your classmates, research these whispering galleries. How does the whispering effect compare and contrast with the scenario in **Example 5.4.7**?

5.5 Hyperbolas

Learning Objectives

- Define a hyperbola in a plane.
- Determine whether an equation represents a hyperbola.
- Graph a hyperbola from a given equation.
- Determine the center, vertices and foci of a hyperbola.
- Find the equation of a hyperbola from a graph or from stated properties.
- Solve applications of hyperbolas.

In the definition of an ellipse, **Definition 5.4**, we fixed two points called foci and looked at points whose distances to the foci always **added** to a constant distance *d*. Those prone to syntactical tinkering may wonder what, if any, curve we'd generate if we replaced **added** with **subtracted**. The answer is a hyperbola.

The Definition of a Hyperbola

Definition 5.6. Given two distinct points F_1 and F_2 in the plane and a fixed distance *d*, a **hyperbola** is the set of all points (x, y) in the plane such that the absolute value of the difference of the distance from F_1 to (x, y) and the distance from F_2 to (x, y) is *d*. The points F_1 and F_2 are called the **foci** of the hyperbola.

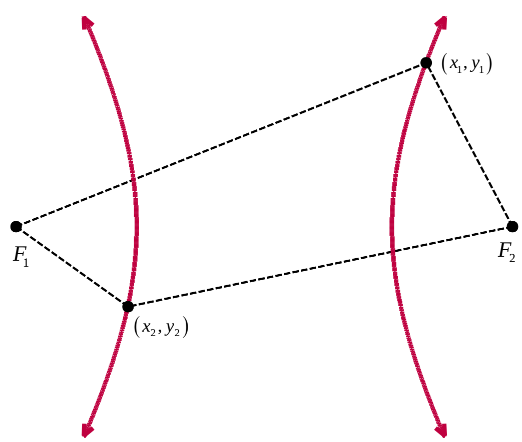


Figure 5.5. 1

In the figure,

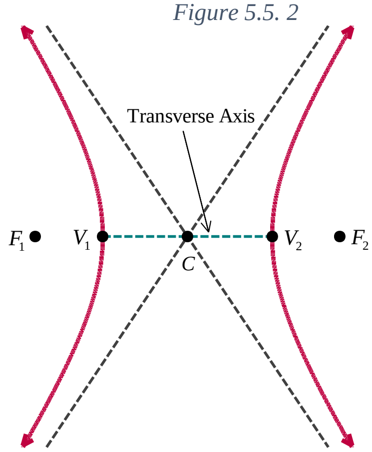
$$\left[\text{the distance from } F_1 \text{ to } (x_1, y_1)\right] - \left[\text{the distance from } F_2 \text{ to } (x_1, y_1)\right] = d$$

and

$$\left[\text{the distance from } F_2 \text{ to } (x_2, y_2)\right] - \left[\text{the distance from } F_1 \text{ to } (x_2, y_2)\right] = d$$

Note that the hyperbola has two parts, called **branches**. The **center** of the hyperbola is the midpoint of the line segment connecting the two foci. The **transverse axis** of the hyperbola is the line segment that lies on the line containing the center and foci and connects the two branches of the hyperbola. The **vertices** of a hyperbola are the points of the hyperbola which lie on the transverse axis. In addition, we will show momentarily that there are lines called **asymptotes** which the branches of the hyperbola approach for large *x* and *y* values. They serve as guides to the graph, as shown below.

Figure 5.5. 2



A hyperbola with center *C*, foci F_1 and F_2, vertices V_1 and V_2

Before we derive the standard equation of the hyperbola, we need to discuss one further parameter, the **conjugate axis** of the hyperbola. The conjugate axis of a hyperbola is the line segment through the center which is perpendicular to the transverse axis and has the same length as the line segment, also perpendicular to the transverse axis, which connects the asymptotes by passing through a vertex.

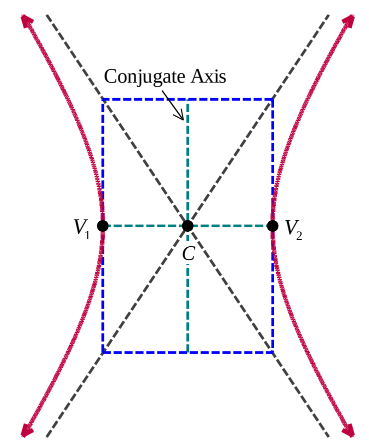


Figure 5.5. 3

Note that in the diagram, we can construct a rectangle using line segments with lengths equal to the lengths of the transverse and conjugate axes whose center is the center of the hyperbola and whose diagonals are contained in the asymptotes. This **guide rectangle** will aid us in graphing hyperbolas.

The Equation of a Hyperbola with Center at $(0,0)$

Suppose we wish to derive the equation of a hyperbola. For simplicity, we shall assume that the center is $(0,0)$, the vertices are $(a,0)$ and $(-a,0)$ and the foci are $(c,0)$ and $(-c,0)$. We label the endpoints of the conjugate axis $(0,b)$ and $(0,-b)$. (Although b does not enter into our derivation, we will justify this choice later.) As before, we assume a, b and c are all positive numbers. Schematically, we have the following.

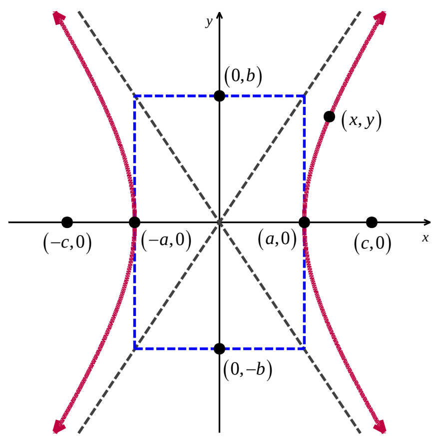


Figure 5.5. 4

Since $(a,0)$ is on the hyperbola, it must satisfy the conditions of **Definition 5.6**. That is, the distance from $(-c,0)$ to $(a,0)$ minus the distance from $(c,0)$ to $(a,0)$ must equal the fixed distance d. Since all these points lie on the x-axis, we have

$$\begin{aligned} \left[\text{distance from } (-c,0) \text{ to } (a,0)\right] - \left[\text{distance from } (c,0) \text{ to } (a,0)\right] &= d \\ (a+c)-(c-a) &= d \\ 2a &= d \end{aligned}$$

In other words, the fixed distance d from the definition of the hyperbola is actually the length of the transverse axis! (Where have we seen that type of coincidence before?) Now consider a point (x,y) on the hyperbola. Applying **Definition 5.6**, we get

$$\left[\text{distance from } (-c,0) \text{ to } (x,y)\right]-\left[\text{distance from } (c,0) \text{ to } (x,y)\right]=2a \quad \textbf{use } \boldsymbol{d = 2a}$$
$$\sqrt{\left(x-(-c)\right)^2+(y-0)^2}-\sqrt{(x-c)^2+(y-0)^2}=2a \quad \textbf{apply distance formula}$$
$$\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=2a$$

Using the same arsenal of Intermediate Algebra weaponry we used in deriving the standard formula of an ellipse, **Equation 5.5**, we arrive at the following.[13]

$$\left(a^2-c^2\right)x^2+a^2y^2=a^2\left(a^2-c^2\right)$$

What remains is to determine the relationship between *a*, *b* and *c*. To that end, we note that since *a* and *c* are both positive numbers with $a<c$, we get $a^2<c^2$ so that a^2-c^2 is a negative number. Hence, c^2-a^2 is a positive number. For reasons which will become clear soon, we re-write the equation by solving for $\frac{y^2}{x^2}$ to get

$$\left(a^2-c^2\right)x^2+a^2y^2=a^2\left(a^2-c^2\right)$$
$$-\left(c^2-a^2\right)x^2+a^2y^2=-a^2\left(c^2-a^2\right) \quad \textbf{factor to obtain positive values}$$
$$a^2y^2=\left(c^2-a^2\right)x^2-a^2\left(c^2-a^2\right)$$
$$\frac{a^2y^2}{a^2x^2}=\frac{\left(c^2-a^2\right)x^2}{a^2x^2}-\frac{a^2\left(c^2-a^2\right)}{a^2x^2}$$
$$\frac{y^2}{x^2}=\frac{\left(c^2-a^2\right)}{a^2}-\frac{\left(c^2-a^2\right)}{x^2}$$

As *x* and *y* attain very large numbers, the quantity $\frac{\left(c^2-a^2\right)}{x^2}\to 0$ so that $\frac{y^2}{x^2}\to\frac{\left(c^2-a^2\right)}{a^2}$. By setting $b^2=c^2-a^2$ we get $\frac{y^2}{x^2}\to\frac{b^2}{a^2}$. This shows that $y\to\pm\frac{b}{a}x$ as $|x|$ grows large. Thus $y=\pm\frac{b}{a}x$ are the asymptotes to the graph as predicted and our choice of labels for the endpoints of the conjugate axis is justified. In our equation of the hyperbola we can substitute $a^2-c^2=-b^2$ as follows.

$$\left(a^2-c^2\right)x^2+a^2y^2=a^2\left(a^2-c^2\right)$$
$$-b^2x^2+a^2y^2=-a^2b^2$$
$$\frac{-b^2x^2}{-a^2b^2}+\frac{a^2y^2}{-a^2b^2}=\frac{-a^2b^2}{-a^2b^2}$$
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

[13] It is a good exercise to actually work this out.

The equation above is for a hyperbola whose center is the origin, and which opens to the left and right. We refer to a hyperbola that opens to the left and right as a **horizontal hyperbola.**

Equation 5.7. The Standard Equation of a Horizontal Hyperbola with Center (0,0): For positive numbers *a* and *b*, the equation of a horizontal hyperbola with center $(0,0)$ is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

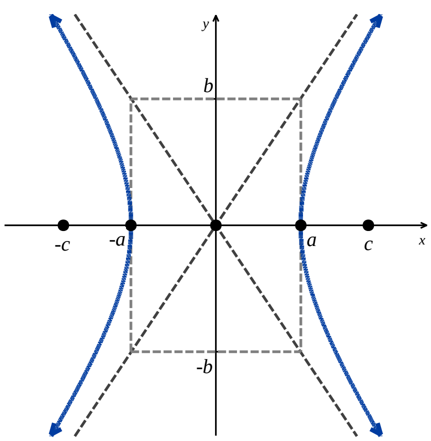
If the roles of *x* and *y* were interchanged, then the hyperbola's branches would open upward and downward and we would get a **vertical hyperbola**.

Equation 5.8. The Standard Equation of a Vertical Hyperbola with Center (0,0): For positive numbers *a* and *b*, the equation of a vertical hyperbola with center $(0,0)$ is

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$$

The values of *a* and *b* determine how far in the *x* and *y* directions, respectively, one counts from the center to determine the rectangle through which the asymptotes pass. In both cases, the distance from the center to the foci, *c*, as seen in the derivation, can be found by the formula $c=\sqrt{a^2+b^2}$.

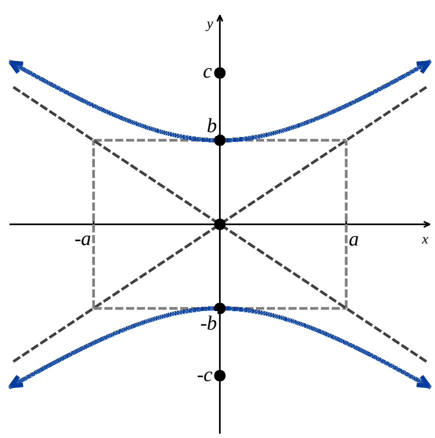
Figure 5.5. 5



Horizontal hyperbola with center $(0,0)$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Figure 5.5. 6



Vertical hyperbola with center $(0,0)$

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$$

Lastly, note that we can quickly distinguish the equation of a hyperbola from that of a circle or ellipse because the hyperbola formula involves a **difference** of squares where the circle and ellipse formulas both involve the **sum** of squares.

Example 5.5.1. Identify the vertices and the foci of the hyperbola given by the equation $\frac{y^2}{49}-\frac{x^2}{32}=1$.

Graph the hyperbola.

Solution. We first see that this equation is given to us in the standard form of **Equation 5.8**. Hence, we have a vertical hyperbola centered at $(0,0)$. We see that $a^2=32$ so $a=4\sqrt{2}$ and $b^2=49$ so $b=7$. This means we move $4\sqrt{2}$ units left and right of the center and 7 units up and down from the center to arrive at points on a guide rectangle. The asymptotes pass through the center of the hyperbola as well as the corners of the rectangle. This yields the following set up.

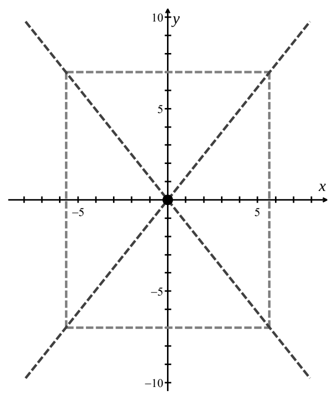


Figure 5.5. 7

Seeing that the x^2 term is being subtracted from the y^2 term, we know that the branches of the hyperbola open upward and downward, and thus the transverse axis lies along the *y*-axis. The vertices of the hyperbola are where the hyperbola intersects the transverse axis. In this case our vertices are $(0,\pm 7)$. To find the foci, we need *c*.

$$\begin{aligned} c &= \sqrt{a^2+b^2} \\ &= \sqrt{32+49} \\ &= \sqrt{81} \\ &= 9 \end{aligned}$$

Since the foci lie on the transverse axis, we move 9 units upward and downward from the center $(0,0)$ to arrive at foci of $(0,\pm 9)$. Putting it all together, we get the following.

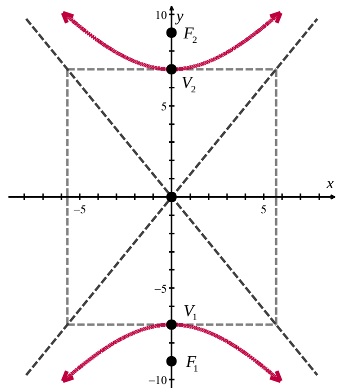


Figure 5.5. 8

□

Example 5.5.2. Find the standard form of the equation of the hyperbola that has vertices $(\pm 6, 0)$ and foci $(\pm 2\sqrt{10}, 0)$.

Solution. Since the vertices and foci are on the *x*-axis, the equation for the hyperbola will have the form of **Equation 5.7**, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. From the vertices, $(\pm 6, 0)$, we see that $a = 6$ so $a^2 = 36$. The foci, $(\pm 2\sqrt{10}, 0)$, give us $c = 2\sqrt{10}$ so we have $c^2 = 40$. We next find b^2.

$$\begin{aligned} b^2 &= c^2 - a^2 \\ &= 40 - 36 \\ &= 4 \end{aligned}$$

Finally, substituting $a^2 = 36$ and $b^2 = 4$ into **Equation 5.8** yields $\frac{x^2}{36} - \frac{y^2}{4} = 1$, as graphed below.

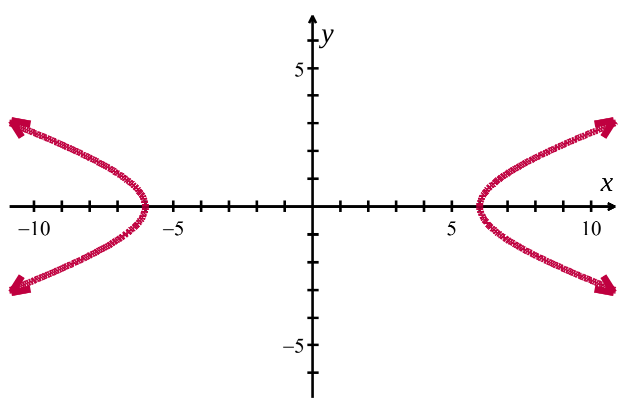


Figure 5.5. 9

□

The Equation of a Hyperbola with Center at (h,k)

For a hyperbola centered at the point (h,k), we would get the following equation for a horizontal hyperbola whose branches open to the left and right.

Equation 5.9. The Standard Equation of a Horizontal Hyperbola with Center (*h*,*k*): For positive numbers *a* and *b*, the equation of a horizontal hyperbola with center (h,k) is

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

Exchanging the roles of *x* and *y* results in a vertical hyperbola, whose branches open upward and downward.

Equation 5.10. The Standard Equation of a Vertical Hyperbola with Center (*h*,*k*): For positive numbers *a* and *b*, the equation of a vertical hyperbola with center (h,k) is

$$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$$

Example 5.5.3. Find the center, the vertices, the foci and the equations of the asymptotes for the hyperbola given by the equation $\frac{(x-2)^2}{4}-\frac{y^2}{25}=1$. Graph the hyperbola.

Solution. We first see that this equation is given to us in the standard form of **Equation 5.9**. Here $x-h$ is $x-2$ so $h=2$, and $y-k$ is *y* so $k=0$. Hence, our hyperbola is centered at $(2,0)$. We see that $a^2=4$ so $a=2$, and $b^2=25$ so $b=5$. This means we move 2 units to the left and right of the center and 5 units up and down from the center to arrive at points on the guide rectangle. The asymptotes pass through the center of the hyperbola as well as the corners of the rectangle, as displayed below.

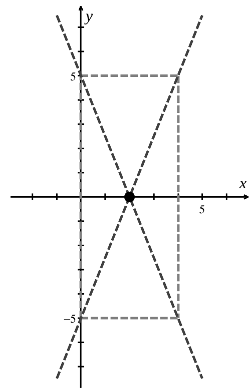


Figure 5.5. 10

Since the y^2 term is being subtracted from the x^2 term, we know that the branches of the hyperbola open to the left and right. This means that the transverse axis lies along the *x*-axis. Knowing that the vertices of the hyperbola are the points where the hyperbola intersects the transverse axis, we get that the vertices are 2 units to the left and right of $(2,0)$, at $(0,0)$ and $(4,0)$.

To find the foci, we determine *c*.

$$\begin{aligned} c &= \sqrt{a^2+b^2} \\ &= \sqrt{4+25} \\ &= \sqrt{29} \end{aligned}$$

Since the foci lie on the transverse axis, we move $\sqrt{29}$ units to the left and right of $(2,0)$ to arrive at $(2-\sqrt{29},0)$ and $(2+\sqrt{29},0)$, approximately $(-3.39,0)$ and $(7.39,0)$, respectively.

We next determine the equations of the asymptotes, recalling that the asymptotes go through the center of the hyperbola, $(2,0)$, as well as the corners of the guide rectangle, so they have slopes of $\pm\frac{b}{a}=\pm\frac{5}{2}$.

Using the point-slope equation of a line yields $y-0=\pm\frac{5}{2}(x-2)$, so we get $y=\frac{5}{2}x-5$ and $y=-\frac{5}{2}x+5$.

Putting all of this together, we get the following graph.

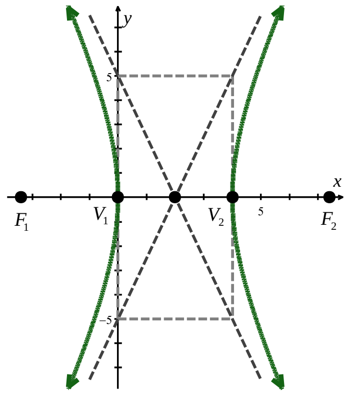


Figure 5.5. 11

□

Example 5.5.4. Find the equation of the hyperbola with asymptotes $y = \pm 2x$ and vertices $(\pm 5, 0)$.

Solution. Plotting the data given to us, we have the following.

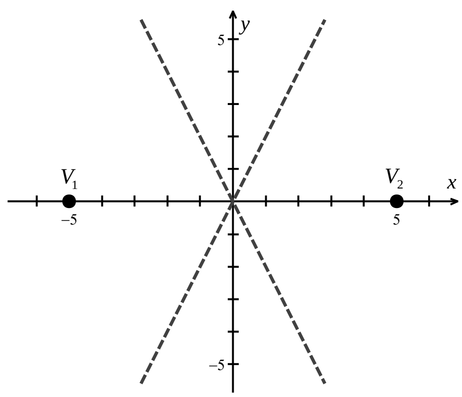


Figure 5.5. 12

This graph not only tells us that the branches of the hyperbola open to the left and to the right, it also tells us that the center is $(0,0)$. Hence, our standard form is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. From vertices of $(\pm 5, 0)$ we have $a = 5$, so $a^2 = 25$. In order to determine b^2, we recall that the slopes of the asymptotes are $\pm \frac{b}{a}$. Since $a = 5$ and the slope of the line $y = 2x$ is 2, we have that $\frac{b}{5} = 2$, so $b = 10$. Hence, $b^2 = 100$ and our final answer is $\frac{x^2}{25} - \frac{y^2}{100} = 1$.

□

As with the other conic sections, an equation whose graph is a hyperbola may not be given in either of the standard forms. To rectify that, we have the following.

To Write the Equation of a Hyperbola in Standard Form

1. Position all terms containing variables on one side of the equation, grouping terms with 'like' variables together. Position the constant on the other side.
2. Complete the square in both variables as needed.
3. Divide both sides of the equation by the lone constant term value, resulting in a lone constant term equal to 1.

Example 5.5.5. Put the equation $9y^2 - x^2 - 6x = 10$ into standard form. Find the center, the vertices, the foci and the equations of the asymptotes. Graph the hyperbola.

Solution. We need only complete the square on *x*.

$$9y^2 - x^2 - 6x = 10$$
$$9y^2 - 1(x^2 + 6x) = 10$$
$$9y^2 - 1(x^2 + 6x + 9) = 10 - 1(9) \quad \text{complete the square in } x$$
$$9y^2 - (x+3)^2 = 1 \quad \text{factor and simplify}$$
$$\frac{y^2}{\frac{1}{9}} - \frac{(x+3)^2}{1} = 1 \quad \text{write in format of standard equation for vertical hyperbola}$$

Now that this equation is in the standard form of **Equation 5.10**, we see that $x - h$ is $x + 3$ so $h = -3$, and $y - k$ is *y* so $k = 0$. Hence, our hyperbola is centered at $(-3, 0)$. We find that $a^2 = 1$ so $a = 1$, and $b^2 = \frac{1}{9}$ so $b = \frac{1}{3}$. This means that we move 1 unit to the left and right of the center and $\frac{1}{3}$ unit up and down from the center to arrive at points on the guide rectangle.

Since the x^2 term is being subtracted from the y^2 term, we know the branches of the hyperbola open upward and downward. This means the transverse axis lies along the vertical line $x = -3$ and the conjugate axis lies along the *x*-axis. Since the vertices of the hyperbola are where the hyperbola intersects the transverse axis, we get that the vertices are $\frac{1}{3}$ of a unit above and below $(-3, 0)$ at $\left(-3, \frac{1}{3}\right)$ and $\left(-3, -\frac{1}{3}\right)$. To find the foci, we first determine *c*.

$$\begin{aligned} c &= \sqrt{a^2 + b^2} \\ &= \sqrt{\frac{1}{9} + 1} \\ &= \frac{\sqrt{10}}{3} \end{aligned}$$

The foci lie on the transverse axis, so we move $\frac{\sqrt{10}}{3}$ units above and below $(-3, 0)$ to arrive at foci of $\left(-3, \frac{\sqrt{10}}{3}\right)$ and $\left(-3, -\frac{\sqrt{10}}{3}\right)$.

To determine the asymptotes, recall that the asymptotes go through the center of the hyperbola, $(-3, 0)$, as well as the corners of the guide rectangle, so they have slopes of $\pm\frac{b}{a} = \pm\frac{1}{3}$. We use the point-slope equation of a line to find the asymptotes.

$$\begin{aligned} y - 0 &= \frac{1}{3}(x - (-3)) \\ y &= \frac{1}{3}x + 1 \end{aligned}$$

$$\begin{aligned} y - 0 &= -\frac{1}{3}(x - (-3)) \\ y &= -\frac{1}{3}x - 1 \end{aligned}$$

Putting it all together, we have the following graph.

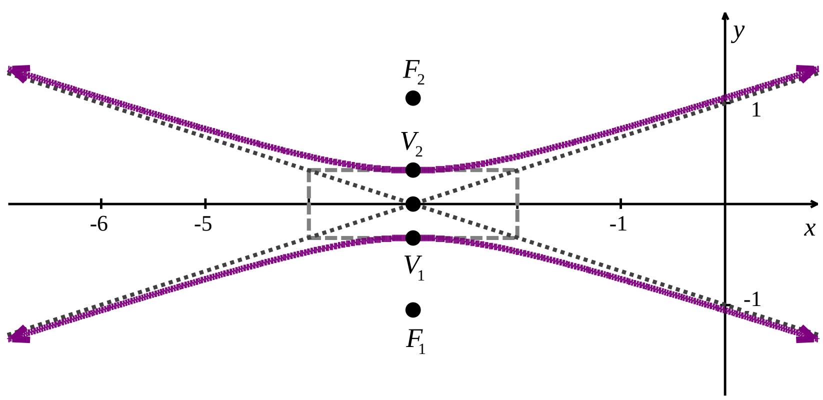


Figure 5.5. 13

□

Applications of Hyperbolas

Hyperbolas have real-world applications in many fields. They can be used to model the paths of comets, supersonic booms, ancient Grecian pillars and natural draft cooling towers. The design efficiency of hyperbolic cooling towers is particularly interesting. Cooling towers are used to transfer waste heat to the atmosphere and are often touted for their ability to generate power efficiently. Because of their hyperbolic form, these structures are able to withstand extreme winds while requiring less material than any other forms of their size and strength.

Figure 5.5. 14

Example 5.5.6. The design layout of a hyperbolic cooling tower is shown below. The tower stands 179.6 meters tall. The diameter of the top is 72 meters. At their closest, the sides of the tower are 60 meters apart. Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola is the origin of the coordinate plane. Round final values to four decimal places.

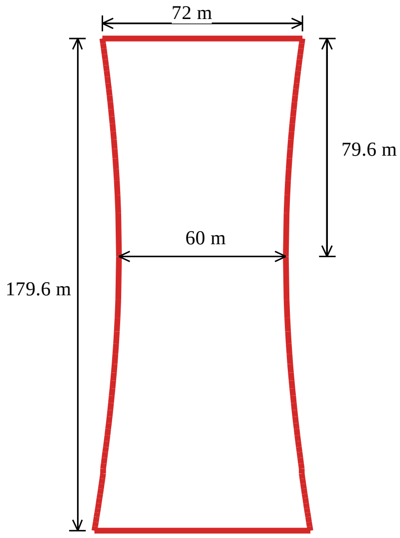


Figure 5.5. 15

Solution. We assume the center of the tower is the origin, and that the branches open to the left and right of the center. Thus, we can use the standard equation of a horizontal hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

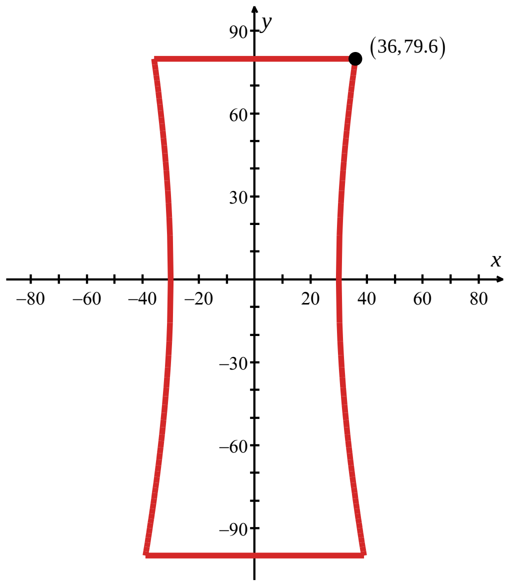


Figure 5.5. 16

We first determine *a* by finding the length of the transverse axis. Since the the transverse axis connects the two sides of the cooling tower where their sides are closest, the length of the transverse axis is 60 meters, from which $2a=60$, so $a=30$ and $a^2=900$.

To solve for b^2, we need to substitute for *x* and *y* in our equation using a known point. To do this, we can use the dimensions of the tower to find a point (x,y) that lies on the hyperbola. We will use the top right corner of the tower to represent that point. Since the *y*-axis bisects the tower, our *x*-value can be represented by the radius of the top, or 36 meters. The *y*-value is represented by the distance from the origin to the top, which is given as 79.6 meters.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \textbf{standard equation of horizontal hyperbola centered at origin}$$

$$b^2=\frac{y^2}{\frac{x^2}{a^2}-1} \qquad \textbf{isolate } b^2$$

$$b^2=\frac{(79.6)^2}{\frac{(36)^2}{900}-1} \qquad \textbf{substitute for } a^2\textbf{, } x \textbf{ and } y$$

$$b^2\approx 14400.3636 \quad \textbf{round to four decimal places}$$

The sides of the tower can be modeled by the hyperbolic equation $\dfrac{x^2}{900}-\dfrac{y^2}{14400.3636}=1$.

□

Identifying Conic Sections

Each of the conic sections we have studied in this chapter results from graphing equations of the form $Ax^2 + By^2 + Cx + Dy + E = 0$ for different choices of *A*, *B*, *C*, *D*, and *E*. While we've seen examples demonstrate how to convert an equation from this general form to one of the standard forms, we close this chapter with some advice about which standard form to choose.

Strategies for Identifying Conic Sections

Suppose the graph of equation $Ax^2 + By^2 + Cx + Dy + E = 0$ is a non-degenerate conic section.

If just one variable is squared, the graph is a parabola.

- Put the equation in the form of Equation 5.4: $(x-h)^2 = 4p(y-k)$ if *x* is squared or $(y-k)^2 = 4p(x-h)$ if *y* is squared.

If both variables are squared, look at the coefficients of x^2 and y^2.

- If $A = B$, the graph is a circle. Put the equation in the form of Equation 5.1: $(x-h)^2 + (y-k)^2 = r^2$.
- If $A \neq B$ but *A* and *B* have the same sign, the graph is an ellipse. Put the equation in the form of Equation 5.6: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.
- If *A* and *B* have different signs, the graph is a hyperbola. Put the equation in the form of Equation 5.9, $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, or Equation 5.10, $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$.

5.5 Exercises

1. Define a hyperbola in terms of its foci.

2. What can we conclude about a hyperbola if its asymptotes intersect at the origin?

3. If the transverse axis of a hyperbola is vertical, what do we know about the graph?

In Exercises 4 – 8, determine whether the given equation represents a hyperbola. If it does, rewrite the equation in standard form.

4. $3y^2 + 2x = 6$

5. $\frac{x^2}{36} - \frac{y^2}{9} = 1$

6. $5y^2 + 4x^2 = 6x$

7. $25x^2 - 16y^2 = 400$

8. $-9x^2 + 18x + y^2 + 4y - 14 = 0$

In Exercises 9 – 20, find the center, the vertices, the foci and the equations of the asymptotes. Graph the hyperbola.

9. $\frac{x^2}{16} - \frac{y^2}{9} = 1$

10. $\frac{y^2}{9} - \frac{x^2}{16} = 1$

11. $\frac{x^2}{49} - \frac{y^2}{16} = 1$

12. $\frac{y^2}{9} - \frac{x^2}{25} = 1$

13. $\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1$

14. $\frac{(y-3)^2}{11} - \frac{(x-1)^2}{10} = 1$

15. $\frac{(x+4)^2}{16} - \frac{(y-4)^2}{1} = 1$

16. $\frac{(x+1)^2}{9} - \frac{(y-3)^2}{4} = 1$

17. $\frac{(y+2)^2}{16} - \frac{(x-5)^2}{20} = 1$

18. $\frac{(x-4)^2}{8} - \frac{(y-2)^2}{18} = 1$

19. $\frac{(y+5)^2}{9} - \frac{(x-4)^2}{25} = 1$

20. $\frac{(y-3)^2}{9} - \frac{(x-3)^2}{9} = 1$

In Exercises 21 – 30, put the equation into standard form. Find the center, the vertices and the foci. Graph the hyperbola.

21. $12x^2 - 3y^2 + 30y - 111 = 0$

22. $18y^2 - 5x^2 + 72y + 30x - 63 = 0$

23. $9x^2 - 25y^2 - 54x - 50y - 169 = 0$

24. $-6x^2 + 5y^2 - 24x + 40y + 26 = 0$

25. $-4x^2 + 16y^2 - 8x - 32y - 52 = 0$

26. $x^2 - 25y^2 - 8x - 100y - 109 = 0$

27. $-x^2 + 4y^2 + 8x - 40y + 88 = 0$

28. $64x^2 - 9y^2 + 128x - 72y - 656 = 0$

29. $16x^2 - 4y^2 + 64x - 8y - 4 = 0$

30. $-100x^2 + y^2 + 1000x - 10y - 2575 = 0$

In Exercises 31 – 42, find the standard form of the equation of the hyperbola which has the given properties.

31. Vertices $(0, \pm 5)$, Foci $(0, \pm 8)$

32. Vertices $(\pm 3, 0)$, Focus $(5, 0)$

33. Vertices $(0, \pm 6)$, Focus $(0, -8)$

34. Center $(0, 0)$, Vertex $(0, -13)$, Focus $(0, \sqrt{313})$

35. Foci $(\pm 5, 0)$ with conjugate axis having length of 6

36. Center $(3, 7)$, Vertex $(3, 3)$, Focus $(3, 2)$

37. Center $(4, 2)$, Vertex $(9, 2)$, Focus $(4 + \sqrt{26}, 2)$

38. Center $(3, 5)$, Vertex $(3, 11)$, Focus $(3, 5 + 2\sqrt{10})$

39. Vertices $(0, 1)$ and $(8, 1)$, Focus $(-3, 1)$

40. Vertices $(1, 1)$ and $(11, 1)$, Focus $(12, 1)$

41. Vertices $(3, 2)$ and $(13, 2)$ with endpoints of the conjugate axis $(8, 4)$ and $(8, 0)$

42. Vertex $(-10, 5)$, Asymptotes $y = \pm \frac{1}{2}(x - 6) + 5$

In Exercises 43 – 47, given the graph of the hyperbola, determine its equation.

43.

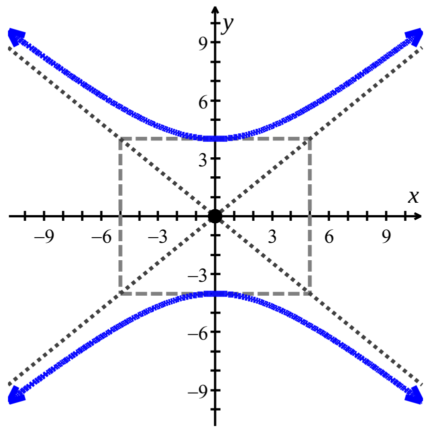


Figure 5.5. 17

44.

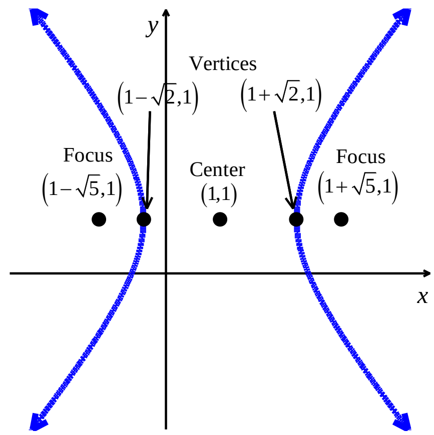


Figure 5.5. 18

45.

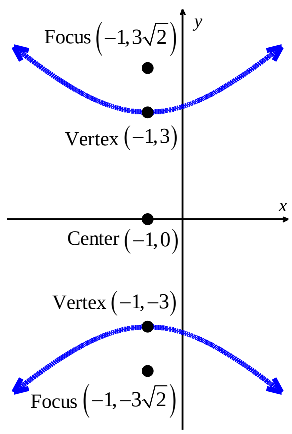


Figure 5.5. 19

46.

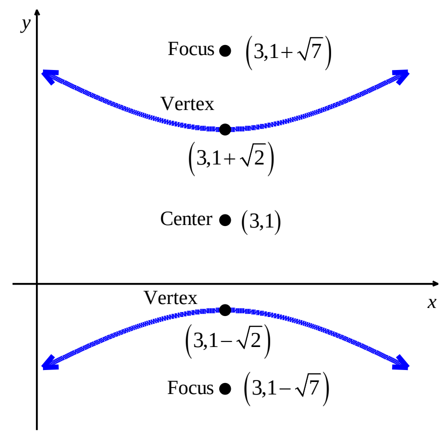


Figure 5.5. 20

47.

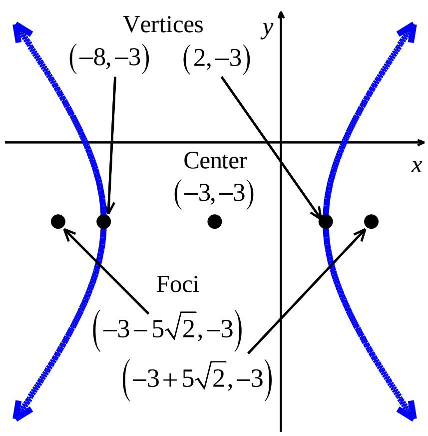


Figure 5.5. 21

48. The design layout of a hyperbolic cooling tower is shown below. The tower stands 167.082 meters tall. The diameter of the top is 60 meters. At their closest, the sides of the tower are 40 meters apart. Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola is the origin of the coordinate plane. Round final values to four decimal places.

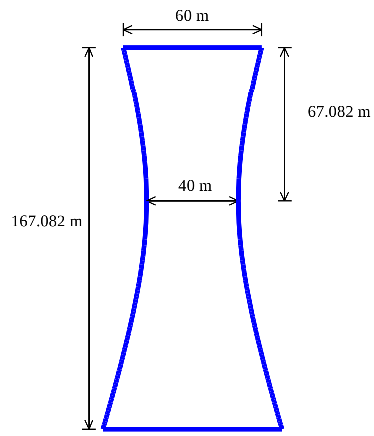


Figure 5.5. 22

49. The cross section of a hyperbolic cooling tower is shown below. Suppose the tower is 450 feet wide at the base, 275 feet wide at the top, and 220 feet at its narrowest point (which occurs 330 feet above the ground.) Determine the height of the tower to the nearest foot.

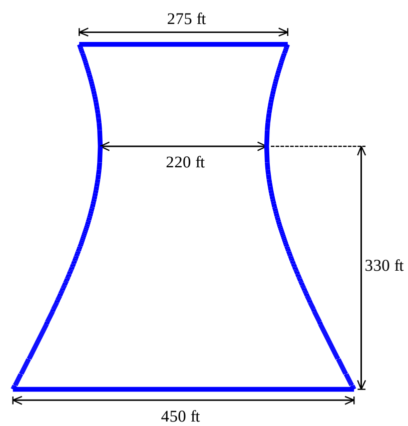


Figure 5.5. 23

50. A hedge is to be constructed in the shape of a hyperbola near a fountain at the center of a yard. The hedge will follow the asymptotes $y = x$ and $y = -x$, and its closest distance to the center fountain is 5 yards. Find the equation of the hyperbola and sketch the graph.

51. A hedge is to be constructed in the shape of a hyperbola near a fountain at the center of a yard. The hedge will follow the asymptotes $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$, and its closest distance to the center fountain is 20 yards. Find the equation of the hyperbola and sketch the graph.

52. With the help of your classmates, show that if $Ax^2 + By^2 + Cx + Dy + E = 0$ determines a non-degenerate conic[14] then

- $AB < 0$ means that the graph is a hyperbola
- $AB = 0$ means that the graph is a parabola
- $AB > 0$ means that the graph is an ellipse or circle

In Exercises 53 – 62, find the standard form of the equation using the Strategies for Identifying Conic Sections and then graph the conic section.

53. $x^2 - 2x - 4y - 11 = 0$

54. $x^2 + y^2 - 8x + 4y + 11 = 0$

55. $9x^2 + 4y^2 - 36x + 24y + 36 = 0$

56. $9x^2 - 4y^2 - 36x - 24y - 36 = 0$

57. $y^2 + 8y - 4x + 16 = 0$

58. $4x^2 + y^2 - 8x + 4 = 0$

59. $4x^2 + 9y^2 - 8x + 54y + 49 = 0$

60. $x^2 + y^2 - 6x + 4y + 14 = 0$

61. $2x^2 + 4y^2 + 12x - 8y + 25 = 0$

62. $4x^2 - 5y^2 - 40x - 20y + 160 = 0$

[14] Recall that this means its graph is either a circle, parabola, ellipse or hyperbola.

Key Equations

Circle: $\left(x-h\right)^2+\left(y-k\right)^2=r^2$ where (h,k) is the center of the circle and r is the radius

Midpoint Formula: $M=\left(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2}\right)$

Unit Circle: $x^2+y^2=1$

Parabola:

$(x-h)^2=4p(y-k)$ (Vertical Parabola)

$(y-k)^2=4p(x-h)$ (Horizontal Parabola)

where (h,k) is the vertex and $|p|$ is the focal length

Ellipse: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ where (h,k) is the center, a and b are positive unequal numbers

Distance from center to focus in an ellipse:

$c=\sqrt{a^2-b^2}$ or $c=\sqrt{b^2-a^2}$

Eccentricity of an Ellipse:

$e=\frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}}$

Hyperbola:

$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ (opens left and right)

$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$ (opens up and down)

where (h,k) is the center, a and b are positive numbers

Distance from center to focus in a hyperbola:

$c=\sqrt{a^2+b^2}$

Key Terms

Circle: Set of all points in a plane that are a fixed distance r from a fixed point (h,k); (h,k) is the center of the circle, and r is the radius

Conic Sections: Cross sections of a cone; figure formed by the intersection of a right circular cone and a plane

Conjugate Axis of a Hyperbola: Line containing the center of the hyperbola that is perpendicular to the transverse axis.

Directrix: A fixed line used to describe a parabola

Eccentricity of an Ellipse: Roundness or flatness of an ellipse

Ellipse: Set of all points (x,y) in a plane such that the sum of the distances from (x,y) to two fixed points (foci) is a constant

Focus: A fixed point or points used to describe a parabola, ellipse, or hyperbola

Hyperbola: Set of all points (x,y) in a plane such that the absolute value of the difference of the distances from (x,y) to two fixed points (foci) is a constant.

Parabola: Set of all points in a plane that are equidistant from a fixed point F (focus) and a fixed line D (directrix) not containing F

Paraboloid: Solid formed by rotating a parabola about its axis of symmetry

Transverse Axis of a hyperbola: Line containing the center and foci of the hyperbola

Unit Circle: circle centered at (0,0) with radius 1

CHAPTER 6
SYSTEMS OF EQUATIONS AND MATRICES

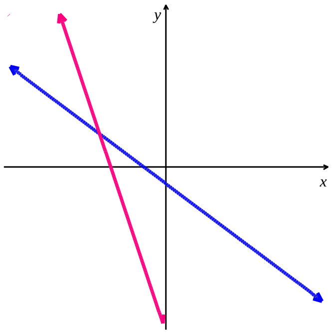


Figure 6.0. 1

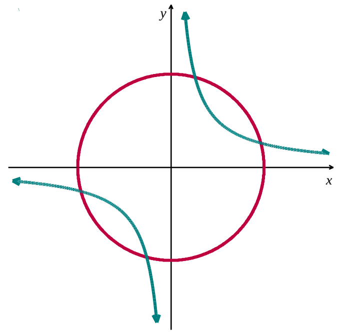


Figure 6.0. 2

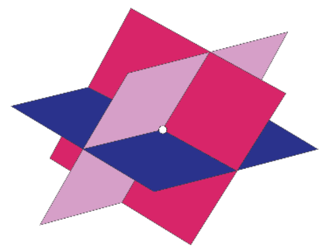


Figure 6.0. 3

Chapter Outline

Introduction

In this chapter we extend ideas for solving 2×2 systems of linear equations to solving non-linear systems, systems of 3×3 linear equations, and developing ideas about augmented matrix systems. We then transition to matrix arithmetic where you will see that it shares some properties with real number arithmetic; most importantly, some square matrices have inverses and inverses may be used to solve matrix equations (in much the same way we use inverse operations in arithmetic to solve algebraic equations.) Additionally, you will learn a little about determinants of square matrices, what they tell you about a system, and how they may be used to solve systems.

Section 6.1 starts with a review of solving 2×2 systems of linear equations. The goal of the review is to remind you of both the graphical meaning of a solution and the two basic analytic processes for finding solutions (substitution and elimination.) We build on these analytic processes to solve systems of non-linear equations. Throughout the section, however, it will be helpful to think about what the equations in the system represent. For example, if you are asked to find the point(s) of intersection for a circle and a parabola, it will be helpful to visualize all the possibilities of their intersections (0, 1, 2, 3, or 4 in this case.) Analytically, as with systems of linear equations, you will start by finding solution(s) for one of the variables and then substituting it/them in to an equation to find the corresponding value for the other. You may check the accuracy of your answer by testing both equations for the solution point(s) you find.

Section 6.2 also builds on your previous understandings for solving 2×2 systems, but now with 3×3 systems. Again, you will be looking for intersections; geometrically this means looking for intersections among three planes. Analytically, the process is very much like the elimination and substitution processes for 2×2 systems, but with an extra variable, so there is an extra elimination step. This process will be extended in the next section and is a useful tool (by itself) in future mathematics.

In Section 6.3 you will learn how to simplify the process of 6.2 by using augmented matrices with a process called Gaussian Elimination. The method allows you to get the system in row-echelon and/or reduced row-echelon form. Know that there are, in general, many ways to do this. Also note that often when solving a system, completely reducing is not necessary.

In Section 6.4 you are introduced to matrix operations, namely addition, subtraction, scalar multiplication, and matrix multiplication. Further, you will see that for matrix addition, the commutative, associative, identity and inverse properties hold. For matrix multiplication, associativity holds with matrices and with scalar multiplication. Also, there is an identity matrix for multiplication, and the distributive property holds. Arithmetic manipulations with matrices will be useful for solving systems in this chapter and for future mathematics.

In Section 6.5 you will learn how to find the inverse of a 2×2 and a 3×3 matrix and how it is useful for solving a system. Section 6.6 progresses with the ideas in 6.5 by introducing determinants, how to find them for 2×2 and 3×3 matrices, and then how to use them for solving systems using Cramer's Rule. In the example with the 2×2 systems, you will see that if the slope of the two lines is the same, the determinant will be 0 and there will be no inverse to the matrix.

Section 6.7 deals with the decomposition of fractions. In this section, you will be looking for the two or more rational expressions that could have been added to result in the rational expression with which you started. This skill will be helpful when you take Calculus and relies on the skills you learned in this chapter.

6.1 Systems of Linear and Non-Linear Equations

Learning Objectives

- Solve systems of two linear equations in two variables using substitution.
- Solve systems of two linear equations in two variables using elimination.
- Interpret solutions to 2×2 systems of linear equations.
- Solve systems of two non-linear equations in two variables using elimination.
- Solve systems of two non-linear equations in two variables using substitution.
- Solve and interpret solutions to 2×2 systems of non-linear equations.

We begin our study of systems of equations by reviewing the definition of a linear equation in two variables.

Definition 6.1. A **linear equation in two variables** *x* and *y* is an equation of the form $ax+by=c$ where *a*, *b* and *c* are constants and at least one of *a* and *b* is nonzero.

The key to identifying linear equations is to note that the variables involved are to the first power and that the coefficients of the variables are numbers. Some examples of equations that are non-linear are $x^2+y=1$, $xy=5$ and $e^{2x}+\ln(y)=1$. We leave it to the reader to explain why these equations do not satisfy **Definition 6.1**.

From graphing linear equations in prior math classes, you will recall that the graph of a linear equation is a line. If we couple two linear equations, in two variables, together in order to find the points of intersection of two lines, we obtain a **2×2 system of linear equations**. We read '2×2' as 'two by two'. The first 2 designates the number of equations and the second 2 is the number of variables.

Solving 2×2 Systems of Linear Equations

While the following examples may be review, they provide a good starting place for our study of systems of equations and matrices.

Example 6.1.1. Use the substitution method to solve the system of equations.

$$\begin{cases} 2x-y=1 \\ -4x+y=-5 \end{cases}$$

Solution. We solve the second equation for y in terms of x.

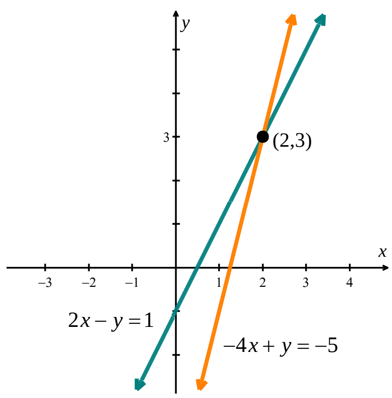
$$\begin{aligned} -4x + y &= -5 \\ y &= 4x - 5 \end{aligned}$$

We **substitute** this expression, $y = 4x - 5$, for y in the first equation.

$$\begin{aligned} 2x - y &= 1 && \text{1st equation} \\ 2x - (4x - 5) &= 1 && \text{substitute } y = 4x - 5 \text{ from 2nd equation} \\ 2x - 4x + 5 &= 1 \\ -2x &= -4 \end{aligned}$$

We find $x = 2$, and we use this result along with the equation $y = 4x - 5$ to get $y = 4(2) - 5 = 3$. Our solution to the system is $(2,3)$. As shown in the following graph, this solution is the point of intersection of the lines $2x - y = 1$ and $-4x + y = -5$.

Figure 6.1. 1



□

We may also check solutions algebraically by substituting the x and y values in the solution into the original equations to see that they are satisfied.

Example 6.1.2. Use the elimination method to solve the following systems of equations.

1. $\begin{cases} 3x + 4y = -2 \\ -3x - y = 5 \end{cases}$ 2. $\begin{cases} 2x - 4y = 6 \\ 3x - 6y = 9 \end{cases}$ 3. $\begin{cases} 6x + 3y = 9 \\ 4x + 2y = 12 \end{cases}$

Solution.

1. To solve the system of equations, we use **addition** to **eliminate** the variable x. We take the two equations as given and add them together.

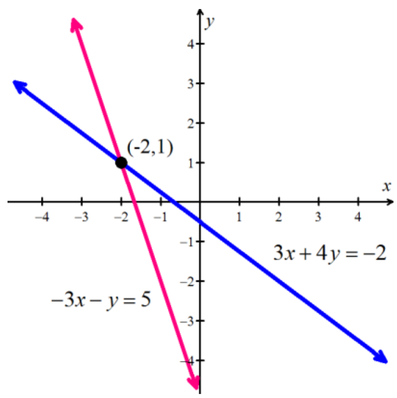
$$\begin{aligned} 3x + 4y &= -2 \\ \underline{-3x - y} &= \underline{5} \\ 3y &= 3 \end{aligned}$$

This gives us $y=1$, which we may substitute in either of the two equations to solve for *x*. We select the first equation, $3x+4y=-2$.

$$\begin{aligned} 3x+4(1)&=-2 \\ 3x&=-6 \\ x&=-2 \end{aligned}$$

Our solution is $(-2,1)$. Below, the graphs of the two equations verify that $(-2,1)$ is the point of intersection.

Figure 6.1. 2

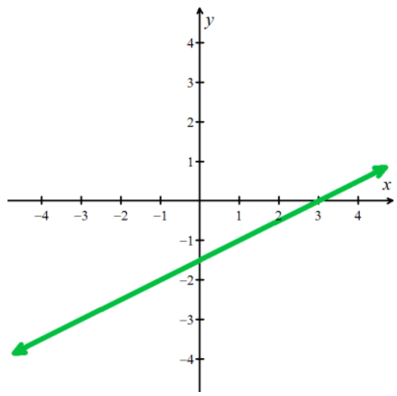


2. Adding the two equations, $2x-4y=6$ and $3x-6y=9$, directly fails to eliminate either of the variables, but we note that if we multiply both sides of the first equation by 3 and both sides of the second equation by −2, we will be in a position to eliminate the *x* term.

$$\begin{array}{r} 6x-12y=\ \ 18 \\ -6x+12y=-18 \\ \hline 0\ =\ \ 0 \end{array}$$

We eliminated not only the *x* term, but the *y* term as well, and we are left with the identity $0=0$. This means that the two different linear equations are, in fact, equivalent. In other words, if an ordered pair (x,y) satisfies the equation $2x-4y=6$, it automatically satisfies the equation $3x-6y=9$. One way to describe the solution set to this system is to use the roster method and write $\{(x,y)\,|\,2x-4y=6\}$. Geometrically, $2x-4y=6$ and $3x-6y=9$ are the same line, which means that they intersect at every point on their graphs.

Figure 6.1. 3



$$2x - 4y = 6$$
$$3x - 6y = 9$$

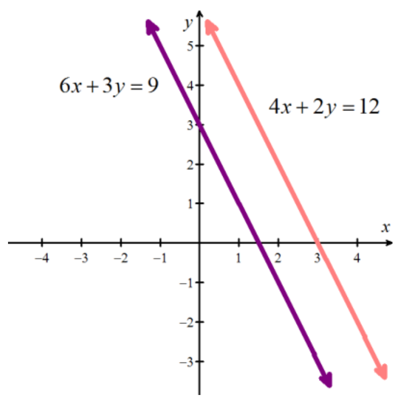
(same line)

3. Multiplying both sides of the first equation, $6x + 3y = 9$, by 2 and both sides of the second equation, $4x + 2y = 12$, by −3, we set the stage to eliminate *x*.

$$\begin{array}{r} 12x + 6y = \ \ 18 \\ -12x - 6y = -36 \\ \hline 0 \ = -18 \end{array}$$

As in the previous problem, both *x* and *y* dropped out of the equation, but we are left with an irrevocable contradiction, $0 = -18$. This tells us that it is impossible to find a pair (x, y) that satisfies both equations; in other words, the system has no solution. Graphically, the lines $6x + 3y = 9$ and $4x + 2y = 12$ are distinct and parallel, so they do not intersect. Note that if two lines have the same slope they will either represent the same line as in part 2, or they will be parallel as in this example.

Figure 6.1. 4



□

If a system of equations, linear or nonlinear, has no solutions it is called **inconsistent**. Systems, linear or nonlinear, with at least one solution are called **consistent**. The systems in **Example 6.1.1** and **Example 6.1.2**, parts 1 and 2, are consistent. The system in **Example 6.1.2**, part 3, is inconsistent.

In the case of linear systems, we can divide the consistent systems into two categories. A consistent linear system is called **independent** if it has a single solution, as in **Example 6.1.1** and **Example 6.1.2**, part 1. Geometrically, this solution is the point of intersection of two lines having different slopes. A consistent linear system is called **dependent** if it has an infinite number of solutions, as the system in **Example 6.1.2**, part 2. Geometrically, the lines are coincident and every coordinate pair on the line is a solution to both equations.

We move on to the challenge of solving systems of non-linear equations, in which we apply the above techniques while addressing additional issues such as extraneous solutions.

Solving 2×2 Systems of Non-Linear Equations

By now, we have seen many nonlinear equations in two variables, such as polynomials. These equations represent curves in the plane. If we couple two nonlinear equations in two variables together, in effect to find the points of intersection of two curves, we obtain a 2×2 **system of nonlinear equations**. There is no general method for solving nonlinear systems. We will try to use the substitution method or the elimination method, or a combination, to reduce the system to one equation in one variable.

Example 6.1.3. Solve the following systems of equations.

1. $\begin{cases} x^2+y^2=4 \\ 4x^2+9y^2=36 \end{cases}$

2. $\begin{cases} x^2+y^2=4 \\ 4x^2-9y^2=36 \end{cases}$

Solution. We can solve each system by either substitution or elimination. To demonstrate both, we will solve the first system by the substitution method and the second by the elimination method.

1. We can solve for x^2 in the first equation to get $x^2=4-y^2$, and substitute this result in the second equation to find *y* value(s) at the point(s) of intersection.

$$\begin{aligned} 4x^2+9y^2&=36 \\ 4\left(4-y^2\right)+9y^2&=36 \\ 16-4y^2+9y^2&=36 \\ 5y^2&=20 \end{aligned}$$

We find $y^2=4$, from which $y=\pm\ 2$. To find the associated *x* values, we substitute each value of *y* into one of the original equations. Here, we choose $x^2+y^2=4$.

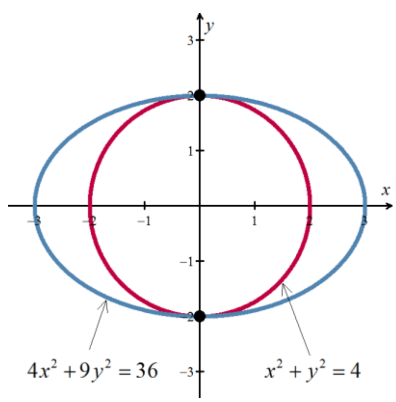
$$x^2 + (-2)^2 = 4 \quad \text{set } y = -2 \qquad\qquad x^2 + (2)^2 = 4 \quad \text{set } y = 2$$
$$x^2 = 0 \qquad\qquad x^2 = 0$$
$$x = 0 \qquad\qquad x = 0$$

Our potential solutions are the ordered pairs $(0,-2)$ and $(0,2)$. To check these algebraically, we need to show that both points satisfy both of the original equations.

(x,y)	First equation: $x^2+y^2=4$	Second equation: $4x^2+9y^2=36$
$(0,-2)$	Left side is $(0)^2+(-2)^2=4$	Left side is $4(0)^2+9(-2)^2=36$.
$(0,2)$	Left side is $(0)^2+(2)^2=4$.	Left side is $4(0)^2+9(2)^2=36$

For each potential solution, substituting the values of the ordered pair in the left side of each equation gives the value of the right side of the equation, so both potential solutions are acceptable. To check our answer graphically, we sketch both equations and look for their points of intersection. The graph of $x^2+y^2=4$ is a circle centered at $(0,0)$ with a radius of 2, whereas the graph of $4x^2+9y^2=36$, when written in the standard form $\frac{x^2}{9}+\frac{y^2}{4}=1$, is easily recognized as an ellipse centered at $(0,0)$ with a major axis along the *x*-axis of length 6 and a minor axis along the *y*-axis of length 4.

Figure 6.1. 5



We see from the graph that the two curves intersect only at their *y*-intercepts, $(0,-2)$ and $(0,2)$.

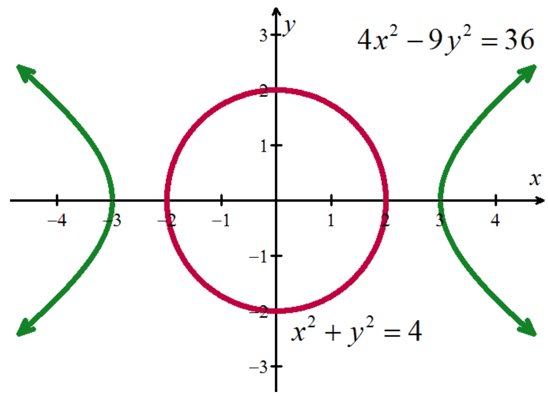
We have verified, both algebraically and geometrically, that the solution is $\{(0,-2),(0,2)\}$.

2. We apply the elimination method here. We multiply the equation $x^2+y^2=4$ by −4, then add the result to $4x^2-9y^2=36$.

$$\begin{aligned} -4x^2 - 4y^2 &= -16 \\ 4x^2 - 9y^2 &= 36 \\ \hline -13y^2 &= 20 \end{aligned}$$

Since the equation $-13y^2 = 20$ does not have any real solutions, the system is inconsistent. Thus, there is no solution. To verify this graphically, we note that $x^2 + y^2 = 4$ is the same circle as before, but when writing the second equation in standard form, $\frac{x^2}{9} - \frac{y^2}{4} = 1$, we find a hyperbola centered at $(0,0)$ opening to the left and right with a transverse axis of length 6 and a conjugate axis of length 4.

Figure 6.1. 6



We see that the circle and the hyperbola have no points in common.

□

Following is a strategy for solving a system of two nonlinear equations in two variables.

Solving a 2×2 System of Nonlinear Equations

1. Apply substitution, elimination, or a combination, to obtain equation(s) in one variable.
   a) In applying substitution, you may solve an equation for a variable, an expression, or a combination of variables.
   b) In applying elimination, you may also multiply or divide equations by variables.
2. Solve the new equation(s).
3. Substitute solutions from part 2, if any, in one of original equations to obtain corresponding values of the other variable.
4. Check each potential ordered pair solution obtained in part 3.

If there is no solution in part 2, the system has no solution.[1] Recall that such a system is called inconsistent. Graphically, this means the two curves do not intersect. We may have one or more solutions after checking the potential solutions in part 4, in which case you may recall that such a system is called consistent. Each solution corresponds to a point of intersection of the two curves.

Checking of potential solutions is not necessary for every nonlinear system. However, rather than specifying the type of nonlinear systems, or instances, for which checking potential solutions is essential, we will check all potential solutions.

Example 6.1.4. Solve the system of equations.

$$\begin{cases} x^2 + y^2 = 5 \\ y - 2x = 0 \end{cases}$$

Solution. We observe that it is easy to solve for one of the variables in the second equation, while it is not obvious how to combine the two equations to eliminate a variable, so we will apply the substitution method. We begin by solving $y - 2x = 0$ to get $y = 2x$, and substitute $y = 2x$ in $x^2 + y^2 = 5$ to get $x^2 + (2x)^2 = 5$. We proceed by solving for *x*.

$$\begin{aligned} x^2 + (2x)^2 &= 5 \\ 5x^2 &= 5 \\ x^2 &= 1 \end{aligned}$$

We find $x = \pm 1$, which we substitute in the first equation, $x^2 + y^2 = 5$, to determine corresponding y values.

$$\begin{aligned} (-1)^2 + y^2 &= 5 \quad \text{set } x = -1 \\ y^2 &= 4 \\ y &= \pm 2 \end{aligned} \qquad \begin{aligned} (1)^2 + y^2 &= 5 \quad \text{set } x = 1 \\ y^2 &= 4 \\ y &= \pm 2 \end{aligned}$$

Our potential solutions are the ordered pairs, $(-1,-2)$, $(-1,2)$, $(1,-2)$ and $(1,2)$. We check these potential solutions. Since we already know they satisfy the first equation, we only need to check them in the second equation, $y - 2x = 0$.

[1] We may also say the solution set is the empty set.

(x,y)	Second Equation: $y-2x=0$
$(-1,-2)$	Left side is $(-2)-2(-1)=0$, so left side equals right side.
$(-1,2)$	Left side is $(2)-2(-1)=4$, so left side is NOT equal to right side.
$(1,-2)$	Left side is $(-2)-2(1)=-4$, so left side is NOT equal to right side.
$(1,2)$	Left side is $(2)-2(1)=0$, so left side equals right side.

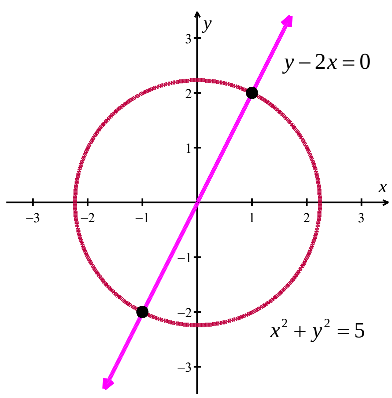
Both $(-1,2)$ and $(1,-2)$ are extraneous solutions. The solution is $\{(-1,-2),(1,2)\}$.

□

In the previous example, if we had plugged $x=\pm 1$ into the second equation, $y-2x=0$, to find *y* values, we would not have had any extraneous solutions. It is not always easy to tell which scenario will result in extraneous solutions, but checking all potential solutions eliminates the necessity for that determination.

To visualize the solutions in **Example 6.1.4**, we graph the circle $x^2+y^2=5$ and the line $y-2x=0$ below. Their two points of intersection are our two solutions.

Figure 6.1. 7



There is no general method for solving nonlinear systems, and at times it may take several steps to obtain an equation in one variable. Some creativity may be helpful, as demonstrated in the following examples.

Example 6.1.5. Solve the system of equations.

$$\begin{cases} x^2+x-y=0 \\ \dfrac{y^2}{x}-\dfrac{y}{x}+1=0 \end{cases}$$

Solution. We use substitution by solving the first equation for y to get $y = x^2 + x$, and then substituting this result in the second equation: $\frac{(x^2+x)^2}{x} - \frac{x^2+x}{x} + 1 = 0$. Now that we have an equation in one variable, we can solve this equation for x by, first of all, multiplying both sides by x.

$$\begin{aligned} (x^2+x)^2 - (x^2+x) + x &= 0 \\ x^4 + 2x^3 + x^2 - x^2 - x + x &= 0 \\ x^4 + 2x^3 &= 0 \\ x^3(x+2) &= 0 \end{aligned}$$

We get $x^3 = 0$ or $x+2=0$, resulting in $x=0$ or $x=-2$. We substitute each of these x values in the first equation, $x^2 + x - y = 0$, to find corresponding y values.

$$\begin{aligned} (0)^2 + (0) - y &= 0 \quad \text{set } x = 0 \\ 0 &= y \\ y &= 0 \end{aligned}$$

$$\begin{aligned} (-2)^2 + (-2) - y &= 0 \quad \text{set } x = -2 \\ 4 - 2 &= y \\ y &= 2 \end{aligned}$$

Our potential solutions are $(0,0)$ and $(-2,2)$. We know these potential solutions satisfy the first equation, and proceed by checking that they also satisfy the second equation.

(x,y)	Second Equation: $\frac{y^2}{x} - \frac{y}{x} + 1 = 0$
$(0,0)$	Left side is $\frac{(0)^2}{(0)} - \frac{(0)}{(0)} + 1$, which is undefined due to zero in denominator.
$(-2,2)$	Left side is $\frac{(2)^2}{(-2)} - \frac{(2)}{(-2)} + 1 = -2 + 1 + 1 = 0$, so left side equals right side.

The only solution is $(-2,2)$.

Alternate Solution. An alternate way to arrive at the x values of 0 and −2 in **Example 6.1.5** is to begin by multiplying the second equation, $\frac{y^2}{x} - \frac{y}{x} + 1 = 0$, through by $-x$. We then have the system

$$\begin{cases} x^2 + x - y = 0 \\ -y^2 + y - x = 0 \end{cases}$$

Adding these two equations gives us $x^2 - y^2 = 0$, which we solve by factoring as $(x-y)(x+y) = 0$ to get $y = x$ or $y = -x$. We substitute these results into the first equation, $x^2 + x - y = 0$.

$$\begin{aligned} x^2+x-(-x)&=0 \quad \text{set } y=-x \\ x^2+2x&=0 \\ x(x+2)&=0 \end{aligned}$$

$$\begin{aligned} x^2+x-(x)&=0 \quad \text{set } y=x \\ x^2&=0 \\ x&=0 \end{aligned}$$

We find $x=0$ or $x=-2$ and proceed to solve for y as in the original solution.

□

Example 6.1.6. Solve the system of equations.

$$\begin{cases} x^2-3xy+2y^2=0 \\ x^2+xy=6 \end{cases}$$

Solution. It is not obvious how to eliminate a variable so we will try the substitution method. Although solving for one variable in terms of the other is not immediate, we can do this is we factor the first equation.

$$\begin{aligned} x^2-3xy+2y^2&=0 \\ (x-y)(x-2y)&=0 \end{aligned}$$

Setting each factor equal to zero, we get $y=x$ or $y=\frac{x}{2}$. We plug each of these into the second equation, $x^2+xy=6$, and solve for x.

$$\begin{aligned} x^2+x(x)&=6 \quad \text{set } y=x \\ 2x^2&=6 \\ x^2&=3 \end{aligned}$$

$$\begin{aligned} x^2+x\left(\frac{x}{2}\right)&=6 \quad \text{set } y=\frac{x}{2} \\ \frac{3}{2}x^2&=6 \\ x^2&=4 \end{aligned}$$

The result is $x=\pm\sqrt{3}$ or $x=\pm 2$. Now we use the second equation, $x^2+xy=6$, to solve for y values.

Set $x=-\sqrt{3}$:

$$\begin{aligned} \left(-\sqrt{3}\right)^2+\left(-\sqrt{3}\right)y&=6 \\ 3-\sqrt{3}y&=6 \\ y&=\frac{3}{-\sqrt{3}} \\ y&=-\sqrt{3} \end{aligned}$$

Set $x=\sqrt{3}$:

$$\begin{aligned} \left(\sqrt{3}\right)^2+\left(\sqrt{3}\right)y&=6 \\ 3+\sqrt{3}y&=6 \\ y&=\frac{3}{\sqrt{3}} \\ y&=\sqrt{3} \end{aligned}$$

Set $x=-2$:

$$\begin{aligned} (-2)^2+(-2)y&=6 \\ 4-2y&=6 \\ y&=\frac{2}{-2} \\ y&=-1 \end{aligned}$$

Set $x=2$:

$$\begin{aligned} (2)^2+(2)y&=6 \\ 4+2y&=6 \\ y&=\frac{2}{2} \\ y&=1 \end{aligned}$$

We have the potential solutions $\left(-\sqrt{3},-\sqrt{3}\right)$, $\left(\sqrt{3},\sqrt{3}\right)$, $(-2,-1)$ and $(2,1)$. We check each of these in the first equation, $x^2-3xy+2y^2=0$, since we already know they satisfy the second.

(x, y)	First Equation: $x^2 - 3xy + 2y^2 = 0$
$(-\sqrt{3}, -\sqrt{3})$	Left Side is $(-\sqrt{3})^2 - 3(-\sqrt{3})(-\sqrt{3}) + 2(-\sqrt{3})^2 = 3 - 9 + 6 = 0$, same as right side.
$(\sqrt{3}, \sqrt{3})$	Left side is $(\sqrt{3})^2 - 3(\sqrt{3})(\sqrt{3}) + 2(\sqrt{3})^2 = 3 - 9 + 6 = 0$, same as right side.
$(-2, -1)$	Left side is $(-2)^2 - 3(-2)(-1) + 2(-1)^2 = 4 - 6 + 2 = 0$, same as right side.
$(2, 1)$	Left side is $(2)^2 - 3(2)(1) + 2(1)^2 = 4 - 6 + 2 = 0$, same as right side.

There are four solutions: $(-\sqrt{3}, -\sqrt{3})$, $(\sqrt{3}, \sqrt{3})$, $(-2, -1)$ and $(2, 1)$.

□

6.1 Exercises

1. Can a system of linear equations have exactly two solutions? Explain why or why not.

2. Can a system of two non-linear equations have exactly two solutions? What about exactly three? In not, explain why not. If so, sketch the graph of such a system.

In Exercises 3 – 16, solve the given system using substitution and/or elimination. Classify each system as consistent independent, consistent dependent, or inconsistent.

3. $\begin{cases} x+2y=5 \\ x=6 \end{cases}$

4. $\begin{cases} 2y-3x=1 \\ y=-3 \end{cases}$

5. $\begin{cases} x+3y=5 \\ 2x+3y=4 \end{cases}$

6. $\begin{cases} x-2y=3 \\ -3x+6y=-9 \end{cases}$

7. $\begin{cases} 3x-2y=18 \\ 5x+10y=-10 \end{cases}$

8. $\begin{cases} 4x+2y=-10 \\ 3x+9y=0 \end{cases}$

9. $\begin{cases} -2x+5y=-42 \\ 7x+2y=30 \end{cases}$

10. $\begin{cases} 6x-5y=-34 \\ 2x+6y=4 \end{cases}$

11. $\begin{cases} -x+2y=-1 \\ 5x-10y=6 \end{cases}$

12. $\begin{cases} 5x+9y=16 \\ x+2y=4 \end{cases}$

13. $\begin{cases} \dfrac{x+2y}{4}=-5 \\ \dfrac{3x-y}{2}=1 \end{cases}$

14. $\begin{cases} \dfrac{1}{2}x-\dfrac{1}{3}y=-1 \\ 2y-3x=6 \end{cases}$

15. $\begin{cases} x+4y=6 \\ \dfrac{1}{12}x+\dfrac{1}{3}y=\dfrac{1}{2} \end{cases}$

16. $\begin{cases} 3y-\dfrac{3}{2}x=-\dfrac{15}{2} \\ \dfrac{1}{2}x-y=\dfrac{3}{2} \end{cases}$

In Exercises 17 – 20, graph the system of equations and state whether the system has one solution, no solution, or infinite solutions.

17. $\begin{cases} -x+2y=4 \\ 2x-4y=1 \end{cases}$

18. $\begin{cases} x+2y=7 \\ 2x+6y=12 \end{cases}$

19. $\begin{cases} 3x-5y=7 \\ x-2y=3 \end{cases}$

20. $\begin{cases} 3x-2y=5 \\ -9x+6y=-15 \end{cases}$

In Exercises 21 – 32, solve the given system of non-linear equations. Sketch the graph of both equations on the same set of axes to verify the solution set.

21. $\begin{cases} x+y=4 \\ x^2+y^2=9 \end{cases}$

22. $\begin{cases} y=x-3 \\ x^2+y^2=9 \end{cases}$

23. $\begin{cases} y=x \\ x^2+y^2=9 \end{cases}$

24. $\begin{cases} y=-x \\ x^2+y^2=9 \end{cases}$

25. $\begin{cases} x=2 \\ x^2-y^2=9 \end{cases}$

26. $\begin{cases} 4x^2-9y^2=36 \\ 4x^2+9y^2=36 \end{cases}$

27. $\begin{cases} x^2+y^2=25 \\ x^2-y^2=1 \end{cases}$

28. $\begin{cases} x^2-y=4 \\ x^2+y^2=4 \end{cases}$

29. $\begin{cases} x^2+y^2=4 \\ x^2-y=5 \end{cases}$

30. $\begin{cases} x^2+y^2=16 \\ 16x^2+4y^2=64 \end{cases}$

31. $\begin{cases} x^2+y^2=16 \\ 9x^2-16y^2=144 \end{cases}$

32. $\begin{cases} x^2+y^2=16 \\ \frac{1}{9}y^2-\frac{1}{16}x^2=1 \end{cases}$

In Exercises 33 – 40, solve the given system of non-linear equations.

33. $\begin{cases} x^2+y^2=16 \\ x-y=2 \end{cases}$

34. $\begin{cases} x^2-y^2=1 \\ x^2+4y^2=4 \end{cases}$

35. $\begin{cases} x+2y^2=2 \\ x^2+4y^2=4 \end{cases}$

36. $\begin{cases} (x-2)^2+y^2=1 \\ x^2+4y^2=4 \end{cases}$

37. $\begin{cases} x^2+y^2=25 \\ y-x=1 \end{cases}$

38. $\begin{cases} x^2+y^2=25 \\ x^2+(y-3)^2=10 \end{cases}$

39. $\begin{cases} y=x^3+8 \\ y=10x-x^2 \end{cases}$

40. $\begin{cases} x^3-10x+y=5 \\ x-y=-5 \end{cases}$

6.2 Systems of Linear Equations and Applications

Learning Objectives

- Solve systems of three linear equations in three variables.
- Interpret solutions to 3×3 systems of linear equations.
- Solve applications of linear equations in three variables.

We begin this section with the definition of a linear equation in three variables.

Definition 6.2. A **linear equation in three variables** x, y and z is an equation of the form $ax + by + cz = d$ where a, b, c and d are constants and at least one of a, b and c is nonzero.

Just as equations involving the variables x and y describe graphs of one-dimensional lines and curves in the two-dimensional plane, equations involving x, y and z describe objects called **surfaces** in three-dimensional space. Linear equations, as described above, represent planes in three-space.

Coupling more than one linear equation in three variables results in a **system of linear equations in three variables**. If we couple three linear equations, in three variables, together in order to find the points of intersection of three planes, we obtain a **3×3 system of linear equations**.

Solving 3×3 Systems of Linear Equations

When solving systems of equtions in more than two variables, it becomes increasingly important to keep track of what operations are performed to which equations and to develop a strategy based on the manipulations we've already employed. To this end, we identify a strategy that can be used in solving 3×3 systems of linear equations.

Solving a 3×3 System of Linear Equations

1. Target one of the three variables for elimination.
2. Eliminate the targeted variable from one pair of equations.
3. Eliminate the targeted variable from a different pair of equations.
4. Solve the resulting 2×2 system for both variables.
5. Back-substitute known values into one of original equations to find value of targeted variable.

The idea of systems being consistent or inconsistent carries over from **Section 6.1**. We say a 3×3 system of linear equations is inconsistent if it has no solutions. A system with at least one solution is consistent. A system with a single solution is said to be independent while a system having infinitely many solutions is said to be dependent.

Example 6.2.1. Solve the system, if possible. Classify the system as consistent independent, consistent dependent or inconsistent.

$$\begin{cases} 3x - y + z = 3 \\ 2x - 4y + 3z = 16 \\ x - y + z = 5 \end{cases}$$

Solution. We begin by labeling the equations:

$$\begin{aligned} 3x - y + z &= 3 \quad (1) \\ 2x - 4y + 3z &= 16 \quad (2) \\ x - y + z &= 5 \quad (3) \end{aligned}$$

We choose z as the variable [2] we will eliminate, and equations (1) and (3) as the pair [3] we will start with. After multiplying equation (3) by −1, we add the resulting equations.

$$\begin{array}{ll} 3x - y + z = 3 & (1) \\ \underline{-x + y - z = -5} & (3) \text{ multiplied by } -1 \\ 2x \qquad\quad = -2 & \end{array}$$

Now we choose equations [4] (1) and (2), multiplying equation (1) by −3 in an attempt to eliminate z.

$$\begin{array}{ll} -9x + 3y - 3z = -9 & (1) \text{ multiplied by } -3 \\ \underline{\quad 2x - 4y + 3z = 16} & (2) \\ -7x - y \qquad\quad = 7 & \end{array}$$

We solve the resulting system for x and y.

$$\begin{cases} 2x = -2 \\ -7x - y = 7 \end{cases}$$

From the first equation, $2x = -2$, we get $x = -1$. After substituting $x = -1$ in the second equation, we find $y = 0$. We back-substitute these values into equation (1) to find the value of z.

$$\begin{aligned} 3x - y + z &= 3 \\ 3(-1) - (0) + z &= 3 \\ -3 + z &= 3 \end{aligned}$$

---

[2] We could just as easily choose x or y.

[3] Any pair will work here – simply a matter of choice.

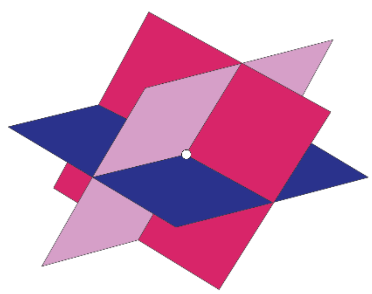
[4] Any pair is okay as long as it's different from the first pair we used.

We get $z=6$. We leave it to the reader to check that substituting the respective values for *x*, *y* and *z* into the original system results in three identities. Since we have found a solution, the system is consistent; since each variable is assigned a distinct value, it is independent.

□

It is desirable for us to write the solution to the system in **Example 6.2.1** by extending the usual (x,y) notation to (x,y,z) and list our solution as $(-1,0,6)$. The question quickly becomes what does an 'ordered triple' like $(-1,0,6)$ represent? Just as ordered pairs are used to locate points on the two-dimensional plane, ordered triples can be used to locate points in space. Geometrically, the ordered triple $(-1,0,6)$ is the intersection, or common point, of the three planes represented by the three equations. If you imagine three sheets of notebook paper, each representing a portion of these planes, you will start to see the complexities involved in how three such planes intersect. Below is a sketch of three planes intersecting in a single point.

Figure 6.2. 1



Example 6.2.2. Solve the system, if possible. Classify the system as consistent independent, consistent dependent or inconsistent.

$$\begin{cases} 2x+3y-z=1 \\ 10x-z=2 \\ 4x-9y+2z=5 \end{cases}$$

Solution. We label the equations.

$$\begin{aligned} 2x+3y-z&=1 \quad (1) \\ 10x-z&=2 \quad (2) \\ 4x-9y+2z&=5 \quad (3) \end{aligned}$$

We see that there is no *y* in equation (2), so we already have *y* eliminated from one equation. We need another equation with no *y* term, so we work with equations (1) and (3), multiplying equation (1) by 3.

$$\begin{array}{rl} 6x+9y-3z=3 & \text{(1) multiplied by 3} \\ \underline{4x-9y+2z=5} & \text{(3)} \\ 10x \qquad -z=8 & \end{array}$$

We have the resulting system

$$\begin{cases} 10x-z=2 \\ 10x-z=8 \end{cases}$$

Now we eliminate *x*, after multiplying the second equation by −1.

$$\begin{array}{rl} 10x-z=2 & \\ \underline{-10x+z=-8} & \text{after multiplying } 10x-z=8 \text{ by } -1 \\ 0=-6 & \end{array}$$

By eliminating *x*, we have also eliminated *z* with the resulting equation $0=-6$. This is a contradiction so the system has no solution. Thus, we have an inconsistent system.

□

There are three different geometric possibilities for systems of three linear equations that have no solutions. As seen in the first figure, below, the planes may intersect with each other, but not at a common point. In the second figure, two of the planes are parallel and intersect with the third plane, but not with each other. The third figure illustrates three parallel planes that do not intersect.

Figure 6.2. 2

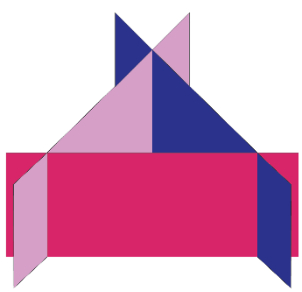


Figure 6.2. 3

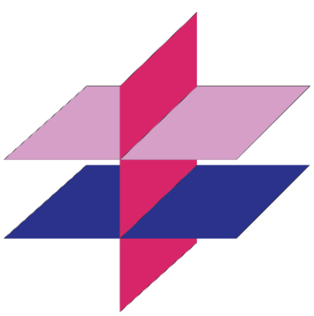
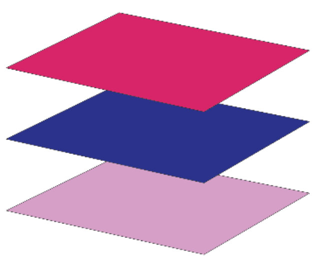


Figure 6.2. 4



Example 6.2.3. Solve the system, if possible. Classify the system as consistent independent, consistent dependent or inconsistent.

$$\begin{cases} 2x+3y-6z=1 \\ -4x-6y+12z=-2 \\ x+2y+5z=10 \end{cases}$$

Solution. Once again, we start by numbering the equations.

$$\begin{array}{rl} 2x+3y-6z=1 & (1) \\ -4x-6y+12z=-2 & (2) \\ x+2y+5z=10 & (3) \end{array}$$

It looks like x is easiest to eliminate. We begin with equations (1) and (2), multiplying equation (1) by 2.

$$\begin{array}{rl} 4x+6y-12z=2 & \text{(1) multiplied by 2} \\ \underline{-4x-6y+12z=-2} & \text{(2)} \\ 0=0 & \end{array}$$

We can also try equations (2) and (3), multiplying equation (3) by 4.

$$\begin{array}{rl} -4x-6y+12z=-2 & \text{(2)} \\ \underline{4x+8y+20z=40} & \text{(3) multiplied by 4} \\ 2y\ \ +32z=38 & \end{array}$$

These two elimination result in the following system of equations.

$$\begin{cases} 0=0 \\ 2y+32z=38 \end{cases}$$

The first equation, $0=0$, is always true. We have a consistent dependent system. We **parametrize** the solution set with a parameter t by letting $z=t$. Substituting $z=t$ in the second equation, we have

$$\begin{aligned} 2y+32t&=38 \\ y+16t&=19 \\ y&=-16t+19 \end{aligned}$$

Substituting $y=-16t+19$ and $z=t$ in the first equation, $2x+3y-6z=1$, gives us

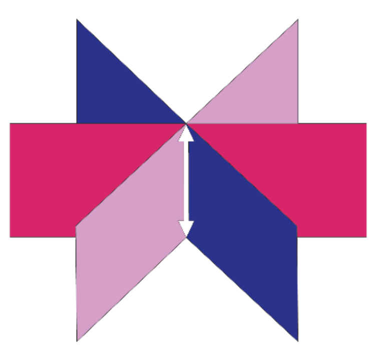
$$\begin{aligned} 2x+3(-16t+19)-6t&=1 \\ 2x-48t+57-6t&=1 \\ 2x&=54t-56 \\ x&=27t-28 \end{aligned}$$

Our solution is the set $\{27t-28, -16t+19,\ t\}$ for all real numbers t.

□

The solution set can be thought of geometrically as three planes intersecting in a single line, as the following illustration demonstrates.

Figure 6.2. 5



Applications of Linear Equations in Three Variables

Example 6.2.4. John received an inheritance of $12,000 that he divided into three parts and invested in three ways: a money-market fund paying 3% annual interest; municipal bonds paying 4% annual interest; and mutual funds paying 7% annual interest. John invested $4,000 more in municipal funds than in municipal bonds. He earned $670 in interest the first year. How much did John invest in each type of fund?

Solution. To solve this problem, we use all of the information given and set up three equations. First, we assign a variable to each of the three investment amounts:

$$\begin{aligned} x &= \text{amount invested in money-market fund} \\ y &= \text{amount invested in municipal bonds} \\ z &= \text{amount invested in mutual funds} \end{aligned}$$

Our first equation indicates that the sum of the three investments is $12,000: $x + y + z = 12000$. We use the information that John invested $4,000 more in mutual funds than he invested in municipal bonds for a second equation: $z = y + 4000$. The third equation, $0.03x + 0.04y + 0.07z = 670$, shows that the total amount earned from the three funds equals $670. We write these three equations as a system.

$$\begin{cases} x + y + z = 12000 \\ z = y + 4000 \\ 0.03x + 0.04y + 0.07z = 670 \end{cases}$$

After reordering the variables in the second equation and multiplying the third equation by 100, to simplify calculations, we assign numbers to the equations.

$$\begin{aligned} x + y + z &= 12000 \quad (1) \\ -y + z &= 4000 \quad (2) \\ 3x + 4y + 7z &= 67000 \quad (3) \end{aligned}$$

Since there is no *x* in equation (2), we choose *x* as our target variable to eliminate, and proceed with eliminating *x* from equations (1) and (3).

$$\begin{array}{rl} -3x - 3y - 3z = -36000 & (1) \text{ multiplied by } -3 \\ \underline{3x + 4y + 7z = \quad 67000} & (3) \\ y + 4z = \quad 31000 & \end{array}$$

We have the resulting system

$$\begin{cases} -y + z = 4000 & \text{equation (2)} \\ y + 4z = 31000 \end{cases}$$

Now we eliminate *y* from this 2×2 system.

$$\begin{array}{r} -y+z=4000 \\ \underline{y+4z=31000} \\ 5z=35000 \end{array}$$

So $z=7000$ and we find y by substituting into equation (2), $-y+z=4000$.

$$\begin{aligned} -y+(7000)&=4000 \\ -y&=-3000 \\ y&=3000 \end{aligned}$$

After substituting $y=3000$ and $z=7000$ into equation (1), $x+y+z=12000$, we have

$$\begin{aligned} x+(3000)+(7000)&=12000 \\ x&=2000 \end{aligned}$$

To summarize, John invested $2,000 in a money-market fund, $3,000 in municipal bonds and $7,000 in mutual funds.

□

Example 6.2.5. Find the quadratic function passing through the points $(1,6)$, $(-1,10)$ and $(2,13)$.

Solution. Recall that a quadratic function has the form $f(x)=ax^2+bx+c$ where $a\neq 0$. Our goal is to find *a*, *b* and *c* so that the three given points are on the graph of f. If $(1,6)$ is on the graph, then $f(1)=6$ or $a(1)^2+b(1)+c=6 \Rightarrow a+b+c=6$. Since the point $(-1,10)$ is also on the graph of f, then $f(-1)=10$ which gives us the equation $a-b+c=10$. Lastly, the point $(2,13)$ being on the graph of f gives us $4a+2b+c=13$. We have the following system, with equations numbered.

$$\begin{cases} a+b+c=6 & (1) \\ a-b+c=10 & (2) \\ 4a+2b+c=13 & (3) \end{cases}$$

Let's make *b* the target variable to eliminate and start with equations (1) and (2).

$$\begin{array}{rl} a+b+c=6 & (1) \\ \underline{a-b+c=10} & (2) \\ 2a\quad +2c=16 & \end{array}$$

We can eliminate *b* from equations (2) and (3) by multiplying equation (2) by 2.

$$\begin{array}{rl} 2a-2b+2c=20 & \text{(2) multiplied by 2} \\ \underline{4a+2b+c=13} & (3) \\ 6a\quad +3c=33 & \end{array}$$

The resulting 2×2 system is

$$\begin{cases} 2a+2c=16 \\ 6a+3c=33 \end{cases}$$

We multiply the first equation by −3 to eliminate *a*.

$$\begin{array}{r} -6a-6c=-48 \text{ after multiplying } 2a+2c=16 \text{ by } -3 \\ \underline{6a+3c=\ \ 33} \\ -3c=-15 \end{array}$$

We find $c=5$ and substitute into the equation $6a+3c=33$ to get $a=3$. Then, substituting both $a=3$ and $c=5$ into equation (1), $a+b+c=6$, we find $b=-2$. We plug these three values into $f(x)=ax^2+bx+c$ and have the quadratic function $f(x)=3x^2-2x+5$ that passes through the points $(1,6)$, $(-1,10)$ and $(2,13)$. We leave it to the reader to check this solution.

□

6.2 Exercises

1. Can a linear system of three equations have exactly two solutions? Explain why or why not.

2. What is the geometric interpretation of a system of linear equations in three variables that is independent? How many solutions are there?

In Exercises 3 – 16, solve the system, if possible. Classify each system as consistent independent, consistent dependent, or inconsistent.

3. $\begin{cases} x+y+z=3 \\ 2x-y+z=0 \\ -3x+5y+7z=7 \end{cases}$

4. $\begin{cases} 4x-y+z=5 \\ 2y+6z=30 \\ x+z=5 \end{cases}$

5. $\begin{cases} 4x-y+z=5 \\ 2y+6z=30 \\ x+z=6 \end{cases}$

6. $\begin{cases} x+y+z=-17 \\ y-3z=0 \end{cases}$

7. $\begin{cases} x-2y+3z=7 \\ -3x+y+2z=-5 \\ 2x+2y+z=3 \end{cases}$

8. $\begin{cases} 3x-2y+z=-5 \\ x+3y-z=12 \\ x+y+2z=0 \end{cases}$

9. $\begin{cases} 2x-y+z=-1 \\ 4x+3y+5z=1 \\ 5y+3z=4 \end{cases}$

10. $\begin{cases} x-y+z=-4 \\ -3x+2y+4z=-5 \\ x-5y+2z=-18 \end{cases}$

11. $\begin{cases} 2x-4y+z=-7 \\ x-2y+2z=-2 \\ -x+4y-2z=3 \end{cases}$

12. $\begin{cases} 2x-y+z=1 \\ 2x+2y-z=1 \\ 3x+6y+4z=9 \end{cases}$

13. $\begin{cases} x-3y-4z=3 \\ 3x+4y-z=13 \\ 2x-19y-19z=2 \end{cases}$

14. $\begin{cases} x+y+z=4 \\ 2x-4y-z=-1 \\ x-y=2 \end{cases}$

15. $\begin{cases} x-y+z=8 \\ 3x+3y-9z=-6 \\ 7x-2y+5z=39 \end{cases}$

16. $\begin{cases} 2x-3y+z=-1 \\ 4x-4y+4z=-13 \\ 6x-5y+7z=-25 \end{cases}$

17. You inherit one million dollars. You invest it all in three accounts for one year. The first account pays 3% compounded annually, the second account pays 4% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $34,000 in interest. If you invest four times the money into the account that pays 3% compared to 2%, how much did you invest in each account?

18. You inherit one hundred thousand dollars. You invest it all in three accounts for one year. The first account pays 4% compounded annually, the second account pays 3% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $3,650 in interest. If you invest five times the money into the account that pays 4% compared to 3%, how much did you invest in each account?

19. Find the quadratic function passing through the points $(-1,-4)$, $(1,6)$ and $(3,0)$.

20. Find the quadratic function passing through the points $(1,-1)$, $(3,-1)$ and $(-2,14)$.

21. The top three countries in oil consumption in a certain year are as follows: the United States, Japan and China. In millions of barrels per day, the top three countries consumed 39.8% of the world's consumed oil. The United States consumed 0.7% more than four times China's consumption. The United States consumed 5% more than triple Japan's consumption. What percent of the world oil consumption did the United States, Japan and China consume?[5]

22. The top three countries in oil production in the same year are Saudi Arabia, the United States and Russia. In millions of barrels per day, the top three countries produced 31.4% of the world's produced oil. Saudi Arabia and the United States combined for 22.1% of the world's production, and Saudi Arabia produced 2% more oil than Russia. What percent of the world oil production did Saudi Arabia, the United States and Russia produce?[6]

23. The top three sources of oil imports for the United States in the same year were Saudi Arabia, Mexico and Canada. These top three countries accounted for 47% of oil imports. The United States imported 1.8% more from Saudi Arabia than they did from Mexico, and 1.7% more from Saudi Arabia than they did from Canada. What percentage of the United States oil imports were from these three countries?[7]

[5] "Oil reserves, production and consumption in 2001," accessed April 6, 2014, http://scaruffi.com/politics/oil.html.
[6] "Oil reserves, production and consumption in 2001," accessed April 6, 2014, http://scaruffi.com/politics/oil.html.
[7] "Oil reserves, production and consumption in 2001," accessed April 6, 2014, http://scaruffi.com/politics/oil.html.

24. The top three oil producers in the United States in a certain year are the Gulf of Mexico, Texas and Alaska. The three regions were responsible for 64% of the United States oil production. The Gulf of Mexico and Texas combined for 47% of oil production. Texas produced 3% more than Alaska. What percent of United States oil production came from these regions?[8]

25. At a family reunion, there were only blood relatives, consisting of children, parents and grandparents, in attendance. There were 400 people total. There were twice as many parents as grandparents, and 50 more children than parents. How many children, parents and grandparents were in attendance?

26. An animal shelter has a total of 350 animals comprised of cats, dogs and rabbits. If the number of rabbits is 5 less than one-half the number of cats, and there are 20 more cats than dogs, how many of each type of animal are at the shelter?

27. Your roommate, Sarah, offered to buy groceries for you and your other roommate. The total bill was $82. She forgot to save the individual receipts but remembered that your groceries were $0.05 cheaper than half of her groceries, and that your other roommate's groceries wee $2.10 more than your groceries. How much was each of your shares of the groceries?

28. Three co-workers have jobs as warehouse manager, office manager and truck driver. The sum of the annual salaries of the warehouse manager and office manager is $82,000. The office manager makes $4,000 more that the truck driver annually. The annual salaries of the warehouse manager and the truck driver total $78,000. What is the annual salary of each of the co-workers?

29. At a carnival, $2,941.25 in receipts was taken by the end of the day. The cost of a child's ticket was $20.50, an adult ticket was $29.75, and a senior citizen ticket was $15.25. There were twice as many senior citizens as adults in attendance, and 20 more children than senior citizens. How many children, adult and senior citizen tickets were sold?

30. A local band sells out for their concert. The sell all 1,175 tickets for a total purse of $28,112.50. The tickets were priced at $20 for students, $22.50 for children and $29 for adults. If the band sold twice as many adult as child tickets, how many of each type were sold?

31. In a bag, a child has 325 coins worth $19.50. There are three types of coins: pennies, nickels and dimes. If the bag contains the same number of nickels as dimes, how many of each type of coin is in the bag?

[8] "Oil reserves, production and consumption in 2001," accessed April 6, 2014, http://scaruffi.com/politics/oil.html.

6.3 Systems of Linear Equations: Augmented Matrices

Learning Objectives

- Write a system of linear equations as an augmented matrix.
- Perform row operations on a matrix.
- Convert an augmented matrix to row echelon form.
- Convert an augmented matrix to reduced row echelon form.
- Use matrix row operations to solve systems of linear equations.

In **Section 6.1** and **Section 6.2**, we solved systems of linear equations using the substitution method, the elimination method, and combinations of these two methods. The goal of these methods was to rewrite the original equations in a way that would allow us to determine solution values. In this section, our goal is to rewrite a system of equations in a format similar to the following.

$$\begin{cases} x-3y+2z=1 \\ \quad\; y-2z=4 \\ \qquad\quad z=-1 \end{cases}$$

Here, clearly, $z=-1$, and we substitute $z=-1$ into the second equation, $y-2(-1)=4$, to obtain $y=2$. Then we substitute $y=2$ and $z=-1$ into the first equation, $x-3(2)+2(-1)=1$, to get $x=9$. The solution is $(9,2,-1)$.

We note that the reason it was so easy to solve this system is that the third equation is solved for *z* and the second involves only *y* and *z*. Additionally, the coefficient of *y* is 1 in the second equation and the coefficient of *x* is 1 in the first equation. It will be our goal in this section to rewrite systems in this form, referred to as **upper triangular form**.[9]

Row Operations

To write a system of linear equations in triangular form, we may apply the following maneuvers to achieve an equivalent system.[10]

[9] We will sometimes refer to this simply as **triangular form**.
[10] That is, a system with the same solution set.

Theorem 6.1. Given a system of equations, the following moves will result in an equivalent system of equations.

- Interchange the position of any two equations.
- Replace an equation with a nonzero multiple of itself.
- Replace an equation with itself plus a multiple of another equation.

We have seen instances of the second and third operations in examples from the previous two sections. The first operation, while it obviousy admits an equivalent system, seems silly. Our perspective will change as we begin solving systems using this methodology. Our first example is a system we originally solved in **Section 6.2**:

$$\begin{cases} 3x - y + z = 3 \\ 2x - 4y + 3z = 16 \\ x - y + z = 5 \end{cases}$$

As we attempt to write this system in triangular form, we will mimic our moves in a matrix representation of the system. A **matrix** is simply a rectangular array of real numbers, enclosed with square brackets, '[' and ']'. To write our system as a matrix, we include only coefficients and constants. The rows represent equations and columns correspond to the coefficients of specific variables. Since each column corresponds to a specific variable (generally column 1 for *x*, column 2 for *y* and column 3 for *z*), before rewriting a system of equations in matrix format, it is a good idea to check that all equations are written in the form $ax + by + cz = d$. Noting that this is indeed the case in the above system, we write it in matrix format.

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 3 \\ 2 & -4 & 3 & 16 \\ 1 & -1 & 1 & 5 \end{array}\right]$$

The vertical line is not necessary, but its presence tells us that this is an **augmented matrix**; the column containing the constants is 'appended' to the matrix containing the coefficients.

Example 6.3.1. Put the following system of linear equations into triangular form and then solve the system, if possible.

$$\begin{cases} 3x - y + z = 3 \\ 2x - 4y + 3z = 16 \\ x - y + z = 5 \end{cases}$$

Solution. Before getting started, we note that E_1 refers to the top equation, E_2 refers to the middle equation and E_3 refers to the bottom equation. In the matrix, we use R_1 for 'row one', R_2 for 'row two' and R_3 for 'row three', or top row, middle row, bottom row, respectively. Our first step is to get an x having a coefficient of 1 in the first equation, so we begin by interchanging E_1 with E_3.

$$\begin{cases} 3x - y + z = 3 \\ 2x - 4y + 3z = 16 \\ x - y + z = 5 \end{cases} \xrightarrow{\text{Switch } E_1 \text{ and } E_3} \begin{cases} x - y + z = 5 \\ 2x - 4y + 3z = 16 \\ 3x - y + z = 3 \end{cases}$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 3 \\ 2 & -4 & 3 & 16 \\ 1 & -1 & 1 & 5 \end{array}\right] \xrightarrow{\text{Switch } R_1 \text{ and } R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 2 & -4 & 3 & 16 \\ 3 & -1 & 1 & 3 \end{array}\right]$$

Now we need to eliminate x's from the second and third equations, so we replace each with a sum of that equation and a multiple of the first equation. To eliminate x from E_2, we multiply E_1 by −2 then add; to eliminate x from E_3, we multiply E_1 by −3 then add.

$$\begin{cases} x - y + z = 5 \\ 2x - 4y + 3z = 16 \\ 3x - y + z = 3 \end{cases} \xrightarrow[\text{Replace } E_3 \text{ with } -3E_1 + E_3]{\text{Replace } E_2 \text{ with } -2E_1 + E_2} \begin{cases} x - y + z = 5 \\ -2y + z = 6 \\ 2y - 2z = -12 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 2 & -4 & 3 & 16 \\ 3 & -1 & 1 & 3 \end{array}\right] \xrightarrow[\text{Replace } R_3 \text{ with } -3R_1 + R_3]{\text{Replace } R_2 \text{ with } -2R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & -2 & 1 & 6 \\ 0 & 2 & -2 & -12 \end{array}\right]$$

Next, we get the coefficient of y in the second row to be 1. This is accomplished through multiplying the second row by $-\frac{1}{2}$.

$$\begin{cases} x - y + z = 5 \\ -2y + z = 6 \\ 2y - 2z = -12 \end{cases} \xrightarrow{\text{Replace } E_2 \text{ with } -\frac{1}{2}E_2} \begin{cases} x - y + z = 5 \\ y - \frac{1}{2}z = -3 \\ 2y - 2z = -12 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & -2 & 1 & 6 \\ 0 & 2 & -2 & -12 \end{array}\right] \xrightarrow{\text{Replace } R_2 \text{ with } -\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & -\frac{1}{2} & -3 \\ 0 & 2 & -2 & -12 \end{array}\right]$$

We move on to eliminating *y* in the third equation; we multiply the second equation by −2, then add.

$$\begin{cases} x - y + z = 5 \\ y - \frac{1}{2}z = -3 \\ 2y - 2z = -12 \end{cases} \xrightarrow{\text{Replace } E_3 \text{ with } -2E_2 + E_3} \begin{cases} x - y + z = 5 \\ y - \frac{1}{2}z = -3 \\ -z = -6 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & -\frac{1}{2} & -3 \\ 0 & 2 & -2 & -12 \end{array}\right] \xrightarrow{\text{Replace } R_3 \text{ with } -2R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & -\frac{1}{2} & -3 \\ 0 & 0 & -1 & -6 \end{array}\right]$$

Finally, we obtain a coefficient of 1 on *z* by multiplying the third equation by −1.

$$\begin{cases} x - y + z = 5 \\ y - \frac{1}{2}z = -3 \\ -z = -6 \end{cases} \xrightarrow{\text{Replace } E_3 \text{ with } -1 \cdot E_3} \begin{cases} x - y + z = 5 \\ y - \frac{1}{2}z = -3 \\ z = 6 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & -\frac{1}{2} & -3 \\ 0 & 0 & -1 & -6 \end{array}\right] \xrightarrow{\text{Replace } R_3 \text{ with } -1 \cdot R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & -\frac{1}{2} & -3 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

The system is in triangular form. We see $z = 6$, which we plug into the second equation to get $y - \frac{1}{2}(6) = -3$ so that $y = 0$. Then, plugging in values for *y* and *z*, the first equation becomes $x - 0 + 6 = 5$, or $x = -1$. According to **Theorem 6.1**, since this system in triangular form has the solution $(-1, 0, 6)$, so does the original.

□

Row Echelon Form of a Matrix

The matrix equivalent of triangular form is **row echelon form**, as demonstrated above.

Definition 6.3. A matrix is said to be in **row echelon form** provided all of the following conditions hold:

1. The first nonzero entry in each row is 1. This is referred to as a ‘leading 1’.
2. The leading 1 of a given row must be to the right of the leading 1 of the row above it.
3. Any row of all zeros cannot be placed above a row with nonzero entries.

The following matrices are in row echelon form. Note that each matrix fulfills the requirements of **Definition 6.3**. Additionally, the third matrix is in what we refer to as reduced row echelon form, a classification we will discuss later in this section.

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 4 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Below are examples of matrices that are not in row echelon form. Can you see where they fail to pass the criteria of **Definition 6.3**?

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 4 & 5 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 1 & 6 & 7 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \end{array}\right]$$

The strategy used to obtain the row-echelon form of a matrix, as demonstrated in **Example 6.3.1**, is known as Gaussian Elimination, named after the prolific German mathematician Carl Friedrich Gauss. It provides us with a systematic way to solve systems of linear equations, though not always the shortest way. To solve a system of linear equations using an augmented matrix, we encode the system into an augmented matrix and apply Gaussian Elimination to get the matrix into row-echelon form. We then decode the matrix and solve the system using back-substitution.

The connection between equations in a system and rows in a matrix allows us to move easily from the maneuvers allowed in **Theorem 6.1** to the row operations allowed in the following theorem.

Theorem 6.2. Row Operations: Given an augmented matrix for a system of linear equations, the following row operations produce an augmented matrix that corresponds to an equivalent system of linear equations.

- Interchange any two rows.
- Replace a row with a nonzero multiple of itself.[11]
- Replace a row with itself plus a multiple of another row.[12]

Example 6.3.2. Solve the system of equations, if possible, by writing the system as an augmented matrix and putting the augmented matrix in row echelon form.

$$\begin{cases} 2x+3y=6 \\ x-y=\dfrac{1}{2} \end{cases}$$

[11] That is, the row obtained by multiplying each entry in the row by the same nonzero number.
[12] Where we add entries in corresponding columns.

Solution. We begin by encoding the system into an augmented matrix.

$$\left[\begin{array}{cc|c} 2 & 3 & 6 \\ 1 & -1 & \frac{1}{2} \end{array}\right]$$

We now attempt to get the matrix into row echelon form. First of all, we get a 1 in the first row, first column, by interchanging the first and second rows.

$$\left[\begin{array}{cc|c} 2 & 3 & 6 \\ 1 & -1 & \frac{1}{2} \end{array}\right] \xrightarrow{\text{Switch } R_1 \text{ and } R_2} \left[\begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 2 & 3 & 6 \end{array}\right]$$

Next, dropping down to the second row of the first column, we obtain a 0 by multiplying the first row by −2, then adding the result to the second row.

$$\left[\begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 2 & 3 & 6 \end{array}\right] \xrightarrow{\text{Replace } R_2 \text{ with } -2R_1+R_2} \left[\begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 0 & 5 & 5 \end{array}\right]$$

To get a 1 in the second column of the second row, we multiply the second row by $\frac{1}{5}$.

$$\left[\begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 0 & 5 & 5 \end{array}\right] \xrightarrow{\text{Replace } R_2 \text{ with } \frac{1}{5}R_2} \left[\begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 0 & 1 & 1 \end{array}\right]$$

Now that our matrix is in row echelon form, we decode by writing the corresponding system of equations.

$$\begin{cases} x - y = \frac{1}{2} \\ y = 1 \end{cases}$$

We back-substitute $y = 1$ into the equation $x - y = \frac{1}{2}$ to get $x - 1 = \frac{1}{2}$, or $x = \frac{3}{2}$. The solution is $\left(\frac{3}{2}, 1\right)$.

□

Now that we've worked through a couple of examples, you may notice that we have followed the same order in getting the 1's and 0's in the required positions to achieve row echelon form. While there is more than one way to reach row echelon form, the following order of operations will prevent 'undoing' a correct placement of 1's or 0's that has already been made.

Figure 6.3. 1

$$\left[\begin{array}{cc|c} \overset{\text{1st}}{1} & \# & \# \\ \overset{\text{2nd}}{0} & \overset{\text{3rd}}{1} & \# \end{array}\right]$$

Figure 6.3. 2

$$\left[\begin{array}{ccc|c} \overset{\text{1st}}{1} & \# & \# & \# \\ \overset{\text{2nd}}{0} & \overset{\text{4th}}{1} & \# & \# \\ \overset{\text{3rd}}{0} & \overset{\text{5th}}{0} & \overset{\text{6th}}{1} & \# \end{array}\right]$$

To the left is the order for a matrix with two rows and to the right is the order for a matrix with three rows. Keep in mind that rows of zeros may appear through this process and should be moved to the bottom row.

Note that it is not necessary to get a matrix fully into row echelon form in order to solve the system of equations. At any point along the way, the matrix can be decoded and the resulting equations solved. Additionally, it is possible to place the triangle of zeros in any corner.

Example 6.3.3. Solve the system of equations, if possible, by writing the system as an augmented matrix and putting the augmented matrix in row echelon form.

$$\begin{cases} -x-2y+z=-1 \\ 2x+3y=2 \\ y-2z=0 \end{cases}$$

Solution. Encoding the system as an augmented matrix, we have

$$\left[\begin{array}{ccc|c} -1 & -2 & 1 & -1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

To obtain a leading coefficient of 1 in the first row, we multiply by -1.

$$\left[\begin{array}{ccc|c} -1 & -2 & 1 & -1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow{\text{Replace } R_1 \text{ with } -1\cdot R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

Now, we want zeros down the first column below row 1. For the second row, we multiply row one by -2 and add the result to row 2. We already have a zero in row 3.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow{\text{Replace } R_2 \text{ with } -2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

The next step requires obtaining a 1 in the second row, second column. We swap row 2 and row 3.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow{\text{Switch } R_2 \text{ and } R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

Now we need a 0 below the 1 in row 2, so we multiply row 2 by 1 and add it to row 3.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \xrightarrow{\text{Replace } R_3 \text{ with } 1\cdot R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The matrix is now in row echelon form. We decode it to arrive at the following system of equations.

$$\begin{cases} x + 2y - z = 1 \\ \quad y - 2z = 0 \\ \quad\quad 0 = 0 \end{cases}$$

The equation $0 = 0$ is always true. We have a consistent dependent system. We select t as our parameter and set $z = t$. Substituting $z = t$ in the second equation, we have

$$\begin{aligned} y - 2t &= 0 \\ y &= 2t \end{aligned}$$

Substituting $y = 2t$ and $z = t$ in the first equation, $x + 2y - z = 1$, gives us

$$\begin{aligned} x + 2(2t) - t &= 1 \\ x + 3t &= 1 \\ x &= -3t + 1 \end{aligned}$$

Our solution is the set $\{-3t+1, 2t, t\}$ for all real numbers t.

□

Reduced Row Echelon Form of a Matrix

To get a matrix into row echelon form, we used row operations to obtain 0's beneath each leading 1. If we also require that 0's are the only numbers above a leading 1, we have what is known as the **reduced row echelon form** of the matrix.

Definition 6.4. A matrix is said to be in **reduced row echelon form** provided both of the following conditions hold:

1. The matrix is in row echelon form.
2. The leading 1's are the only nonzero entry in their respective columns.

Following is a **suggested** order for using row operations to transform a matrix into reduced row echelon form. The first shows the order for any matrix with two rows and the second for a matrix with three rows.

Figure 6.3. 3

$$\left[\begin{array}{cc|c} \overset{\text{1st}}{1} & \overset{\text{4th}}{0} & \# \\ \overset{\text{2nd}}{0} & \overset{\text{3rd}}{1} & \# \end{array}\right]$$

Figure 6.3. 4

$$\left[\begin{array}{ccc|c} \overset{\text{1st}}{1} & \overset{\text{8th}}{0} & \overset{\text{9th}}{0} & \# \\ \overset{\text{2nd}}{0} & \overset{\text{4th}}{1} & \overset{\text{7th}}{0} & \# \\ \overset{\text{3rd}}{0} & \overset{\text{5th}}{0} & \overset{\text{6th}}{1} & \# \end{array}\right]$$

In **Example 6.3.3**, we applied row operations to get the matrix into the following row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

To go a step further and put the matrix in reduced row echelon form, we must have a 0 in place of the 2 in row 1. We achieve this by multiplying row 2 by −2, then adding the result to row 1.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{Replace } R_1 \text{ with } -2R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

While putting this matrix in reduced row echelon form would save a few steps in the decoding process[13], it is even more helpful when a system has an independent solution, as in the following example.

Example 6.3.4. Solve the system of equations, if possible, by writing the system as an augmented matrix and putting the augmented matrix in reduced row echelon form.

$$\begin{cases} x+y+z=4 \\ 2x-y-2z=-1 \\ x-2y-z=1 \end{cases}$$

Solution. We first write the system as an augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & -1 & -2 & -1 \\ 1 & -2 & -1 & 1 \end{array}\right]$$

Since we have a leading 1 in the first row, we proceed with obtaining zeros in rows 2 and 3 of the first column.

---

[13] Try it!

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & -1 & -2 & -1 \\ 1 & -2 & -1 & 1 \end{array}\right] \xrightarrow[\text{Replace } R_3 \text{ with } -1\cdot R_1+R_3]{\text{Replace } R_2 \text{ with } -2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -3 & -4 & -9 \\ 0 & -3 & -2 & -3 \end{array}\right]$$

Now we obtain a leading 1 in row 2 through multiplication by $-\frac{1}{3}$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -3 & -4 & -9 \\ 0 & -3 & -2 & -3 \end{array}\right] \xrightarrow{\text{Replace } R_2 \text{ with } -\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{4}{3} & 3 \\ 0 & -3 & -2 & -3 \end{array}\right]$$

Next, we get a 0 in the third row, second column.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{4}{3} & 3 \\ 0 & -3 & -2 & -3 \end{array}\right] \xrightarrow{\text{Replace } R_3 \text{ with } 3R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{4}{3} & 3 \\ 0 & 0 & 2 & 6 \end{array}\right]$$

We're almost to row echelon form; just need a leading 1 in row 3.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{4}{3} & 3 \\ 0 & 0 & 2 & 6 \end{array}\right] \xrightarrow{\text{Replace } R_3 \text{ with } \frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{4}{3} & 3 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Now that the matrix is in row echelon form, we need to make sure any columns with leading 1's have 0's elsewhere. Row 3 looks good, so we move up to row 2, and see that we need a 0 in the position currently occupied by $\frac{4}{3}$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{4}{3} & 3 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R_2 \text{ with } -\frac{4}{3}R_3+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

In row 1, we need zeros in both column 2 and column 3. To prevent 'undoing' zeros, it is important to move from left to right so we start with column 2.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R_1 \text{ with } -1\cdot R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Now we take care of column 3.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R_1 \text{ with } -1\cdot R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

With the matrix in reduced row echelon form, we decode it to get a system of equations.

$$\begin{cases} x = 2 \\ y = -1 \\ z = 3 \end{cases}$$

To our surprise and delight, when we decoded this matrix, we obtained the solution instantly without having to deal with back-substitution.

□

Before moving on to the next section, we note that it is possible to use matrices in solving systems with any number of variables. Following is an example of a system with the four variables A, B, C and D.

Example 6.3.5. Solve the system of equations, if possible, by writing the system as an augmented matrix and putting the augmented matrix in reduced row echelon form.

$$\begin{cases} A = 1 \\ B = 0 \\ 3A + C = 5 \\ 3B + D = -1 \end{cases}$$

Solution. We begin by writing the system as an augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 5 \\ 0 & 3 & 0 & 1 & -1 \end{array}\right]$$

The first entry we need to work on is the 3 in the third row, first column. We proceed with obtaining a zero in that position.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 5 \\ 0 & 3 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\text{Replace } R_3 \text{ with } -3R_1 + R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 3 & 0 & 1 & -1 \end{array}\right]$$

Next, we get a 0 in row 4, the second column.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 3 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\text{Replace } R_4 \text{ with } -3R_2+R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

Having the matrix in reduced row echelon form, we can decode it to find the solutions.

$$\begin{cases} A=1 \\ B=0 \\ C=2 \\ D=-1 \end{cases}$$

□

We will refer back to the solution in **Example 6.3.5** when we get to partial fraction decomposition in **Section 6.7**. For now, we move on to matrix arithmetic in **Section 6.4**.

6.3 Exercises

1. What are two different row operations that can be used to obtain a leading 1 in the first row of the matrix $\left[\begin{array}{cc|c} 9 & 3 & 0 \\ 1 & -2 & 6 \end{array}\right]$?

2. What does a row of 0's in an augmented matrix tell you about the system of equations it is representing?

In Exercises 3 – 6, state whether the given matrix is in reduced row echelon form, row echelon form only, or neither of these.

3. $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 3 \end{array}\right]$

4. $\left[\begin{array}{ccc|c} 3 & -1 & 1 & 3 \\ 2 & -4 & 3 & 16 \\ 1 & -1 & 1 & 5 \end{array}\right]$

5. $\left[\begin{array}{ccc|c} 1 & 1 & 4 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 \end{array}\right]$

6. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

In Exercises 7 – 10, the matrices are in reduced row echelon form. Determine the solution of the corresponding system of linear equations or state that the system in inconsistent.

7. $\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 7 \end{array}\right]$

8. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 19 \end{array}\right]$

9. $\left[\begin{array}{ccc|c} 1 & 0 & 9 & -3 \\ 0 & 1 & -4 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right]$

10. $\left[\begin{array}{ccc|c} 1 & 0 & 9 & -3 \\ 0 & 1 & -4 & 20 \\ 0 & 0 & 0 & 1 \end{array}\right]$

In Exercises 11 – 26, solve the systems of linear equations using the techniques discussed in this section.

11. $\begin{cases} -5x + y = 17 \\ x + y = 5 \end{cases}$

12. $\begin{cases} 2x - 3y = -9 \\ 5x + 4y = 58 \end{cases}$

13. $\begin{cases} 2x + 3y = 12 \\ 4x + y = 14 \end{cases}$

14. $\begin{cases} 3x + 4y = 12 \\ -6x - 8y = -24 \end{cases}$

15. $\begin{cases} x+y+z=3 \\ 2x-y+z=0 \\ -3x+5y+7z=7 \end{cases}$

16. $\begin{cases} 4x-y+z=5 \\ 2x+6z=30 \\ x+z=5 \end{cases}$

17. $\begin{cases} x-2y+3z=7 \\ -3x+y+2z=-5 \\ 2x+2y+z=3 \end{cases}$

18. $\begin{cases} 3x-2y+z=-5 \\ x+3y-z=12 \\ x+y+2z=0 \end{cases}$

19. $\begin{cases} 2x-y+z=-1 \\ 4x+3y+5z=1 \\ 5y+3z=4 \end{cases}$

20. $\begin{cases} x-y+z=-4 \\ -3x+2y+4z=-5 \\ x-5y+2z=-18 \end{cases}$

21. $\begin{cases} 2x-4y+z=-7 \\ x-2y+2z=-2 \\ -x+4y-2z=3 \end{cases}$

22. $\begin{cases} 2x-y+z=1 \\ 2x+2y-z=1 \\ 3x+6y+4z=9 \end{cases}$

23. $\begin{cases} x-3y-4z=3 \\ 3x+4y-z=13 \\ 2x-19y-19z=2 \end{cases}$

24. $\begin{cases} x+y+z=4 \\ 2x-4y-z=-1 \\ x-y=2 \end{cases}$

25. $\begin{cases} x-y+z=8 \\ 2x+3y-9z=-6 \\ 7x-2y+5z=39 \end{cases}$

26. $\begin{cases} 2x-3y+z=-1 \\ 4x-4y+4z=-13 \\ 6x-5y+7z=-25 \end{cases}$

27. At the local buffet, 22 diners (5 of whom were children) feasted for \$162.25, before taxes. If the kids buffet is \$4.50, the basic buffet is \$7.50, and the deluxe buffet (with crab legs) is \$9.25, how many diners chose the deluxe buffet?

28. Robert wants to make a party mix consisting of almonds (which cost \$7 per pound), cashews (which cost \$5 per pound) and peanuts (which cost \$2 per pound). If he wants to make a 10 pound mix with a budget of \$35, what are the possible combinations of almonds, cashews and peanuts?

29. Find the quadratic function passing through the points $(-2,1)$, $(1,4)$ and $(3,-2)$.

30. Find two different row echelon forms for the matrix $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 12 & 8 \end{array}\right]$.

6.4 Matrix Arithmetic

Learning Objectives

- Find the sum and difference of two matrices.
- Find the scalar multiple of a matrix.
- Find the product of two matrices.

In **Section 6.3**, we used a special class of matrices, the augmented matrices, to assist us in solving systems of linear equations. In this section, we study matrices as mathematical objects of their own accord, temporarily divorced from the systems of linear equations. Recall that a **matrix** is a rectangular array of real numbers, such as the following.

$$\begin{bmatrix} 3 & 0 & -1 \\ 2 & -5 & 10 \end{bmatrix}$$

The **size**, sometimes called the **dimension**, of the matrix shown above is 2×3, read as 'two by three', because it has 2 rows and 3 columns. In general, we say a matrix with *m* rows and *n* columns is of size 'm by n', written as $m\times n$. Before moving on to properties and operations, we note that the individual numbers in a matrix are called its **entries** and that matrices are usually denoted by uppercase letters (A, B, C, etc.). We begin with a definition of what it means for two matrices to be equal.

Definition 6.5. Matrix Equality: Two matrices are said to be **equal** if they are the same size and their corresponding entries are equal.

Essentially, two matrices are equal if they are the same size and they have the same numbers in the same spots. For example, the matrices below are shown as being equal.

$$\begin{bmatrix} x+2 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 6 & y+1 \end{bmatrix}$$

This tells us that we must have $x+2=3$ and $y+1=7$, so $x=1$ and $y=6$. Now that we have an understanding of what it means for two matrices to equal each other, we may begin defining arithmetic operations on matrices. Our first operation is addition.

Matrix Addition

Definition 6.6. Matrix Addition: Given two matrices of the same size, the matrix obtained by adding the corresponding entries of the two matrices is called the **sum** of the two matrices.

Example 6.4.1. Find the sum, $A+B$, for matrices A and B, shown below.

$$A=\begin{bmatrix}2 & 3\\ 4 & -1\\ 0 & -7\end{bmatrix},\ B=\begin{bmatrix}-1 & 4\\ -5 & -3\\ 8 & 1\end{bmatrix}$$

Solution.

$$A+B=\begin{bmatrix}2 & 3\\ 4 & -1\\ 0 & -7\end{bmatrix}+\begin{bmatrix}-1 & 4\\ -5 & -3\\ 8 & 1\end{bmatrix}=\begin{bmatrix}2+(-1) & 3+4\\ 4+(-5) & (-1)+(-3)\\ 0+8 & (-7)+1\end{bmatrix}=\begin{bmatrix}1 & 7\\ -1 & -4\\ 8 & -6\end{bmatrix}$$

□

It is worth the reader's time to think what would have happened had we reversed the order of the summands above. As we would expect, we arrive at the same answer. In general, $A+B=B+A$ for matrices A and B. This is the **commutative property** of matrix addition. Since addition of matrices is done entry by entry, and each entry is a real number, the commutative property of real number addition carries over to matrices.

The **associative property** of matrix addition also holds, inherited in turn from the associative law of real number addition. Specifically, for matrices A, B and C of the same size, $(A+B)+C=A+(B+C)$. In other words, when adding more than two matrices, it doesn't matter how they are grouped. This means we can write $A+B+C$ without parentheses and there is no ambiguity as to what this means.

These properties and more are summarized in the following theorem. The matrix referred to as $\mathbf{0}$ is the **zero matrix**, which is a matrix whose entries are all 0. The zero matrix may be any size. For example, $\mathbf{0}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ is the 2×3 zero matrix.

Theorem 6.3. Properties of Matrix Addition

- **Commutative Property:** For all $m\times n$ matrices, $A+B=B+A$
- **Associative Property:** For all $m\times n$ matrices, $(A+B)+C=A+(B+C)$
- **Identity Property:** For all $m\times n$ matrices, $A+\mathbf{0}=\mathbf{0}+A=A$
- **Inverse Property:** Every $m\times n$ matrix A has a unique **additive inverse**, denoted $-A$, such that $A+(-A)=(-A)+A=\mathbf{0}$

The identity property is easily verified by resorting to the definition of matrix addition; just as the number 0 is the additive identity for real numbers, the matrix comprised of all 0's does the same job for matrices.

To establish the inverse property, given an $m \times n$ matrix A, we are looking for a matrix $-A$, of the same size, such that all corresponding entries in A and $-A$ add to 0. Thus, to get the additive inverse of the matrix A, we must use the additive inverse of each of its entries to form the matrix we refer to as $-A$.

So, if $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & -5 & 10 \end{bmatrix}$, then $-A = \begin{bmatrix} -3 & 0 & 1 \\ -2 & 5 & -10 \end{bmatrix}$ since $\begin{bmatrix} 3 & 0 & -1 \\ 2 & -5 & 10 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 1 \\ -2 & 5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Matrix Subtraction

With the concept of additive inverse well in hand, we may now discuss what is meant by subtracting matrices. You may remember from arithmetic that $a - b = a + (-b)$; that is, subtraction is defined as 'adding the opposite (inverse)'. We extend this concept to matrices. For two matrices A and B of the same size, we define $A - B = A + (-B)$.

Example 6.4.2. Find $A - B$ for $A = \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & -1 \\ 5 & 4 \end{bmatrix}$.

Solution.

$$\begin{aligned} A - B &= A + (-B) \\ &= \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -8 & 1 \\ -5 & -4 \end{bmatrix} \\ &= \begin{bmatrix} (-2)+(-8) & 3+1 \\ 0+(-5) & 1+(-4) \end{bmatrix} \\ &= \begin{bmatrix} -10 & 4 \\ -5 & -3 \end{bmatrix} \end{aligned}$$

□

Scalar Multiplication

Our next task is to define what it means to multiply a matrix by a real number. Thinking back to arithmetic, you may recall that multiplication, at least by a natural number, can be thought of as 'rapid addition'. For example, $2+2+2 = 3 \times 2$. We know from algebra[14] that $3x = x + x + x$, so it seems natural that given a matrix A, we define $3A = A + A + A$. This result is achieved by multiplying each entry by 3, leading to the following definition.

[14] The Distributive Property, in particular.

Definition 6.7. Scalar[15] Multiplication: We define the product of a real number k and a matrix A, denoted as kA, to be the matrix obtained by multiplying each entry of A by k.

As did matrix addition, scalar multiplication inherits some properties from real number arithmetic. We leave the discovery of these properties to the reader, with the exception of the additive inverse property.

Theorem 6.4. Additive Inverse Property: For all $m \times n$ matrices A, $-A = (-1)A$.

This property is easily verified, and adds clarity to our earlier attempt at finding inverse matrices since the $-A$ from **Theorem 6.4** is indeed the additive inverse of the matrix A.

Example 6.4.3. For matrices A and B, given below, find $2A-3B$.

$$A = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix}$$

Solution.

$$\begin{aligned} 2A-3B &= 2A + (-3)B \\ &= 2\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 4 \end{bmatrix} + (-3)\begin{bmatrix} 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2(-1) & 2\cdot 0 & 2\cdot 3 \\ 2\cdot 2 & 2(-1) & 2\cdot 4 \end{bmatrix} + \begin{bmatrix} (-3)\cdot 2 & (-3)\cdot 1 & (-3)(-3) \\ (-3)\cdot 4 & (-3)\cdot 0 & (-3)\cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 0 & 6 \\ 4 & -2 & 8 \end{bmatrix} + \begin{bmatrix} -6 & -3 & 9 \\ -12 & 0 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -2+(-6) & 0+(-3) & 6+9 \\ 4+(-12) & -2+0 & 8+(-3) \end{bmatrix} \\ &= \begin{bmatrix} -8 & -3 & 15 \\ -8 & -2 & 5 \end{bmatrix} \end{aligned}$$

□

Matrix Multiplication

We now turn our attention to **matrix multiplication** – that is, multiplying a matrix by another matrix. We begin by demonstrating the procedure for finding the product of a row and a column. Consider the two matrices A and B below.

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 5 \\ -1 & 2 & 4 \end{bmatrix}$$

[15] The word 'scalar' here refers to real numbers. 'Scalar multiplication' in this context means we are multiplying a matrix by a real number (a scalar).

Let R_1 denote the first row of A and C_1 denote the first column of B. To find the 'product' of R_1 with C_1, denoted R_1C_1, we find the product of the first entry in R_1 and the first entry in C_1. Next, we add to that the product of the second entry in R_1 and the second entry in C_1. We can visualize this as follows.

$$R_1C_1 = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = (1)(3) + (2)(-1) = 1$$

To find R_2C_3 we proceed similarly.

$$R_2C_3 = \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = (-2)(5) + (3)(4) = 2$$

Note that to multiply a row by a column, the number of entries in the row must match the number of entries in the column. We are now in the position to define matrix multiplication.

Definition 6.8. Matrix Multiplication: Suppose A is an $m \times p$ matrix and B is a $p \times n$ matrix. Let R_i denote the i^{th} row of A and C_j denote the j^{th} column of B. The **product of *A* and *B***, denoted AB, is the matrix defined by

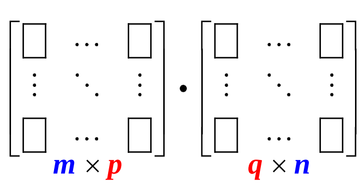
$$AB = \begin{bmatrix} R_1C_1 & R_1C_2 & \cdots & R_1C_n \\ R_2C_1 & R_2C_2 & \cdots & R_2C_n \\ \vdots & \vdots & & \vdots \\ R_mC_1 & R_mC_2 & \cdots & R_mC_n \end{bmatrix}$$

There are a number of subtleties in **Definition 6.8** which warrant closer inspection.

- To find the product of a row in A and a column in B, the number of entries in the rows of A must match the number of entries in the columns of B. This means the number of columns of A must match[16] the number of rows of B.
- When multiplying a matrix A of size $m \times \boldsymbol{p}$ with a matrix B of size $\boldsymbol{p} \times n$, the resulting matrix AB will have size $m \times n$.

It may help to remember these results with the following.

Figure 6.4. 1



Must have $p = q$

Dimension of final product: $m \times n$

[16] The reader is encouraged to think this through carefully.

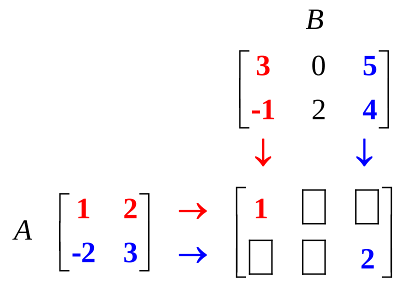
Returning to our example matrices, shown again below, we see that A is a $2\times\mathbf{2}$ matrix and B is a $\mathbf{2}\times 3$ matrix. This means that the product matrix AB is defined and will be a 2×3 matrix.

$$A=\begin{bmatrix}1 & 2\\ -2 & 3\end{bmatrix},\quad B=\begin{bmatrix}3 & 0 & 5\\ -1 & 2 & 4\end{bmatrix}$$

Using R_i to denote the i^{th} row of A and C_j to denote the j^{th} column of B, we form AB according to **Definition 6.8**.

$$AB=\begin{bmatrix}R_1C_1 & R_1C_2 & R_1C_3\\ R_2C_1 & R_2C_2 & R_2C_3\end{bmatrix}$$

We have already determined that $R_1C_1=1$ and $R_2C_3=2$, as shown in the following diagram.



We compute the remaining entries:

$$\begin{aligned}R_1C_2&=(1)(0)+(2)(2)=4\\ R_1C_3&=(1)(5)+(2)(4)=13\\ R_2C_1&=(-2)(3)+(3)(-1)=-9\\ R_2C_2&=(-2)(0)+(3)(2)=6\end{aligned}$$

Thus, $AB=\begin{bmatrix}1 & 4 & 13\\ -9 & 6 & 2\end{bmatrix}$.

Note that the product BA is not defined, since B is a $2\times\mathbf{3}$ matrix while A is a $\mathbf{2}\times 2$ matrix; B has more columns than A has rows, and so it is not possible to multiply a row of B by a column of A.

Even when the dimensions of A and B are compatible such that AB and BA are both defined, the products AB and BA aren't necessarily equal.[17] In other words, AB may not equal BA. For example, if

$A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}2 & 0\\ 3 & -1\end{bmatrix}$, we have

[17] They may not even have the same dimensions. For example, if A is a 2×3 matrix and B is a 3×2 matrix, then AB is defined and is a 2×2 matrix while BA is also defined, but is a 3×3 matrix!

$$AB = \begin{bmatrix} 8 & -2 \\ 18 & -4 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}$$

Although there is no commutative property of matrix multiplication in general, several other real number properties are inherited by matrix multiplication, as stated in our next theorem.

Theorem 6.5. Properties of Matrix Multiplication Let A, B and C be matrices such that all of the matrix products below are defined and let r be a real number.

- **Associative Property of Matrix Multiplication:** $(AB)C = A(BC)$
- **Associative Property with Scalar Multiplication:** $r(AB) = (rA)B = A(rB)$
- **Identity Property:** For a natural number k, the $\boldsymbol{k \times k}$ **identity matrix**, denoted I_k, is a square matrix containing 1's down the main diagonal and 0's elsewhere. For a $m \times n$ matrix A, $I_m A = AI_n = A$.
- **Distributive Property of Matrix Multiplication over Matrix Addition:**

$$A(B \pm C) = AB \pm AC \text{ and } (A \pm B)C = AC \pm BC$$

The one property in **Theorem 6.5** that begs further discussion is the identity property. The **main diagonal** that is mentioned refers to positions in the matrix in which the number of the row and the number of the column match. As stated, all entries along the main diagonal are 1's while the matrix contains 0's in all other positions. A few examples of identity matrices are presented below.

$$\underset{I_1}{[1]} \qquad \underset{I_2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \qquad \underset{I_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \qquad \underset{I_4}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

The identity matrix being a **square matrix** means that it has the same number of rows as columns. Note that in order to verify that the identity matrix acts as a multiplicative identity, some care must be taken depending on the order of the multiplication. For example, take the 2×3 matrix A from earlier.

$$A = \begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix}$$

In order for the product $I_k A$ to be defined, $k = 2$; for AI_k to be defined, $k = 3$. We leave it to the reader to verify that $I_2 A = A$ and $AI_3 = A$. In other words,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix}$$

While the proofs of the properties in **Theorem 6.5** are computational in nature, the notation becomes quite involved very quickly, so they are left to a course in Linear Algebra. The following examples provide some practice with matrix multiplication and its properties.

Example 6.4.4. Matrices A, B, C and D are defined as follows:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix},\ B = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 \\ -2 & 3 \\ 1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & -1 \\ -2 & 3 \end{bmatrix}$$

Perform the indicated arithmetic operations, if defined, for the given matrices.

(a) $A+D$ (b) $A+B$ (c) BC (d) CB (e) BD (f) DB

Solution.

(a) $A+D$

$$\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1+0 & 2+(-1) \\ 0+(-2) & -3+3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}$$

(b) $A+B$ is not defined since A is size 2×2 and B is size 2×3. It is only possible to add matrices of the same size.

(c) BC

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & 3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (1)(0)+(-1)(-2)+(2)(1) & (1)(1)+(-1)(3)+(2)(0) \\ (0)(0)+(2)(-2)+(1)(1) & (0)(1)+(2)(3)+(1)(0) \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -2 \\ -3 & 6 \end{bmatrix}$$

(d) CB

$$\begin{bmatrix} 0 & 1 \\ -2 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} (0)(1)+(1)(0) & (0)(-1)+(1)(2) & (0)(2)+(1)(1) \\ (-2)(1)+(3)(0) & (-2)(-1)+(3)(2) & (-2)(2)+(3)(1) \\ (1)(1)+(0)(0) & (1)(-1)+(0)(2) & (1)(2)+(0)(1) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 2 & 1 \\ -2 & 8 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

(e) BD is not defined since B is size 2×3 and D is size 2×2; the number of columns in B do not match the number of rows in D.

(f) DB

$$\begin{bmatrix} 0 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} (0)(1)+(-1)(0) & (0)(-1)+(-1)(2) & (0)(2)+(-1)(1) \\ (-2)(1)+(3)(0) & (-2)(-1)+(3)(2) & (-2)(2)+(3)(1) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -2 & -1 \\ -2 & 8 & -1 \end{bmatrix}$$

□

The next example introduces us to polynomials involving matrices.

Example 6.4.5. Find $C^2 - 5C + 10I_2$ for $C = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$.

Solution. Just as x^2 means x times itself, C^2 denotes the matrix C times itself. We get

$$C^2 - 5C + 10I_2 = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}^2 - 5\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} + 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -5 & 10 \\ -15 & -20 \end{bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$
$$= \begin{bmatrix} -5 & -10 \\ 15 & 10 \end{bmatrix} + \begin{bmatrix} 5 & 10 \\ -15 & -10 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

□

When we started this section, we mentioned that we would temporarily consider matrices as their own entities. Now, the algebra we have developed here will ultimately allow us to solve systems of linear equations. To that end, consider the system

$$\begin{cases} 2x + 3y + z = 4 \\ 3x + 3y + z = 2 \\ 2x + 4y + z = 5 \end{cases}$$

We may encode this system into the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 4 \\ 3 & 3 & 1 & 2 \\ 2 & 4 & 1 & 5 \end{array}\right]$$

Recall that the entries to the left of the vertical line come from the coefficients of the variables in the system, while those on the right comprise the associated constants. For that reason, we may form the **coefficient matrix** A, the **variable matrix** X and the **constant matrix** B as shown below.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}, \ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

We now consider the matrix equation $AX = B$.

$$AX = B$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2x+3y+z \\ 3x+3y+z \\ 2x+4y+z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

We see that finding a solution (x, y, z) to the original system corresponds to finding a solution X for the matrix equation $AX = B$. If we think about solving the real number equation $ax = b$, we would simply 'divide' both sides by *a*. Is it possible to 'divide' both sides of the matrix equation $AX = B$ by the matrix A? This is the central topic of **Section 6.5**.

6.4 Exercises

1. Can we add any two matrices together? If so, explain why; if not, explain why not and give an example of two matrices that cannot be added together.

2. Can any two matrices of the same size be multiplied? If so, explain why; if not, explain why not and give an example of two matrices of the same size that cannot be multiplied together.

For each pair of matrices A and B In Exercises 3 – 9, find the following, if defined.

(a) $3A$ (b) $-B$ (c) A^2

(d) $A-2B$ (e) AB (f) BA

3. $A=\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 5 & -2 \\ 4 & 8 \end{bmatrix}$

4. $A=\begin{bmatrix} -1 & 5 \\ -3 & 6 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 10 \\ -7 & 1 \end{bmatrix}$

5. $A=\begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 7 & 0 & 8 \\ -3 & 1 & 4 \end{bmatrix}$

6. $A=\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$, $B=\begin{bmatrix} -1 & 3 & -5 \\ 7 & -9 & 11 \end{bmatrix}$

7. $A=\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

8. $A=\begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix}$, $B=\begin{bmatrix} -5 & 1 & 8 \end{bmatrix}$

9. $A=\begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -2 \\ -7 & 1 & -1 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 2 & 1 \\ 17 & 33 & 19 \\ 10 & 19 & 11 \end{bmatrix}$

In Exercises 10 – 23, use the following matrices to compute the indicated operation or state that the indicated operation is undefined.

$$A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B=\begin{bmatrix} 0 & -3 \\ -5 & 2 \end{bmatrix} \quad C=\begin{bmatrix} 10 & -\frac{11}{2} & 0 \\ \frac{3}{5} & 5 & 9 \end{bmatrix} \quad D=\begin{bmatrix} 7 & -13 \\ -\frac{4}{3} & 0 \\ 6 & 8 \end{bmatrix} \quad E=\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & -9 \\ 0 & 0 & -5 \end{bmatrix}$$

10. $7B-4A$
11. AB
12. BA
13. $E+D$
14. ED
15. $CD+2I_2A$
16. $A-4I_2$
17. A^2-B^2
18. $(A+B)(A-B)$
19. $A^2-5A-2I_2$
20. $E^2+5E-36I_3$
21. EDC

22. CDE 23. $ABCEDI_2$

24. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ and $E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$. Compute E_1A, E_2A and E_3A. What effect did each of the E_i matrices have on the rows of A? Create E_4 so that its effect on A is to multiply the bottom row by −6. How would you extend this idea to matrices with more than two rows?

25. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -3 \\ -5 & 2 \end{bmatrix}$. Compare $(A+B)^2$ to $A^2 + 2AB + B^2$. Discuss with your classmates what constraints must be placed on two arbitrary matrices A and B so that both $(A+B)^2$ and $A^2 + 2AB + B^2$ exist. When will $(A+B)^2 = A^2 + 2AB + B^2$? In general, what is the correct formula for $(A+B)^2$?

In Exercises 26 – 30, consider the following definitions. A square matrix is said to be an **upper triangular matrix** if all of its entries below the main diagonal are zero and it is said to be a **lower triangular matrix** if all of its entries above the main diagonal are zero. For example,

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & -9 \\ 0 & 0 & -5 \end{bmatrix}$$

is an upper triangular matrix whereas

$$F = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

is a lower triangular matrix. (Zeros are allowed on the main diagonal.) Discuss the following questions with your classmates.

26. Give an example of a matrix which is neither upper triangular nor lower triangular.

27. Is the product of two $n \times n$ upper triangular marices always upper triangular?

28. Is the product of two $n \times n$ lower triangular matrices always lower triangular?

29. Given the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, write A as $L \cdot U$ where L is a lower triangular matrix and U is an upper triangular matrix.

30. Are there any matrices which are simultaneously upper and lower triangular?

6.5 Systems of Linear Equations: Matrix Inverses

Learning Objectives

- Find the inverse of a 2×2 or a 3×3 matrix.
- Solve a system of linear equations using an inverse matrix.

We concluded **Section 6.4** by showing how we can rewrite a system of linear equations as the matrix equation $AX = B$, where A and B are known matrices and the variable matrix X of the matrix equation corresponds to the solution of the system. In this section, we develop the method for solving such an equation. To that end, consider the system

$$\begin{cases} 2x - 3y = 16 \\ 3x + 4y = 7 \end{cases}$$

To write this system as a matrix equation, we follow the procedure outlined at the end of **Section 6.4**. We find the coefficient matrix A, the variable matrix X and the constant matrix B to be

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}, \ X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 16 \\ 7 \end{bmatrix}$$

In order to motivate how we solve a matrix equation like $AX = B$, we revisit a similar equation, $ax = b$, where a and b are real numbers. In particular, consider the equation $3x = 5$. To solve this equation, we simply divide both sides by 3, which is the equivalent of multiplying by $\frac{1}{3}$.

$$\begin{aligned} 3x &= 5 \\ \frac{1}{3} \cdot 3x &= \frac{1}{3} \cdot 5 \\ 1 \cdot x &= \frac{1}{3} \cdot 5 \\ x &= \frac{1}{3} \cdot 5 \end{aligned}$$

Here, $\frac{1}{3}$ is the multiplicative inverse of 3 and 1 is the multiplicative identity. In general, for $ax = b$, $x = \frac{1}{a} \cdot b$ or, if we think of $\frac{1}{a}$ as the multiplicative inverse for the real nonzero number a, we can say $a^{-1} = \frac{1}{a}$ so that we have $x = a^{-1}b$. By paying attention to the multiplicative properties of matrices, we can define an analogous process for solving a matrix equation of the form $AX = B$. We refer to the multiplicative inverse of a matrix A as A^{-1}, and to the multiplicative identity of a matrix as I_n.[18]

$$\begin{aligned} AX &= B \\ A^{-1}(AX) &= A^{-1}B && \text{multiply by } A^{-1} \text{ on same side; multiplication not commutative!} \\ (A^{-1}A)X &= A^{-1}B && \text{Associative Property} \\ I_nX &= A^{-1}B && \text{value of } n \text{ depends on size of } A \\ X &= A^{-1}B && \text{Identity Property} \end{aligned}$$

The matrix A^{-1}, read as 'A-inverse', is the multiplicative inverse for matrices, just as $\frac{1}{a}$ is the multiplicative inverse for real numbers. We have no guarantees that A^{-1} actually exists, and if it does, what it looks like. We will return to this shortly. The identity matrix, I_n, was defined in **Section 6.4** to be a square matrix with 1's down the main diagonal and 0's elsewhere. Like $(a)\left(\frac{1}{a}\right) = \left(\frac{1}{a}\right)(a) = 1$ for real numbers, we have the relationship $AA^{-1} = A^{-1}A = I_n$. We note that, for AA^{-1} to be equivalent to $A^{-1}A$, A and A^{-1} must be square matrices of the same size.[19]

The Inverse of a Matrix

We begin with a formal definition of an invertible matrix.

> **Definition 6.9.** Two $n \times n$ matrices A and B are called **inverses** if $AB = BA = I_n$. If B is the inverse of A, we denote B as A^{-1}. If A has an inverse, we refer to A as **invertible**.

Since not all matrices are square, not all matrices are invertible. However, just because a matrix is square doesn't guarantee it is invertible. Following is an example of two matrices that are inverses.

Example 6.5.1. Verify that $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$ are inverses.

Solution. We must show that $AB = I_2$ and $BA = I_2$.

---

[18] Don't be confused by the switch from 'k' in **Section 6.4** to 'n' here. We think of 'n' as being the more universal notation and no longer have the issues that required our use of 'k' in **Theorem 6.5**.

[19] Think about this, referring back to **Section 6.4** if necessary.

$$AB=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}-2 & 1\\ \frac{3}{2} & -\frac{1}{2}\end{bmatrix}$$
$$=\begin{bmatrix}(1)(-2)+(2)\left(\frac{3}{2}\right) & (1)(1)+(2)\left(-\frac{1}{2}\right)\\ (3)(-2)+(4)\left(\frac{3}{2}\right) & (3)(1)+(4)\left(-\frac{1}{2}\right)\end{bmatrix}$$
$$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$BA=\begin{bmatrix}-2 & 1\\ \frac{3}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$$
$$=\begin{bmatrix}(-2)(1)+(1)(3) & (-2)(2)+(1)(4)\\ \left(\frac{3}{2}\right)(1)+\left(-\frac{1}{2}\right)(3) & \left(\frac{3}{2}\right)(2)+\left(-\frac{1}{2}\right)(4)\end{bmatrix}$$
$$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Since $AB=I_2$ and $BA=I_2$, A and B are inverses.

□

In fact, since a matrix has a unique inverse, we can say $A^{-1}=B$ and $B^{-1}=A$. We return to the matrix $A=\begin{bmatrix}2 & -3\\ 3 & 4\end{bmatrix}$ from the beginning of this section and work at finding its inverse. We know that the inverse will also be size 2×2, so let's use the variables *f*, *g*, *h* and *j* to stand in for the missing numbers:

$$A^{-1}=\begin{bmatrix}f & g\\ h & j\end{bmatrix}$$

Now, since $AA^{-1}=I_2$, we have

$$\begin{bmatrix}2 & -3\\ 3 & 4\end{bmatrix}\begin{bmatrix}f & g\\ h & j\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
$$\begin{bmatrix}2f-3h & 2g-3j\\ 3f+4h & 3g+4j\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

This gives rise to two systems of linear equations.

$$\begin{cases}2f-3h=1\\ 3f+4h=0\end{cases} \qquad \begin{cases}2g-3j=0\\ 3g+4j=1\end{cases}$$

We encode each system into an augmented matrix.

$$\left[\begin{array}{cc|c}2 & -3 & 1\\ 3 & 4 & 0\end{array}\right] \qquad \left[\begin{array}{cc|c}2 & -3 & 0\\ 3 & 4 & 1\end{array}\right]$$

At this point, we could use Gaussian Elimination to solve each system. Instead, we notice that the left sides of the augmented matrices are equal. This allows us to combine the two augmented matrices into a single matrix, as shown below, and hence solve both systems at the same time.

$$\left[\begin{array}{cc|cc}2 & -3 & 1 & 0\\ 3 & 4 & 0 & 1\end{array}\right]$$

We use row operations to put our augmented matrix into reduced row echelon form.

Replace R_1 with $\frac{1}{2}R_1 \rightarrow$ $\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & \frac{1}{2} & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$

Replace R_2 with $-3R_1 + R_2 \rightarrow$ $\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & \frac{17}{2} & -\frac{3}{2} & 1 \end{array}\right]$

Replace R_2 with $\frac{2}{17}R_2 \rightarrow$ $\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{17} & \frac{2}{17} \end{array}\right]$

Replace R_1 with $\frac{3}{2}R_2 + R_1 \rightarrow$ $\left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{17} & \frac{3}{17} \\ 0 & 1 & -\frac{3}{17} & \frac{2}{17} \end{array}\right]$

Separating out the solutions of the original systems, we have the two matrices.

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{4}{17} \\ 0 & 1 & -\frac{3}{17} \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{17} \\ 0 & 1 & \frac{2}{17} \end{array}\right]$$

From the first, we get $f = \frac{4}{17}$ and $h = -\frac{3}{17}$. The second matrix tells us that $g = \frac{3}{17}$ and $j = \frac{2}{17}$. The resulting inverse matrix is

$$A^{-1} = \begin{bmatrix} f & g \\ h & j \end{bmatrix} = \begin{bmatrix} \frac{4}{17} & \frac{3}{17} \\ -\frac{3}{17} & \frac{2}{17} \end{bmatrix}$$

We leave it to the reader to check that $AA^{-1} = A^{-1}A = I_2$. Now, returning to our discussion at the beginning of this section, we know

$$X = A^{-1}B = \begin{bmatrix} \frac{4}{17} & \frac{3}{17} \\ -\frac{3}{17} & \frac{2}{17} \end{bmatrix} \begin{bmatrix} 16 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

Since $X = \begin{bmatrix} x \\ y \end{bmatrix}$, our final solution to the system is $(x, y) = (5, -2)$.

We have the following strategy for finding inverses of square matrices, noting that some square matrices do not have inverses. Matrices without inverses are called **singular**, while matrices with inverses are referred to as **nonsingular**.

Finding the Inverse of an $n \times n$ Matrix A

1. Form the augmented matrix $[A \mid I_n]$.
2. Use row operations to transform the matrix to reduced row echelon form.
3. If the identity matrix does not appear to the left of the vertical bar, the matrix does not have an inverse. Otherwise, the matrix is invertible and the inverse matrix will appear to the right of the vertical bar: $[I_n \mid A^{-1}]$.

Example 6.5.2. Given the 2×2 matrix A, find the inverse, if it exists.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$$

Solution. We form the matrix $[A \mid I_2]$ and proceed with row operations to transform the matrix to reduced row echelon form.

$$\text{Matrix } [A \mid I_2] \rightarrow \quad \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right]$$

$$\text{Replace } R_2 \text{ with } -2R_1 + R_2 \rightarrow \quad \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$\text{Replace } R_1 \text{ with } 2R_2 + R_1 \rightarrow \quad \left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

Noting that the identity matrix appears to the left of the vertical bar, we have $A^{-1} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$.

□

Example 6.5.3. Given the 2×2 matrix A, find the inverse, if it exists.

$$A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$$

Solution. Beginning with the matrix $[A \mid I_2]$, we proceed with row operations, in an attempt to rewrite the matrix in reduced row echelon form.

$$\text{Matrix } [A \mid I_2] \rightarrow \quad \left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$$

$$\text{Switch } R_1 \text{ and } R_2 \rightarrow \quad \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 3 & 6 & 1 & 0 \end{array}\right]$$

$$\text{Replace } R_2 \text{ with } -3R_1 + R_2 \rightarrow \quad \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

The matrix is in reduced row echelon form, but the identity matrix does not appear to the left of the vertical bar. Thus, the inverse matrix does not exist.

□

Before trying a 3×3 matrix, we develop a formula that can be used in finding inverses of 2×2 matrices. Assume we begin with the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then follow the procedure, as above, in determining the inverse.

$$\text{Matrix } [A \mid I_2] \rightarrow \quad \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\text{Replace}^{20}\ R_1 \text{ with } \frac{1}{a}R_1 \rightarrow \quad \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\text{Replace}^{21}\ R_2 \text{ with } -cR_1 + R_2 \rightarrow \quad \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

$$\text{Replace}^{22}\ R_2 \text{ with } \frac{a}{ad-bc} \cdot R_2 \rightarrow \quad \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\text{Replace } R_1 \text{ with } -\frac{b}{a} \cdot R_2 + R_1 \rightarrow \quad \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

We now have a formula for A^{-1}:

---

[20] If $a \neq 0$.
[21] If $ad - bc = 0$, we will not get the identity matrix on the left side.
[22] If $ad - bc \neq 0$.

$$A^{-1}=\begin{bmatrix}\dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc}\\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc}\end{bmatrix}$$ or, more simply, $A^{-1}=\dfrac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$ if $ad-bc\neq 0$.

Before moving on, let's understand the meaning of $ad-bc\neq 0$ from a geometric perspective. As we saw earlier, if we can find A^{-1} then the system $\begin{cases}ax+by=m\\ cx+dy=n\end{cases}$ has the unique solution $\begin{bmatrix}x\\ y\end{bmatrix}=A^{-1}\begin{bmatrix}m\\ n\end{bmatrix}$. Solving this system is equivalent to finding the point(s) of intersection of two lines. The slope of the line $ax+by=m$ is $-\dfrac{a}{b}$ since $y=-\dfrac{a}{b}x+\dfrac{m}{a}$. Similarly, the slope of the line $cx+dy=n$ is $-\dfrac{c}{d}$. These two lines have a unique point of intersection if and only if their slopes are not equal: $-\dfrac{a}{b}\neq-\dfrac{c}{d}$. So, A^{-1} exists if and only if the two lines have a unique point of intersection, or if $ad-bc\neq 0$.

We will revisit the formula for A^{-1} in our next section as it ties in nicely with determinants. For now, let's look at an example of finding the inverse of a 3×3 matrix before moving on to solving systems of equations with inverse matrices.

Example 6.5.4. Given the 3×3 matrix A, find the inverse, if it exists.

$$A=\begin{bmatrix}2 & 3 & 1\\ 3 & 3 & 1\\ 2 & 4 & 1\end{bmatrix}$$

Solution. We start with the matrix $[A \mid I_3]$ and apply row operations to get it into reduced row echelon form.

Matrix $[A \mid I_3]\rightarrow$ $$\left[\begin{array}{ccc|ccc}2 & 3 & 1 & 1 & 0 & 0\\ 3 & 3 & 1 & 0 & 1 & 0\\ 2 & 4 & 1 & 0 & 0 & 1\end{array}\right]$$

Replace R_1 with $\dfrac{1}{2}R_1\rightarrow$ $$\left[\begin{array}{ccc|ccc}1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0\\ 3 & 3 & 1 & 0 & 1 & 0\\ 2 & 4 & 1 & 0 & 0 & 1\end{array}\right]$$

$$\text{Replace } R_2 \text{ with } -3R_1+R_2 \text{ and } R_3 \text{ with } -2R_1+R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{3}{2} & -\frac{1}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\text{Switch } R_2 \text{ and } R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & -\frac{3}{2} & -\frac{1}{2} & -\frac{3}{2} & 1 & 0 \end{array}\right]$$

$$\text{Replace } R_3 \text{ with } \frac{3}{2}R_2+R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -\frac{1}{2} & -3 & 1 & \frac{3}{2} \end{array}\right]$$

$$\text{Replace } R_3 \text{ with } -2R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right]$$

$$\text{Replace } R_1 \text{ with } -\frac{3}{2}R_2+R_1 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 2 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right]$$

$$\text{Replace } R_1 \text{ with } -\frac{1}{2}R_3+R_1 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right]$$

With the matrix in reduced row echelon form, and I_3 to the left of the vertical bar, we find the inverse is

$$A^{-1}=\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

□

Using Matrix Inverses to Solve Systems of Linear Equations

Recall from the beginning of this section that $AX=B \Rightarrow X=A^{-1}B$, only if A^{-1} exists. If an $n\times n$ linear system has a unique solution, we can apply this method.

Example 6.5.5. Solve the system of equations, if possible, using an inverse matrix.

$$\begin{cases} 3x+8y=5 \\ 4x+11y=7 \end{cases}$$

Solution. We write the system in terms of the coefficient matrix A, the variable matrix X and the constant matrix B.

$$A=\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix},\ X=\begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B=\begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Now that we have the matrix equation $AX=B$, we solve for X using the formula $X=A^{-1}B$. First, we need to calculate A^{-1}, for which we use the formula we derived earlier to calculate the inverse of a 2×2 matrix. We note that $A=\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\begin{aligned} A^{-1} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{3(11)-8(4)}\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \\ &= \frac{1}{1}\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \end{aligned}$$

Now we are ready to solve.

$$\begin{aligned} X &= A^{-1}B \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

The solution is $x=-1$ and $y=1$, or $(-1,1)$.

□

Recall from the earlier work we did in determining $X=A^{-1}B$ that A^{-1} must be to the left of B since matrix multiplication is not commutative. We move on to a last example. Part (a) may look familiar. This is the system of equations presented at the end of **Section 6.4**.

Example 6.5.6 Solve the systems of equations, if possible, using an inverse matrix.

(a) $\begin{cases} 2x+3y+z=4 \\ 3x+3y+z=2 \\ 2x+4y+z=5 \end{cases}$

(b) $\begin{cases} 2x+3y+z=11 \\ 3x+3y+z=-2 \\ 2x+4y+z=15 \end{cases}$

Solution. We note that the coefficient matrix is the same for each of these systems:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

In **Example 6.5.4**, we found the inverse of this matrix to be

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

The only difference between the system in part (a) and the system in part (b) is the constants in the matrix B for the associated matrix equation $AX = B$. We solve each system using the formula $X = A^{-1}B$.

(a) $X = A^{-1}B = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$. Our solution is $(-2,1,5)$.

(b) $X = A^{-1}B = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} 11 \\ -2 \\ 15 \end{bmatrix} = \begin{bmatrix} -13 \\ 4 \\ 25 \end{bmatrix}$. We get $(-13,4,25)$.

□

6.5 Exercises

1. In a previous section, we showed examples in which matrix multiplication is not commutative, that is, $AB \neq BA$. Explain why matrix multiplication is commutative for matrix inverses, that is, $A^{-1}A = AA^{-1}$.

2. Does every 2×2 matrix have an inverse? Explain why or why not. Explain what condition is necessary for an inverse to exist.

In Exercises 3 – 8, verify that the matrix A is the inverse of the matrix B.

3. $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

5. $A = \begin{bmatrix} 4 & 5 \\ 7 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & \frac{1}{7} \\ \frac{1}{5} & -\frac{4}{35} \end{bmatrix}$

6. $A = \begin{bmatrix} -2 & \frac{1}{2} \\ 3 & -1 \end{bmatrix}, B = \begin{bmatrix} -2 & -1 \\ -6 & -4 \end{bmatrix}$

7. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}, B = \frac{1}{2}\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 2 \\ 1 & 6 & 9 \end{bmatrix}, B = \frac{1}{4}\begin{bmatrix} 6 & 0 & -2 \\ 17 & -3 & -5 \\ -12 & 2 & 4 \end{bmatrix}$

In Exercises 9 – 22, find the inverse of the matrix or state that the matrix is singular (not invertible).

9. $A = \begin{bmatrix} 3 & -2 \\ 1 & 9 \end{bmatrix}$

10. $B = \begin{bmatrix} -2 & 2 \\ 3 & 1 \end{bmatrix}$

11. $C = \begin{bmatrix} -3 & 7 \\ 9 & 2 \end{bmatrix}$

12. $D = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

13. $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

14. $F = \begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$

15. $G = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix}$

16. $H = \begin{bmatrix} 2 & -1 \\ 16 & -9 \end{bmatrix}$

17. $J = \begin{bmatrix} 1 & 0 & 6 \\ -2 & 1 & 7 \\ 3 & 0 & 2 \end{bmatrix}$

18. $K = \begin{bmatrix} 0 & 1 & -3 \\ 4 & 1 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

19. $L = \begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 1 \\ -2 & -4 & -5 \end{bmatrix}$

20. $M = \begin{bmatrix} 3 & 0 & 4 \\ 2 & -1 & 3 \\ -3 & 2 & -5 \end{bmatrix}$

21. $N = \begin{bmatrix} 4 & 6 & -3 \\ 3 & 4 & -3 \\ 1 & 2 & 6 \end{bmatrix}$

22. $P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 11 \\ 3 & 4 & 19 \end{bmatrix}$

In Exercises 23 – 30, use a matrix inverse to solve the system of linear equations.

23. $\begin{cases} 3x + 7y = 26 \\ 5x + 12y = 39 \end{cases}$

24. $\begin{cases} 3x + 7y = 0 \\ 5x + 12y = -1 \end{cases}$

25. $\begin{cases} 3x + 7y = -7 \\ 5x + 12y = 5 \end{cases}$

26. $\begin{cases} 5x - 6y = -61 \\ 4x + 3y = -2 \end{cases}$

27. $\begin{cases} 8x + 4y = -100 \\ 3x - 4y = 1 \end{cases}$

28. $\begin{cases} 3x - 2y = 6 \\ -x + 5y = -2 \end{cases}$

29. $\begin{cases} -3x - 4y = 9 \\ 12x + 4y = -6 \end{cases}$

30. $\begin{cases} -2x + 3y = \dfrac{3}{10} \\ -x + 5y = \dfrac{1}{2} \end{cases}$

In Exercises 31 – 33, use the inverse of M from **Exercise 20** to solve the system of linear equations.

31. $\begin{cases} 3x + 4z = 1 \\ 2x - y + 3z = 0 \\ -3x + 2y - 5z = 0 \end{cases}$

32. $\begin{cases} 3x + 4z = 0 \\ 2x - y + 3z = 1 \\ -3x + 2y - 5z = 0 \end{cases}$

33. $\begin{cases} 3x + 4z = 0 \\ 2x - y + 3z = 0 \\ -3x + 2y - 5z = 1 \end{cases}$

34. Matrices can be used in cryptography. Suppose we wish to encode the message 'BIGFOOT LIVES'. We start by assigning a number to each letter of the alphabet, say $A = 1$, $B = 2$ and so on. We reserve 0 to act as a space. Hence, our message 'BIGFOOT LIVES' corresponds to the string of numbers '2, 9, 7, 6, 15, 15, 20, 0, 12, 9, 22, 5, 19'. To encode this message, we use an invertible matrix. Any invertible matrix will do, but for this exercise, we choose

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -2 \\ -7 & 1 & -1 \end{bmatrix}$$

Since A is a 3×3 matrix, we encode our message string into a matrix M with 3 rows. To do this, we take the first three numbers, 2, 9, 7, and make them our first column, the next three numbers, 6, 15, 15, and make them our second column, and so on. We put 0's to round out the matrix.

$$M = \begin{bmatrix} 2 & 6 & 20 & 9 & 19 \\ 9 & 15 & 0 & 22 & 0 \\ 7 & 15 & 12 & 5 & 0 \end{bmatrix}$$

To encode the message, we find the product AM.

$$AM = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -2 \\ -7 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 6 & 20 & 9 & 19 \\ 9 & 15 & 0 & 22 & 0 \\ 7 & 15 & 12 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 12 & 42 & 100 & -23 & 38 \\ 1 & 3 & 36 & 39 & 57 \\ -12 & -42 & -152 & -46 & -133 \end{bmatrix}$$

So our coded message is '12, 1, −12, 42, 3, −42, 100, 36, −152, −23, 39, −46, 38, 57, −133'. To decode this message, we start with this string of numbers, construct a message matrix as we did earlier (we should get the matrix AM again) and then multiply by A^{-1}.

(a) Find A^{-1}.

(b) Use A^{-1} to decode the message and check that this method actually works.

(c) Decode the message '14, 37, −76, 128, 21, −151, 31, 65, −140'.

(d) Choose another invertible matrix and encode and decode your own message.

6.6 Systems of Linear Equations: Determinants

Learning Objectives

- Find the determinant of a 2×2 or 3×3 matrix.
- Solve a system of linear equations using Cramer's Rule.

In this section we assign to each square matrix A a real number, called the **determinant of A**, which will lead to another technique for solving consistent independent systems of linear equations. There are two commonly used notations for the determinant of a matrix A: '$\det(A)$' or '$|A|$'. The second notation, $|A|$, can be troublesome when confused with the absolute value, but is useful in providing a quick, easy, method for denoting the determinant of a matrix.

Finding the Determinant of a 2×2 Matrix

While the definition will show up a bit later, to help us get started right away finding determinants, we introduce the following formula for finding determinants of 2×2 matrices.

Formula 6.1. For a 2×2 matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $\det(A)=ad-bc$.

We may also indicate the determinant using the notation $|A|=\begin{vmatrix} a & b \\ c & d \end{vmatrix}=ad-bc$.

Example 6.6.1. Compute the determinant of the matrix $A=\begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}$.

Solution. Using **Formula 6.1**, we find the determinant of A as follows.

$$\begin{aligned} \det(A) &= \begin{vmatrix} 4 & -3 \\ 2 & 1 \end{vmatrix} \\ &= (4)(1)-(-3)(2) \\ &= 4+6 \\ &= 10 \end{aligned}$$

□

Recall that, in **Section 6.5**, we derived a formula for the inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Using **Formula 6.1**, we can now modify that formula. Recall that $\det(A) = ad - bc$

$$\text{If } \det(A) \neq 0 \text{ then } A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Thus, we can use the determinant of a 2×2 matrix in finding its inverse. Note also that when the determinant is zero, the inverse does not exist.

Finding the Determinant of an $n \times n$ Matrix

While the technique discussed here may be applied to any $n \times n$ matrix where $n > 1$, we will focus on 3×3 matrices. We begin by introducing a notation for the entries in a 3×3 matrix A. We denote each entry as a_{ij} where *i* is the row and *j* is the column where that entry resides. For example, a_{23} is the entry in row 2 and column 3, as shown below.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Note that any matrix may be written in a similar manner by increasing or decreasing the number of rows and columns. We move on to a couple of definitions.

Definition 6.10. Given an $n \times n$ matrix A where $n > 1$,

- the **minor** M_{ij} of the entry a_{ij} is the determinant of the $(n-1)\times(n-1)$ matrix formed by deleting row i and column j.
- the **cofactor** C_{ij} of the entry a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

We determine the minor of an entry by deleting the row and column in which that entry appears and then finding the determinant of the resulting matrix. Consider the matrix A, below.

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$$

Entry a_{ij}	Delete row i and column j	M_{ij}
$a_{11}=3$	$\begin{bmatrix} \boxed{\cancel{3}} & \cancel{1} & \cancel{2} \\ \cancel{0} & -1 & 5 \\ \cancel{2} & 1 & 4 \end{bmatrix} \to \begin{bmatrix} -1 & 5 \\ 1 & 4 \end{bmatrix}$	$M_{11}=\det\left(\begin{bmatrix} -1 & 5 \\ 1 & 4 \end{bmatrix}\right)$ $=(-1)(4)-(5)(1)$ $=-9$
$a_{23}=5$	$\begin{bmatrix} 3 & 1 & \cancel{2} \\ \cancel{0} & \cancel{-1} & \boxed{\cancel{5}} \\ 2 & 1 & \cancel{4} \end{bmatrix} \to \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$	$M_{23}=\det\left(\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\right)$ $=(3)(1)-(1)(2)$ $=1$

Note that there are seven additional minors (M_{12}, M_{13}, M_{21}, M_{22}, M_{31}, M_{32} and M_{33}) that can be found in a similar manner. To determine the cofactor, C_{ij}, of entry a_{ij}, we find the minor and multiply it by 1 or −1, depending on whether the sum of i and j is even or odd, respectively. Another way to remember the sign $(-1)^{i+j}$ is with the following 'checkerboard' sign pattern.

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

We find C_{11} and C_{23} as follows.

$$\begin{aligned} C_{11} &= (-1)^{1+1} M_{11} \\ &= (1)(-9) \\ &= -9 \end{aligned} \qquad\qquad \begin{aligned} C_{23} &= (-1)^{2+3} M_{23} \\ &= (-1)(1) \\ &= -1 \end{aligned}$$

Note that, for C_{11}, $(-1)^{1+1}=(-1)^2=+1$ matches the sign is the first row, first column of the sign pattern shown above. For C_{23}, $(-1)^{2+3}=(-1)^5=-1$ matches the sign in the second row, third column.

We are now ready to define the determinant of an $n\times n$ matrix, $n>1$. Because of the recursive[23] nature of the determinant, we must have a starting point and thus define the determinant of a 1×1 matrix as $\det([a_{11}])=a_{11}$. The definition for $n>1$ follows.

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[23] We will talk more about 'recursive' in **Section 7.1**.

Definition 6.11. Given an $n \times n$ matrix A where $n > 1$, the **determinant** of A is the sum of the entries in any row or column multiplied by each entry's respective cofactor.

We refer to the process described in **Definition 6.11** as **expanding along the *i* th row**, or **down the *j* th column**. Although this definition is brief, it is not very convenient to write in general mathematical notation. The best way to understand it is through examples that we show below.

We can now use **Definition 6.11** to verify **Formula 6.1**. We find the determinant of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ by expanding along the first row. Noting that $a_{11} = a$ and $a_{12} = b$, we have

$$\det(A) = \det\left(\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ c & d \end{bmatrix}\right) = \boldsymbol{a} \cdot C_{11} + \boldsymbol{b} \cdot C_{12}$$

Now, $C_{11} = (-1)^{1+1} M_{11} = M_{11}$ and $C_{12} = (-1)^{1+2} M_{12} = -M_{12}$ so $\det(A) = a \cdot M_{11} - b \cdot M_{12}$. We determine M_{11} and M_{12} as follows.

$a_{11} = a$	$\begin{bmatrix} \cancel{a} & \cancel{b} \\ \cancel{c} & d \end{bmatrix} \to [d]$	$M_{11} = \det([d]) = d$	$a_{12} = b$	$\begin{bmatrix} \cancel{a} & \cancel{b} \\ c & \cancel{d} \end{bmatrix} \to [c]$	$M_{12} = \det([c]) = c$

It follows that $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$, verifying **Formula 6.1**.

Example 6.6.2. Compute the determinant of the matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$.

Solution. We choose to expand along the first row. To get a general idea of what that entails, we find

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Noting that $C_{11} = (-1)^{1+1} M_{11} = M_{11}$, $C_{12} = (-1)^{1+2} M_{12} = -M_{12}$ and $C_{13} = (-1)^{1+3} M_3 = M_3$, we have

$$|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

We expand along the first row of our matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$ to get

$$\begin{aligned}|A| &= (3)\begin{vmatrix}-1 & 5\\ 1 & 4\end{vmatrix} - (1)\begin{vmatrix}0 & 5\\ 2 & 4\end{vmatrix} + (2)\begin{vmatrix}0 & -1\\ 2 & 1\end{vmatrix}\\ &= 3\left[(-1)(4)-(5)(1)\right] - \left[(0)(4)-(5)(2)\right] + 2\left[(0)(1)-(-1)(2)\right]\\ &= 3(-9)-(-10)+2(2)\\ &= -13\end{aligned}$$

Alternate Solution. Just to show that we may use any row or column we like, and noting that it often saves time to expand along a row or column containing the most zeros, let's find the determinant by expanding down the first column.

$$\begin{aligned}|A| &= \begin{vmatrix}3 & 1 & 2\\ 0 & -1 & 5\\ 2 & 1 & 4\end{vmatrix}\\ &= +(3)\begin{vmatrix}-1 & 5\\ 1 & 4\end{vmatrix} - (0)\begin{vmatrix}1 & 2\\ 1 & 4\end{vmatrix} + (2)\begin{vmatrix}1 & 2\\ -1 & 5\end{vmatrix}\\ &= 3[-4-5]-0[4-2]+2\left[5-(-2)\right]\\ &= -13\end{aligned}$$

□

Note that we relied on **Formula 6.1** to compute three determinants of 2×2 matrices as part of the solution process in finding the determinant of a 3×3 matrix. Were we to evaluate the determinant of a 4×4 matrix, we would have to compute the determinants of four 3×3 matrices, each of which involves finding determinants of three 2×2 matrices. While this is certainly possible, you can see that our method of evaluating determinants quickly gets out of hand without the aid of technology. There is an alternate method that may be used to find the determinant of a 3×3 matrix, as shown below, with the warning that this method will not work for matrices larger than 3×3.

Say we want to find the determinant of the following matrix.

$$A = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$

We may find the determinant as follows:

1. Augment the matrix A with the first two columns.

$$\left|\begin{matrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{matrix}\right|\begin{matrix}a_{11} & a_{12}\\ a_{21} & a_{22}\\ a_{31} & a_{32}\end{matrix}\right|$$

2. Starting in the upper left corner, multiply the entries down the first diagonal. Moving to the right, add the result to the product of entries down the second diagonal. Add this result to the product of the entries down the third diagonal.

3. Starting in the lower left corner, subtract the product of entries up the first diagonal. Moving to the right, subtract, from this result, the product of entries up the second diagonal. From this result, subtract the product of entries up the third diagonal.

$$\left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right| \qquad \left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right|$$

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \quad -a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

Example 6.6.3. Find the determinant of the matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$ using the alternate method.

Solution. We begin by augmenting the matrix with the first two columns.

$$\left|\begin{array}{ccc|cc} 3 & 1 & 2 & 3 & 1 \\ 0 & -1 & 5 & 0 & -1 \\ 2 & 1 & 4 & 2 & 1 \end{array}\right|$$

We proceed by multiplying then adding entries down diagonals, and multiplying then subtracting entries up diagonals.

$$\begin{aligned} |A| &= (3)(-1)(4)+(1)(5)(2)+(2)(0)(1)-(2)(-1)(2)-(1)(5)(3)-(4)(0)(1) \\ &= -12+10+0+4-15-0 \\ &= -13 \end{aligned}$$

□

Cramer's Rule

We next introduce a theorem that enables us to solve a system of linear equations by means of determinants only. The theorem is stated in full generality, using numbered variables x_1, x_2, etc., instead of the more familiar letters *x*, *y*, *z*, etc. The proof of the general case is best left to a course in Linear Algebra.

Theorem 6.6. Cramer's Rule: Suppose $AX = B$ is the matrix form of a system of n linear equations in n unknowns where A is the coefficient matrix, X is the variable matrix and B is the constant matrix. If $\det(A) \neq 0$, then the corresponding system is consistent and independent and the solution for unknowns $x_1, x_2, \ldots, x_n$ is given by

$$x_j = \frac{\det(A_j)}{\det(A)}$$

where A_j is the matrix A whose jth column has been replaced by the constant matrix B, for $j = 1, 2, \cdots, n$.

In words, Cramer's Rule tells us we can solve for each unknown, one at a time, by finding the ratio of the determinant of A_j to that of the determinant of the coefficient matrix. The matrix A_j is found by replacing the column in the coefficient matrix that holds the coefficients of x_j with the constants of the system. To understand why this works for solving 2×2 systems of equations, consider the following.

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases} \longrightarrow \begin{cases} adx + bdy = ed \\ -bcx - bdy = -bf \end{cases}$$
$$(ad - bc)x = ed - bf$$

Thus, $x = \dfrac{ed - bf}{ad - bc} = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$ and, similarly, $y = \dfrac{af - ce}{ad - bc} = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$ if $ad - bc \neq 0$.

The following two examples demonstrate this method.

Example 6.6.4. Use Cramer's Rule to solve for x_1 and x_2.

$$\begin{cases} 2x_1 - 3x_2 = 4 \\ 5x_1 + x_2 = -2 \end{cases}$$

Solution. Writing this system in matrix form, we find

$$A = \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

To find the matrix A_1, we remove the column of the coefficient matrix A that holds the coefficients of x_1 and replace it with the corresponding entries in B. Likewise, we replace the column of A that corresponds to the coefficients of x_2 with the constants to form the matrix A_2. This yields

$$A_1 = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 2 & 4 \\ 5 & -2 \end{bmatrix}$$

Computing determinants, we get $\det(A) = 17$, $\det(A_1) = -2$ and $\det(A_2) = -24$, so that

$$x_1 = \frac{\det(A_1)}{\det(A)} = -\frac{2}{17} \qquad x_2 = \frac{\det(A_2)}{\det(A)} = -\frac{24}{17}$$

The reader can check that the solution to the system is $\left(-\frac{2}{17}, -\frac{24}{17}\right)$.

□

Example 6.6.5. Use Cramer's Rule to solve for z.

$$\begin{cases} 2x - 3y + z = -1 \\ x - y + z = 1 \\ 3x - 4z = 0 \end{cases}$$

Solution. To get started, we determine the matrices A, X and B.

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 1 \\ 3 & 0 & -4 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Then, in solving for z, we find the matrix[24] A_z that results from replacing the column containing the coefficients of z in the matrix A with the constants in B.

$$A_z = \begin{bmatrix} 2 & -3 & -1 \\ 1 & -1 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

We have $z = \frac{\det(A_z)}{\det(A)} = \frac{-12}{-10} = \frac{6}{5}$.

The reader is encouraged to solve this system for x and y similarly and check the answer.

□

[24] We may use A_z instead of A_3 as alternate notation when variables are x, y, z, etc. instead of x_1, x_2, x_3, etc.

6.6 Exercises

1. Can we always evaluate the determinant of a square matrix? Explain why or why not.

2. Examining Cramer's Rule, explain why there is no unique solution to the system when the determinant of the coefficient matrix is 0. For simplicity, use a 2×2 matrix.

In Exercises 3 – 18, compute the determinant of the given matrix. (Most of these matrices appeared in the **6.5 Exercises**.)

3. $A=\begin{bmatrix} 3 & -2 \\ 1 & 9 \end{bmatrix}$

4. $B=\begin{bmatrix} -2 & 2 \\ 3 & 1 \end{bmatrix}$

5. $C=\begin{bmatrix} -3 & 7 \\ 9 & 2 \end{bmatrix}$

6. $D=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

7. $E=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

8. $F=\begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$

9. $G=\begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix}$

10. $H=\begin{bmatrix} 2 & -1 \\ 16 & -9 \end{bmatrix}$

11. $J=\begin{bmatrix} 1 & 0 & 6 \\ -2 & 1 & 7 \\ 3 & 0 & 2 \end{bmatrix}$

12. $K=\begin{bmatrix} 0 & 1 & -3 \\ 4 & 1 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

13. $L=\begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 1 \\ -2 & -4 & -5 \end{bmatrix}$

14. $M=\begin{bmatrix} 3 & 0 & 4 \\ 2 & -1 & 3 \\ -3 & 2 & -5 \end{bmatrix}$

15. $N=\begin{bmatrix} 4 & 6 & -3 \\ 3 & 4 & -3 \\ 1 & 2 & 6 \end{bmatrix}$

16. $P=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 11 \\ 3 & 4 & 19 \end{bmatrix}$

17. $Q=\begin{bmatrix} x & x^2 \\ 1 & 2x \end{bmatrix}$

18. $R=\begin{bmatrix} i & j & k \\ -1 & 0 & 5 \\ 9 & -4 & -2 \end{bmatrix}$

In Exercises 19 – 34, use Cramer's Rule to solve the system of linear equations. (Some of these systems appeared in the **6.5 Exercises**.)

19. $\begin{cases} 3x+7y=26 \\ 5x+12y=39 \end{cases}$

20. $\begin{cases} 3x+7y=0 \\ 5x+12y=-1 \end{cases}$

21. $\begin{cases} 3x+7y=-7 \\ 5x+12y=5 \end{cases}$

22. $\begin{cases} 5x-6y=-61 \\ 4x+3y=-2 \end{cases}$

23. $\begin{cases} 8x+4y=-100 \\ 3x-4y=1 \end{cases}$

24. $\begin{cases} 3x-2y=6 \\ -x+5y=-2 \end{cases}$

25. $\begin{cases} -3x-4y=9 \\ 12x+4y=-6 \end{cases}$

26. $\begin{cases} -2x+3y=\dfrac{3}{10} \\ -x+5y=\dfrac{1}{2} \end{cases}$

27. $\begin{cases} x+y+z=3 \\ 2x-y+z=0 \\ -3x+5y+7z=7 \end{cases}$

28. $\begin{cases} 3x+y-2z=10 \\ 4x-y+z=5 \\ x-3y-4z=-1 \end{cases}$

29. $\begin{cases} x+2y-4z=-1 \\ 7x+3y+5z=26 \\ -2x-6y+7z=-6 \end{cases}$

30. $\begin{cases} -5x+2y-4z=-47 \\ 4x-3y-z=-94 \\ 3x-3y+2z=94 \end{cases}$

31. $\begin{cases} 4x+5y-z=-7 \\ -2x-9y+2z=8 \\ 5y+7z=21 \end{cases}$

32. $\begin{cases} 4x-3y+4z=10 \\ 5x-2z=-2 \\ 3x+2y-5z=-9 \end{cases}$

33. $\begin{cases} 4x-2y+3z=6 \\ -6x+y=-2 \\ 2x+7y+8z=24 \end{cases}$

34. $\begin{cases} -4x-3y-8z=-7 \\ 2x-9y+5z=\dfrac{1}{2} \\ 5x-6y-5z=-2 \end{cases}$

35. Carl's Sasquatch Attack! game card collection is a mixture of common and rare cards. Each common card is worth \$0.25 while each rare card is worth \$0.75. If his entire 117 card collection is worth \$48.75, how many of each kind of card does he own?

36. Brenda's Exotic Animal Rescue houses snakes, tarantulas and scorpions. When asked how many animals of each kind she boards, Brenda answered: 'We board 49 total animals, and I am responsible for each of their 272 legs and 28 tails.' How many of each animal does the Rescue board? (Recall: Tarantulas have 8 legs and no tails; scorpions have 8 legs and one tail; snakes have no legs and one tail.)

6.7 Partial Fraction Decomposition

Learning Objectives

- Decompose a rational expression with denominator of non-repeated linear factors into a sum of partial fractions.
- Decompose a rational expression with denominator of repeated linear factors into a sum of partial fractions.
- Decompose a rational expression with denominator of non-repeated irreducible quadratic factors into a sum of partial fractions.
- Decompose a rational expression with denominator of repeated irreducible quadratic factors into a sum of partial fractions.

This section uses systems of linear equations to decompose rational expressions into a sum of simpler rational expressions. This process will be useful in Calculus, where simpler rational expressions are often preferred. The following is an example of what we hope to achieve.

$$\frac{x^2-x-6}{x^4+x^2}=\frac{x+7}{x^2+1}-\frac{1}{x}-\frac{6}{x^2}$$

If we are given the expression on the right side of the equation, it is a matter of Intermediate Algebra to determine a common denominator to obtain the expression on the left. The focus of this section is to develop a method by which we start with the expression on the left side and 'resolve it into **partial fractions**' to obtain the expression on the right. Essentially, we want to reverse the least common denominator process.

Since all polynomials can be factored into linear or quadratic factors, the denominators of the rational expressions in this section can be factored into linear and/or quadratic factors. The degree of the numerator is less than the degree of the denominator for all expressions presented here. In Calculus, you will find that long division is a useful tool for decomposing rational expressions with numerators of the same or greater degree than their denominators. We begin with expressions having denominators of non-repeated linear factors.

Non-Repeated Linear Factors

Starting with an example, we look for two rational expressions that, when added together, result in the original expression. In this first example, the denominator is already factored so we have an idea what the partial fraction will look like. Since these are all proper fractions, the degree of the numerator will be less than the degree of the denominator. For now, we use variables A and B to represent the unknown numerators.

Example 6.7.1. Resolve the rational expression $\frac{5x-1}{(x+1)(x-2)}$ into partial fractions.

Solution. We set $\frac{5x-1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$ and attempt to find values for A and B. We multiply through by the least common denominator to eliminate the fractions.

$$\begin{aligned}
\frac{5x-1}{(x+1)(x-2)} \cdot (x+1)(x-2) &= \left(\frac{A}{x+1} + \frac{B}{x-2}\right) \cdot (x+1)(x-2) \\
5x-1 &= \frac{A}{x+1} \cdot (x+1)(x-2) + \frac{B}{x-2} \cdot (x+1)(x-2) \\
5x-1 &= A(x-2) + B(x+1)
\end{aligned}$$

In solving for A and B, we continue by expanding the right side of the equation and collecting like terms.

$$\begin{aligned}
5x-1 &= Ax - 2A + Bx + B \\
5x-1 &= (A+B)x + (-2A+B)
\end{aligned}$$

We note that the corresponding coefficients must be the same on each side of the equation.

$$5x + (-1) = (A+B)x + (-2A+B)$$

From equating coefficients, we get the system

$$\begin{cases} A + B = 5 \\ -2A + B = -1 \end{cases}$$

This system is easily solved using either the substitution or elimination method from **Section 6.1**. We find $A = 2$ and $B = 3$ to arrive at the final answer.

$$\begin{aligned}
\frac{5x-1}{(x+1)(x-2)} &= \frac{A}{x+1} + \frac{B}{x-2} \\
&= \frac{2}{x+1} + \frac{3}{x-2}
\end{aligned}$$

Alternate Solution. Another technique available for decomposing some rational expressions is the Heaviside Method.[25]. For the Heaviside Method, we return to the equation $5x-1 = A(x-2) + B(x+1)$ and look for values of x that will result in an A or B term of zero. Setting $x = 2$ will result in $x - 2 = 0$ and setting $x = -1$ will result in $x + 1 = 0$, so we substitute each of these values for x.

[25] Named after Oliver Heaviside.

$$\begin{aligned} 5x-1 &= A(x-2)+B(x+1) \\ 5(2)-1 &= A(2-2)+B(2+1) \quad \text{set } x=2 \\ 9 &= A(0)+3B \\ B &= 3 \end{aligned}$$

$$\begin{aligned} 5x-1 &= A(x-2)+B(x+1) \\ 5(-1)-1 &= A(-1-2)+B(-1+1) \quad \text{set } x=-1 \\ -6 &= A(-3)+B(0) \\ A &= 2 \end{aligned}$$

Now, we replace A and B with these results to complete the partial fraction decomposition, as shown above.

□

A word of caution in using the Heaviside Method is that this method will not catch a mistake of having initially chosen the wrong type of decomposition. However, in many instances, it is the quicker method. Before presenting the next example, we provide an overview of partial fraction decomposition when denominators consist of non-repeated linear factors.

Partial Fraction Decomposition for Non-Repeated Linear Factors

For the proper rational expression $\frac{p(x)}{q(x)}$ with $q(x)$ factored into the distinct linear factors

$q(x)=(a_1x+b_1)(a_2x+b_2)(a_3x+b_3)\cdots$, $\frac{p(x)}{q(x)}=\frac{A}{a_1x+b_1}+\frac{B}{a_2x+b_2}+\frac{C}{a_3x+b_3}+\cdots$, where A, B, C, etc., are constants yet to be determined.

Example 6.7.2. Resolve the rational expression $\frac{x+5}{2x^2-x-1}$ into partial fractions.

Solution. We begin by factoring the denominator to find $2x^2-x-1=(2x+1)(x-1)$. Using the variables A and B to stand in for missing constants results in

$$\frac{x+5}{2x^2-x-1}=\frac{x+5}{(2x+1)(x-1)}=\frac{A}{2x+1}+\frac{B}{x-1}$$

We multiply through by the least common denominator.

$$\begin{aligned} \frac{x+5}{(2x+1)(x-1)}\cdot(2x+1)(x-1) &= \left(\frac{A}{2x+1}+\frac{B}{x-1}\right)\cdot(2x+1)(x-1) \\ x+5 &= \frac{A}{2x+1}\cdot(2x+1)(x-1)+\frac{B}{x-1}\cdot(2x+1)(x-1) \\ x+5 &= A(x-1)+B(2x+1) \end{aligned}$$

The next step is to expand the right side and collect like terms.

$$x+5=A(x-1)+B(2x+1)$$
$$x+5=Ax-A+2Bx+B$$
$$x+5=(A+2B)x+(-A+B)$$

Now we look for corresponding coefficients.

$$(1)x+(5)=(A+2B)x+(-A+B)$$

From equating coefficients, we get the system

$$\begin{cases} A+2B=1 \\ -A+B=5 \end{cases}$$

This system is easily solved using the elimination method from **Section 6.1**. After adding both equations we get $3B=6$, or $B=2$. Using back substitution, we find $A=-3$.

$$\frac{x+5}{2x^2-x-1}=-\frac{3}{2x+1}+\frac{2}{x-1}$$

Our answer is easily checked by getting a common denominator and adding the fractions.

□

We summarize the steps below for resolving rational expressions into partial fractions.

How to Decompose a Rational Expression into Partial Fractions

1. Create an equation by setting the original expression equal to the appropriate sum of partial fractions, with constants in the numerators to be determined.
2. Multiply both sides of the equation by the least common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Equate coefficients of terms on the left side to coefficients of like terms on the right side. Solve the resulting system of equations.
5. Use the newly found constants to rewrite the original expression as a sum of partial fractions.

Repeated Linear Factors

We may run into rational expressions that contain repeated linear factors. What this means is that, instead of $\frac{5x-1}{(x+1)(x-2)}$, we might have a rational expression such as $\frac{5x-1}{(x+1)^2}$ where a factor in the denominator has a power higher than one. Before going further, we need to think about which individual denominators could contribute to obtain $(x+1)^2$ as the least common denominator. If we think of $(x+1)^2$ as

$(x+1)(x+1)$, a so-called 'repeated' linear factor, it's possible that a term with a denominator of just $x+1$ contributes to the expression as well. We take a closer look in the following example.

Example 6.7.3. Resolve the expression $\frac{5x-1}{(x+1)^2}$ into partial fractions.

Solution. We set $\frac{5x-1}{(x+1)^2}=\frac{A}{x+1}+\frac{B}{(x+1)^2}$ and multiply through by the least common denominator.

$$\begin{aligned}\frac{5x-1}{(x+1)^2}\cdot(x+1)^2&=\frac{A}{x+1}\cdot(x+1)^2+\frac{B}{(x+1)^2}\cdot(x+1)^2\\5x-1&=A(x+1)+B\\5x-1&=Ax+(A+B)\end{aligned}$$

Setting corresponding coefficients equal, we have the system of equations:

$$\begin{cases}A=5\\A+B=-1\end{cases}$$

The first equation gives us $A=5$. Substituting into the second equation, we find $B=-6$.

$$\frac{5x-1}{(x+1)^2}=\frac{5}{x+1}-\frac{6}{(x+1)^2}$$

Alternate Solution. We employ the Heaviside technique, substituting $x=-1$ into the equation $5x-1=A(x+1)+B$ to help us find B.

$$\begin{aligned}5x-1&=A(x+1)+B\\5(-1)-1&=A(-1+1)+B \quad \text{set } x=-1\\-6&=B\end{aligned}$$

In our search for A, there is no additional value of x that will zero out the B term, so we choose any value for x that hasn't already been substituted, say $x=0$, and we also substitute $B=-6$.

$$\begin{aligned}5x-1&=A(x+1)+B\\5(0)-1&=A(0+1)-6 \quad \text{set } x=0,\ B=-6\\-1&=A-6\\5&=A\end{aligned}$$

We have $A=5$ and $B=-6$, as we found with the traditional method.

□

Partial Fraction Decomposition for Repeated Linear Factors

For the proper rational expression $\frac{p(x)}{q(x)}$ with $q(x)=(ax+b)^n$,

$\frac{p(x)}{q(x)}=\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}+\cdots+\frac{N}{(ax+b)^n}$, where $n>1$ is an integer and A, B, $C,\cdots,N$ are constants yet to be determined.

In the process of decomposing rational expressions into partial fractions, we sometimes run across denominators that contain more that one type of factor, as in the following example.

Example 6.7.4. Resolve the expression $\frac{3}{x^3-2x^2+x}$ into partial fractions.

Solution. Factoring the denominator gives $x^3-2x^2+x=(x)(x^2-2x+1)=x(x-1)^2$. Noting that we have both a non-repeated linear factor and a repeated linear factor, we make sure that our sum of partial fractions includes all indicated cases.

$$\frac{3}{x^3-2x^2+x}=\frac{3}{x(x-1)^2}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

We multiply by the least common denominator.

$$\frac{3}{x(x-1)^2}\cdot x(x-1)^2=\left(\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}\right)\cdot x(x-1)^2$$
$$3=A(x-1)^2+Bx(x-1)+Cx$$

Next, we expand and collect like terms.

$$3=A(x^2-2x+1)+B(x^2-x)+Cx$$
$$3=Ax^2-2Ax+A+Bx^2-Bx+Cx$$
$$3=(A+B)x^2+(-2A-B+C)x+A$$

Our system of equations is

$$\begin{cases} A+B=0 \\ -2A-B+C=0 \\ A=3 \end{cases}$$

Substituting $A=3$ into $A+B=0$ gives $B=-3$, and substituting both for A and B in $-2A-B+C=0$ gives $C=3$. Our final answer is

$$\frac{3}{x^3-x^2+x}=\frac{3}{x}-\frac{3}{x-1}+\frac{3}{(x-1)^2}$$

□

Non-Repeated Irreducible Quadratic Factors

So far, our rational expressions have contained linear factors in the denominator. Now we introduce rational expressions having denominators with irreducible quadratic factors, like x^2+1. With linear factors, we used variables A, B, C, etc. to represent constants. For irreducible quadratic factors in the denominator, we must allow for linear expressions in each numerator so will use expressions such as $Ax+B$, $Cx+D$, etc.

Example 6.7.5. Resolve the expression $\dfrac{5x^2-5x+5}{(x-2)(x^2+1)}$ into partial fractions.

Solution. We have one linear factor and one irreducible quadratic factor in the denominator, so we must allow for a constant, represented by A, and a linear expression, represented by $Bx+C$, respectively.

$$\frac{5x^2-5x+5}{(x-2)(x^2+1)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+1}$$

We multiply through by the least common denominator.

$$\frac{5x^2-5x+5}{(x-2)(x^2+1)}\cdot(x-2)(x^2+1)=\left(\frac{A}{x-2}+\frac{Bx+C}{x^2+1}\right)\cdot(x-2)(x^2+1)$$
$$5x^2-5x+5=A(x^2+1)+(Bx+C)(x-2)$$

Now we expand and collect like terms.

$$5x^2-5x+5=A(x^2+1)+(Bx+C)(x-2)$$
$$5x^2-5x+5=Ax^2+A+Bx^2-2Bx+Cx-2C$$
$$5x^2-5x+5=(A+B)x^2+(-2B+C)x+(A-2C)$$

The result is the following system of equations.

$$\begin{cases} A+B=5 \\ -2B+C=-5 \\ A-2C=5 \end{cases}$$

We solve the system of equations using a method from earlier in this chapter, and find $A=3$, $B=2$, and $C=-1$. We can now rewrite the original expression as

$$\frac{5x^2-5x+5}{(x-2)(x^2+1)}=\frac{3}{x-2}+\frac{2x-1}{x^2+1}$$

□

In **Example 6.7.5**, we had a factor of (x^2+1) in the denominator. Note that this is different than the factor $(x+1)^2$. For $(x+1)^2$, since $(x+1)$ is a linear factor, the numerator will be a constant, even though there are two copies of $(x+1)$ in $(x+1)^2$. The factor (x^2+1) has a linear numerator since it is an irreducible quadratic factor. An example of a rational expression that has both is

$$\frac{7x^3+14x^2+9x+6}{(x+1)^2(x^2+1)}=\frac{3}{x+1}+\frac{2}{(x+1)^2}+\frac{4x+1}{x^2+1}$$

Partial Fraction Decomposition for Non-Repeated Irreducible Quadratic Factors

For the proper rational expression $\frac{p(x)}{q(x)}$ with $q(x)$ factored into the distinct irreducible quadratic factors $q(x)=(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)(a_3x^2+b_3x+c_3)\cdots$,

$\frac{p(x)}{q(x)}=\frac{Ax+B}{a_1x^2+b_1x+c_1}+\frac{Cx+D}{a_2x^2+b_2x+c_2}+\frac{Ex+F}{a_3x^2+b_3x+c_3}+\cdots$, where A, B, C, etc., are constants yet to be determined.

Example 6.7.6. Resolve the expression $\frac{3}{x^3-x^2+x}$ into partial fractions.

Solution. The denominator factors as $x(x^2-x+1)$. The quadratic, x^2-x+1, doesn't factor easily but we check that it is irreducible by finding the discriminant. The discriminant is $(-1)^2-4(1)(1)=-3$. Since the discriminant is negative, there are no real zeros, verifying that x^2-x+1 cannot be factored over the real numbers and is indeed irreducible. We rewrite the original expression as follows.

$$\frac{3}{x^3-x^2+x}=\frac{3}{x(x^2-x+1)}=\frac{A}{x}+\frac{Bx+C}{x^2-x+1}$$

Proceeding as usual, we clear denominators and get

$$3=A(x^2-x+1)+(Bx+C)x$$
$$3=Ax^2-Ax+A+Bx^2+Cx$$
$$3=(A+B)x^2+(-A+C)x+A$$

We have the following system of equations.

$$\begin{cases} A+B=0 \\ -A+C=0 \\ A=3 \end{cases}$$

From $A=3$ and $A+B=0$, we get $B=-3$. From $-A+C=0$, we get $C=A=3$. Finally,

$$\frac{3}{x^3-x^2+x}=\frac{3}{x}+\frac{-3x+3}{x^2-x+1}$$

□

Repeated Irreducible Quadratic Factors

Lastly, we look at cases where an irreducible quadratic factor has a power higher than one. We start with an example.

Example 6.7.7. Resolve the expression $\dfrac{x^4+x^3+x^2-x+1}{x\left(x^2+1\right)^2}$ into partial fractions.

Solution. The factors of the denominator are x and $\left(x^2+1\right)^2$. As with linear factors, we can think of $\left(x^2+1\right)^2$ as $\left(x^2+1\right)\left(x^2+1\right)$ and recognize the potential denominator of x^2+1 contributing to the expression. As such, we have

$$\frac{x^4+x^3+x^2-x+1}{x\left(x^2+1\right)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{\left(x^2+1\right)^2}$$

We eliminate fractions by multiplying each term by the least common denominator, $x\left(x^2+1\right)^2$, and proceed to expand the right side and collect like terms.

$$\begin{aligned}
x^4+x^3+x^2-x+1&=A\left(x^2+1\right)^2+\left(Bx+C\right)\left(x\right)\left(x^2+1\right)+\left(Dx+E\right)\left(x\right)\\
x^4+x^3+x^2-x+1&=A\left(x^4+2x^2+1\right)+\left(Bx+C\right)\left(x^3+x\right)+Dx^2+Ex\\
x^4+x^3+x^2-x+1&=Ax^4+2Ax^2+A+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex\\
x^4+x^3+x^2-x+1&=\left(A+B\right)x^4+Cx^3+\left(2A+B+D\right)x^2+\left(C+E\right)x+A
\end{aligned}$$

The resulting system of equations is

$$\begin{cases} A+B=1\\ C=1\\ 2A+B+D=1\\ C+E=-1\\ A=1\end{cases}$$

Right off, we have $A=1$ and $C=1$. Substituting $A=1$ into $A+B=1$, we find $B=0$. From $C=1$ and $C+E=-1$, we get $E=-2$. With $A=1$ and $B=0$, substituting into $2A+B+D=1$ gives us $D=-1$. The resulting decomposition of the original expression is

$$\frac{x^4+x^3+x^2-x+1}{x\left(x^2+1\right)^2}=\frac{1}{x}+\frac{1}{x^2+1}-\frac{x+2}{\left(x^2+1\right)^2}$$

□

Partial Fraction Decomposition for Repeated Irreducible Quadratic Factors

For the proper rational expression $\frac{p(x)}{q(x)}$ with $q(x)=\left(ax^2+bx+c\right)^n$ and ax^2+bx+c irreducible,

$\frac{p(x)}{q(x)}=\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{\left(ax^2+bx+c\right)^2}+\frac{A_3x+B_3}{\left(ax^2+bx+c\right)^3}+\cdots+\frac{A_nx+B_n}{\left(ax^2+bx+c\right)^n}$, where $n>1$ is an integer

and A_1, B_1, A_2, B_2, A_3, B_3, $\cdots$, A_n, B_n are constants yet to be determined.

Example 6.7.8. Resolve the expression $\frac{x^3+5x-1}{x^4+6x^2+9}$ into partial fractions.

Solution. We recognize the denominator, x^4+6x^2+9, as being quadratic in form and factor it as $\left(x^2+3\right)^2$. Since x^2+3 clearly is not factorable over the real numbers, it is irreducible. We write the expression as

$$\frac{x^3+5x-1}{x^4+6x^2+9}=\frac{x^3+5x-1}{\left(x^2+3\right)^2}=\frac{Ax+B}{x^2+3}+\frac{Cx+D}{\left(x^2+3\right)^2}$$

After multiplying each term by the least common denominator of $\left(x^2+3\right)^2$, we find

$$\begin{aligned}x^3+5x-1&=(Ax+B)\left(x^2+3\right)+Cx+D\\x^3+5x-1&=Ax^3+3Ax+Bx^2+3B+Cx+D\\x^3+5x-1&=Ax^3+Bx^2+(3A+C)x+(3B+D)\end{aligned}$$

Our system is

$$\begin{cases}A=1\\B=0\\3A+C=5\\3B+D=-1\end{cases}$$

We solved this system in **Example 6.3.5**, finding $A=1$, $B=0$, $C=2$ and $D=-1$. Our final answer is

$$\frac{x^3+5x-1}{x^4+6x^2+9}=\frac{x}{x^2+3}+\frac{2x-1}{\left(x^2+3\right)^2}$$

□

6.7 Exercises

1. Can any quotient of polynomials be decomposed into at least two partial fractions? If so, explain why, and if not, give an example.
2. How can you check that you decomposed a partial fraction correctly?

In Exercises 3 – 16, find the partial fraction decomposition for the non-repeated linear factors.

3. $\frac{5x+16}{x^2+10x+24}$
4. $\frac{3x-79}{x^2-5x-24}$
5. $\frac{-x-24}{x^2-2x-24}$
6. $\frac{10x+47}{x^2+7x+10}$
7. $\frac{x}{6x^2+25x+25}$
8. $\frac{32x-11}{20x^2-13x+2}$
9. $\frac{x+1}{x^2+7x+10}$
10. $\frac{5x}{x^2-9}$
11. $\frac{10x}{x^2-25}$
12. $\frac{6x}{x^2-4}$
13. $\frac{2x-3}{x^2-6x+5}$
14. $\frac{4x-1}{x^2-x-6}$
15. $\frac{4x+3}{x^2+8x+15}$
16. $\frac{3x-1}{x^2-5x+6}$

In Exercises 17 – 27, find the partial fraction decomposition for the repeated linear factors.

17. $\frac{-5x-19}{(x+4)^2}$
18. $\frac{x}{(x-2)^2}$
19. $\frac{7x+14}{(x+3)^2}$
20. $\frac{-24x-27}{(4x+5)^2}$
21. $\frac{-24x-27}{(6x-7)^2}$
22. $\frac{5-x}{(x-7)^2}$
23. $\frac{5x+14}{2x^2+12x+18}$
24. $\frac{5x^2+20x+8}{2x(x+1)^2}$
25. $\frac{4x^2+55x+25}{5x(3x+5)^2}$
26. $\frac{54x^3+127x^2+80x+16}{2x^2(3x+2)^2}$
27. $\frac{x^3-5x^2+12x+144}{x^2(x^2+12x+36)}$

In Exercises 28 – 40, find the partial fraction decomposition for the non-repeated irreducible quadratic factors.

28. $\frac{4x^2+6x+11}{(x+2)(x^2+x+3)}$
29. $\frac{4x^2+9x+23}{(x-1)(x^2+6x+11)}$
30. $\frac{-2x^2+10x+4}{(x-1)(x^2+3x+8)}$

31. $\dfrac{x^2+3x+1}{(x+1)(x^2+5x-2)}$

32. $\dfrac{4x^2+17x-1}{(x+3)(x^2+6x+1)}$

33. $\dfrac{4x^2}{(x+5)(x^2+7x-5)}$

34. $\dfrac{4x^2+5x+3}{x^3-1}$

35. $\dfrac{-5x^2+18x-4}{x^3+8}$

36. $\dfrac{3x^2-7x+33}{x^3+27}$

37. $\dfrac{x^2+2x+40}{x^3-125}$

38. $\dfrac{4x^2+4x+12}{8x^3-27}$

39. $\dfrac{-50x^2+5x-3}{125x^3-1}$

40. $\dfrac{-2x^3-30x^2+36x+216}{x^4+216x}$

In Exercises 41 – 51, find the partial fraction decomposition for the repeated irreducible quadratic factors.

41. $\dfrac{3x^3+2x^2+14x+15}{(x^2+4)^2}$

42. $\dfrac{x^3+6x^2+5x+9}{(x^2+1)^2}$

43. $\dfrac{x^3-x^2+x-1}{(x^2-3)^2}$

44. $\dfrac{x^2+5x+5}{(x^2+2)^2}$

45. $\dfrac{x^3+2x^2+4x}{(x^2+2x+9)^2}$

46. $\dfrac{x^2+25}{(x^2+3x+25)^2}$

47. $\dfrac{2x^3+11x^2+7x+70}{(2x^2+x+14)^2}$

48. $\dfrac{5x+2}{x(x^2+4)^2}$

49. $\dfrac{x^4+x^3+8x^2+6x+36}{x(x^2+6)^2}$

50. $\dfrac{2x-9}{(x^2-x)^2}$

51. $\dfrac{5x^3-2x+1}{(x^2+2x)^2}$

Key Equations

Linear Equation in Two Variable:
$ax+by=c$

Linear Equation in Three Variables:
$ax+by+cz=d$

Properties of Matrix Addition:

- Commutative Property: $A+B = B+A$
- Associative Property: $(A+B)+C=A+(B+C)$
- Identity Property: $A+\mathbf{0}=\mathbf{0}+A=A$
- Inverse Property: $A+(-A)=(-A)+A=\mathbf{0}$

Properties of Matrix Multiplication:

- Associative Property: $(AB)C=A(BC)$
- Associative Property with Scalar Multiplication: $k(AB)=(kA)B=A(kB)$
- Identity Property: $I_m A=AI_n=A$
- Distributive Property: $A(B\pm C)=AB\pm AC$ and $(A\pm B)C=AC\pm BC$

Determinant of a 2×2 Matrix: If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(A)=ad-bc$

Cofactor of an Entry a_{ij} of a Matrix:
$C_{ij}=(-1)^{i+j}M_{ij}$

Cramer's Rule: $x_j=\dfrac{\det(A_j)}{\det(A)}$ where A is the coefficient matrix and A_j is the matrix whose j^{th} column has been replaced by the constant matrix

Partial Fraction Decomposition:

- Non-Repeated Linear Factors:
$$\frac{p(x)}{q(x)}=\frac{A}{a_1x+b_1}+\frac{B}{a_2x+b_2}+\frac{C}{a_3x+b_3}\cdots$$
- Repeated Linear Factors:
$$\frac{p(x)}{q(x)}=\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}+\cdots+\frac{N}{(ax+b)^n}$$
- Non-Repeated Irreducible Quadratic Factors:
$$\frac{p(x)}{q(x)}=\frac{Ax+B}{a_1x^2+b_1x+c_1}+\frac{Cx+D}{a_2x^2+b_2x+c_2}+\cdots$$
- Repeated Irreducible Quadratic Factors:
$$\frac{p(x)}{q(x)}=\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\frac{A_3x+B_3}{(ax^2+bx+c)^3}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

Key Terms

Augmented Matrix: Matrix with a column containing the constants added to the coefficient matrix of a system of equations

Coefficient Matrix: Matrix whose elements are the coefficients in a system of equations

Consistent: A system of equations with at least one solution

Constant Matrix: A matrix containing the constants of a system of equation

Dependent: A system of equations with an infinite number of solutions

Determinant of a Matrix: the sum of the entries in any row or column multiplied by each entry's respective cofactor

Dimensions of a Matrix: The number of rows by the number of columns

Elimination: Using addition to eliminate one variable in a system of equations

Equal Matrices: Matrices that are the same size and their corresponding entries are equal

Identity Matrix: A square matrix (I_k with 1's down the main diagonal and *0*'s elsewhere

Inconsistent: A system of equations that has no solution

Independent: A system of equations with a single solution

Inverse Matrix: Two $n \times n$ matrices are inverses if $AB = BA = I_n$; the inverse of A is written A^{-1}

Main Diagonal of a Matrix: positions in the matrix in which the row number and column number are the same

Matrix Addition: Adding corresponding elements of two matrices of the same size

Matrix Multiplication: Multiplying the rows of the first matrix by the columns of the second matrix

Matrix: A rectangular array of real numbers

Minor of an Entry a_{ij} of a Matrix: M_{ij}, the determinant of the matrix formed by deleting row *i* and column *j*.

Partial Fractions: A rational function decomposed into a sum of two or more simpler fractions

Reduced Row Echelon form of a Matrix: The matrix is in Row Echelon form and the leading 1's are the only nonzero entries in their respective columns

Row Echelon Form of a Matrix: The first nonzero entry in each row is 1, the leading 1 of any row is to the right of the leading 1 of the row above, and rows with all zeros are placed after any other rows

Scalar Multiplication: Multiplying each entry of a matrix by a scalar, *k*

Square Matrix: A matrix with the same number of rows and columns

System of Equations: Two or more equations with the same variables

Variable Matrix: A matrix containing the variables of a system of equations

Zero Matrix: A matrix whose entries are all zero

CHAPTER 7
SEQUENCES AND SERIES

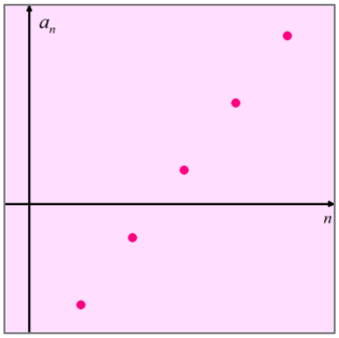


Figure 7.0. 1

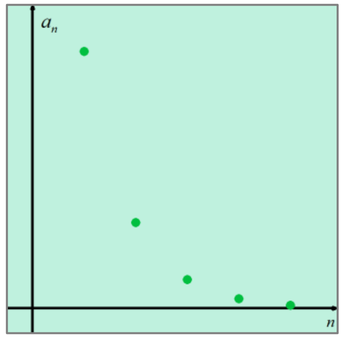


Figure 7.0. 2

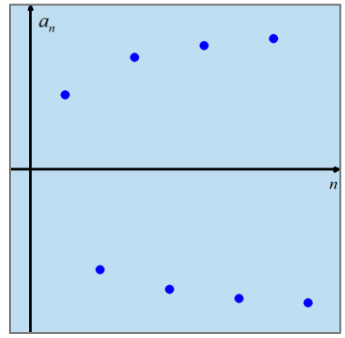


Figure 7.0. 3

Chapter Outline

Introduction

Chapter 7 is brief, but packed with new ideas that will be extended in future math courses. The first two sections deal with sequences (an ordered collection of terms) and series (the finite or infinite sum of a sequence.) We will spend the majority of the two sections focusing on two types of sequences, arithmetic and geometric. By the end of the two sections, you should be able to distinguish between arithmetic and geometric patterns, write formulas for a variety of sequences, find specific terms in a sequence, or find specified sums of arithmetic or geometric sequences. Having an understanding of notation and sequence behavior, along with facility in manipulating series, will be very useful in future courses. The last section focuses on binomial expansion and the binomial theorem.

Section 7.1 introduces the definition of sequences and provides a variety of examples of different types. Special attention should be paid to how 'the next term' is generated, as this is how some sequences are classified. In this section, we will focus special attention on sequences that are generated by adding a fixed amount to the previous term (arithmetic sequences) and those where the next term is generated by multiplying the previous term by a specific factor (geometric sequences.) You will also learn sequence notation, how to find specific terms in sequences, and how to write formulas for arithmetic and geometric sequences, given a few terms or information on how each term is generated along with a specific term.

In Section 7.2 you will learn how to find finite and infinite sums of arithmetic or geometric sequences. Again there will be new notation introduced in this section, and information on how to manipulate it. This is notation you will see in future math courses. Further, you will explore applications of summations, many of which are significantly related to our day-to-day lives (interest accumulation for savings or as part of a debt, for example.)

The last section of the chapter, 7.3, is related to binomial expansion. Here you will see the relationship between the binomial theorem, Pascal's triangle, and the expansion of binomials. While expansion of binomials can be readily done with simple multiplication, this section will show you a faster method and/or a way to find a specific term in the expansion. Ideas in this section relate to probability and other future math courses.

7.1 Sequences

Learning Objectives

- Identify number patterns.
- Recognize and use recursive and explicit formulas for sequences.
- Graph sequences.
- Identify arithmetic and geometric sequences.
- Find formulas for arithmetic and geometric sequences.

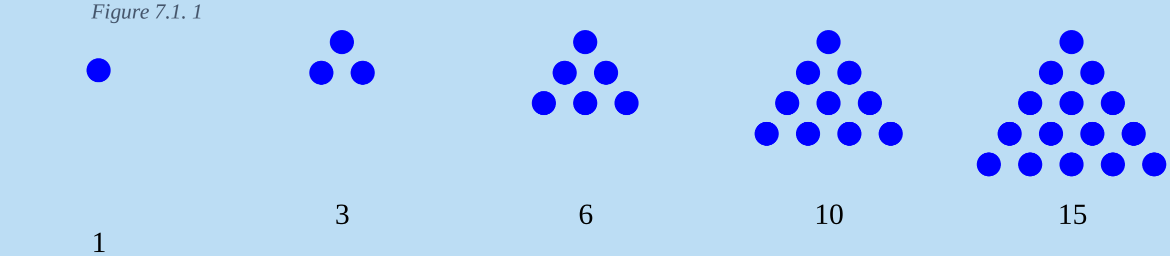
As we perceive time, life is a sequence of events. A typical macro view of a life may be: birth, schooling, job, marriage, kids, retirement and death. Informally, any (countable) ordered collection of events or numbers is called a **sequence**. In this sense, the title of the popular books and movies 'A Series of Unfortunate Events' is mathematically incorrect; it should have been called 'A Sequence of Unfortunate Events'. As you will see later, the words 'sequence' and 'series' have different mathematical meanings.

Number Patterns

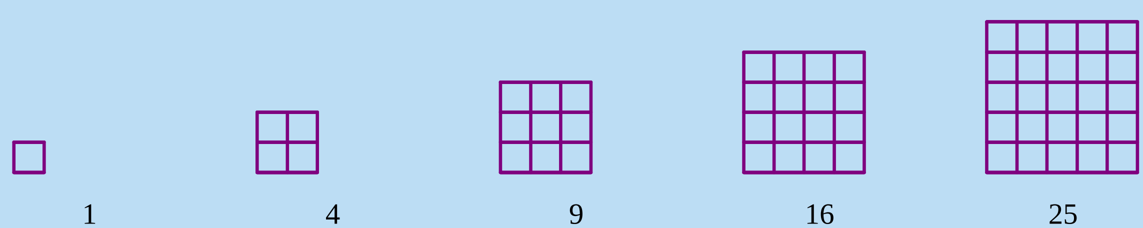
As noted above (informally), a sequence is an ordered collection. There are many sequences or number patterns in the world, some of which we can describe numerically or with a formula. The triangular numbers and the square numbers, shown below, are patterns with geometric names. Can you see why?

Triangular numbers: $1, 3, 6, 10, 15, \cdots$

Figure 7.1. 1



Square numbers: $1, 4, 9, 16, 25, \cdots$



Each number in a sequence is called a **term**. In the sequence $-3, -1, 1, \cdots$, the first term is -3, the second term is -1 and the third term is 1. The three dots at the end indicate this is an **infinite sequence**, meaning it continues indefinitely. It is useful to recognize the pattern of sequences and be able to describe them. The first step in writing a formula for a sequence is to understand how the next number in the sequence is generated.

Example 7.1.1. For each sequence, describe how subsequent terms may be determined. Then, state the following two terms.

a) $1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots$ b) $1, 3, 6, 10, 15, \cdots$ c) $1, 4, 9, 16, 25, \cdots$

d) $-7, -4, -1, 2, 5, 8, \cdots$ e) $64, 32, 16, 8, 4, 2, \cdots$ f) $37, 32, 27, 22, 17, 12, \cdots$

g) $81, 54, 36, 24, 16, \cdots$ h) $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots$ i) $-5, \frac{15}{2}, -\frac{45}{4}, \ldots$

Solution.

a) The numbers $1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots$ are called the **Fibonacci Numbers**.

1	1	2	3	5	8	13	21	…
↘	↓↘	↓↘	↓↘	↓↘	↓↘	↓↘	↓	…
	1+1=2	1+2=3	2+3=5	3+5=8	5+8=13	8+13=21	13+21=34	…

Noting that $1+1=2$, $1+2=3$, $2+3=5$, and so forth, we deduce that adding the two previous terms gives us the next term. So, to find the term after 34, we add 21 and 34 to get 55. Then, to find the term after 55, we add the previous two terms and have $34+55=89$.

Note: The Fibonacci sequence might have been known as early as 200 BCE in India and it appears in nature, like the number of petals of natural flowers. It is named after the Italian mathematician Leonardo of Pisa, also known as Fibonacci, and was the solution to a question he posed in his book *Liber Abaci*, published in 1202 CE. However, the most important contribution of Leonardo of Pisa to Western European mathematics was the concept tof zero which he had learned about from studying the Hindu-Arabic arithmetic system while growing up in North Africa. This sequence has many interesting properties and we encourage the reader to investigate them.

b) The numbers $1, 3, 6, 10, 15, \cdots$ are the triangular numbers. Between the first and second term, 2 is added; between the second and third term, 3 is added; between the third and forth term, 4 is added. We next add 5, then 6, then 7, etc. The two terms following 15 are $15+6=21$ and $21+7=28$.

c) The numbers $1, 4, 9, 16, 25, \cdots$ are the square numbers. The terms in the sequence are the squares of the consecutive natural numbers. The next two terms are $6^2 = 36$ and $7^2 = 49$.

Another way to describe this pattern is to say you start with 1, then add 3, then 5, then 7, e.g. add consecutive odd numbers.

d) For $-7, -4, -1, 2, 5, 8, \cdots$ we are adding the same amount, 3, in going from each term to the next. The two terms following 8 will be 11 and 14.

e) Each term in $64, 32, 16, 8, 4, 2, \cdots$ is half of the previous term. We can think of dividing the previous term by 2 to arrive at the next term, or multiplying by $\frac{1}{2}$. We will see shortly that the latter is more useful. The two terms following 2 are 1 and $\frac{1}{2}$.

f) In the sequence $37, 32, 27, 22, 17, 12, \cdots$ we are adding the same amount each time, like we did in part d, but in this case the amount we are adding is −5. Thus, the next two terms are 7 and 2.

g) For $81, 54, 36, 24, 16, \cdots$ we note that we are not adding the same amount each time as we go from term to term. We look for an amount, *x*, that we multiply by to take us from term to term, starting with the first two terms.

$$81x = 54$$
$$x = \frac{54}{81} = \frac{2}{3}$$

Now we must check each subsequent pair of terms, so we move on to 54 and 36.

$$54x = 36$$
$$x = \frac{36}{54} = \frac{2}{3}$$

We leave it to the reader to confirm for the rest of the terms in this sequence that we get the next term by multiplying the previous term by $\frac{2}{3}$. Following 16, we have $16 \cdot \frac{2}{3} = \frac{32}{3}$ and $\frac{32}{3} \cdot \frac{2}{3} = \frac{64}{9}$.

h) The sequence $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots$ is a bit different than what we've seen so far. The numerators and denominators are increasing by one each time, and the sign is alternating. A sequence in which the sign of its terms is alternating between positive and negative is referred to as an **alternating sequence**. For this sequence, we find the next two terms are $\frac{5}{6}$ and $-\frac{6}{7}$.

i) The sequence $-5, \frac{15}{2}, -\frac{45}{4}, \ldots$ is like parts *e* and *g* in that the absolute value of each term is the previous term multiplied by $\frac{3}{2}$, and it's also like *h* in that it's an alternating sequence. The next two terms in the sequence are $\frac{135}{8}$ and $-\frac{405}{16}$.

□

In the previous example, we have written the next two terms based only on the way we see patterns. The actual pattern may be different. For example, $-3, -1, 1, \cdots$ may continue as $-3, -1, 1, 3, 5, 7, \cdots$ or as $-3, -1, 1, -30, -10, 10, \cdots$, or as any of an infinite number of possibilities.

In general, there are two broad ways formulas are written, recursively and explicitly. A **recursive formula** is a formula that defines each new term by one or more of its previous terms, whereas an **explicit formula** describes how to state any term in the sequence.

The input is usually the set of natural numbers or a subset of the natural numbers, although sometimes it is the whole numbers or even the integers. We can write a sequence using function notation, as we will see later. However, we often use a letter, say *a*, to name the terms. In the sequence $-3, -1, 1, \cdots$, we can write the first term as $a_1 = -3$, the second as $a_2 = -1$, the third as $a_3 = 1$, and so on. That is, $\boldsymbol{a_n}$ is the ***n*th term** of the sequence[1] for $n = 1, 2, 3, \cdots$ Note that we may also talk about the 'starting' term as a_0 (read 'a naught').

Recursive Formulas

Suppose we want to write a formula for the sequence of odd integers starting with −3, e.g. $-3, -1, 1, 3, 5, 7, 9, \cdots$. While we can do so either recursively or explicitly, we begin with a recursive formula. If a_n is the *n*th term, this tells us that the 'previous' term is a_{n-1} and the 'subsequent' term is a_{n+1}. It is important to note here that the words 'previous' and 'subsequent' are relative to the *n*th term. In the case of our sequence $-3, -1, 1, 3, 5, 7, 9, \cdots$, noting that each term is two units larger than the term before it, we have $a_n = a_{n-1} + 2$. To complete our formula, we need the starting term, $a_1 = -3$. Then we will determine a_2 by setting $n = 2$; for a_3, we will set $n = 3$; and so forth. Thus, $n = 2, 3, 4, \cdots$, giving us the recursive formula

$$a_n = a_{n-1} + 2, \; a_1 = -3, \; n = 2, 3, 4, \cdots$$

[1] Of course, the choice of the letter *a* and the index *n* is arbitrary. We could just as well use, for example, b_n or c_k.

This may be read as 'the current term is the previous term plus 2, where we start with −3 and input the natural numbers $2, 3, 4, \cdots$' Using the formula, we generate each new term by adding 2 to the term before.

$$\begin{aligned} a_1 &= -3 \\ a_2 &= a_1 + 2 = -3 + 2 = -1 \quad \text{note: } n = 2 \Rightarrow n - 1 = 1 \\ a_3 &= a_2 + 2 = -1 + 2 = 1 \quad \text{note: } n = 3 \Rightarrow n - 1 = 2 \\ a_4 &= a_3 + 2 = 1 + 2 = 3 \quad \text{note: } n = 4 \Rightarrow n - 1 = 3 \\ a_5 &= a_4 + 2 = 3 + 2 = 5 \quad \text{note: } n = 5 \Rightarrow n - 1 = 4 \end{aligned}$$

When a formula is written recursively, we must know the previous term or terms to find a new term. Thus, to find the 101^{st} term in the recursive sequence from above, we would need the 100^{th} term. Another example of a recursively defined sequence is the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots$ from **Example 7.1.1**. Each of the terms in this sequence, after the first two, is the sum of the previous two terms. We may define this sequence recursively as follows.

$$a_n = a_{n-2} + a_{n-1}, \ a_1 = 1, \ a_2 = 1, \ n = 3, \ 4, \ 5, \cdots$$

Explicit Formulas

Explicit formulas do not rely on knowing a previous term or terms. Rather, they are formulas that allow us to find any term in the sequence. To find an explicit formula for the odd integers, starting with −3, one way we might write the formula is $f(n) = 2n - 5, \ n = 1, \ 2, \ 3, \cdots$. Then, to find any term in the sequence, we evaluate using the formula.

$$\begin{aligned} f(1) &= 2(1) - 5 = -3 \quad \text{the first term} \\ f(2) &= 2(2) - 5 = -1 \quad \text{the second term} \\ f(3) &= 2(3) - 5 = 1 \quad \text{the third term} \\ f(4) &= 2(4) - 5 = 3 \quad \text{the fourth term} \\ f(5) &= 2(5) - 5 = 5 \quad \text{the fifth term} \end{aligned}$$

If we wanted the 102^{nd} term in this sequence, we simply find $f(102) = 2(102) - 5 = 199$. Note that, although we used the traditional function notation $f(n) = 2n - 5$, we could have just as easily written the formula as $a_n = 2n - 5$. Also note that we could have started with any n value. For example, we could define the sequence as $a_n = 2n - 7, \ n = 2, \ 3, \ 4, \cdots$.

Although there is an explicit formula for each sequence, sometimes it is not easy to see. For example, the explicit formula for the Fibonacci sequence is

$$a_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}, \quad n = 1,\ 2,\ 3, \cdots$$

We leave it to the reader to verify that this equation does indeed produce the Fibonacci sequence. In either way of expressing a sequence, we refer to a_n as the **general term** or the ***n*th term formula**.

Sequence Definition and Notation

Definition 7.1. A **sequence** is a function $f(n)$ with the domain of all natural numbers or a subset of consecutive natural numbers.[2] The subscript notation a_n is often used for function values, and is referred to as the general term or the nth term formula. It is customary to write a_n with n values starting at $n = 1$.

We may also write a sequence as $\{a_n\}_{n=\text{starting value}}^{\text{end value or } \infty}$ and if there is no confusion about the domain (for example, the entire set of natural numbers) we may drop the *n* values and just write $\{a_n\}$. For example, we can write the sequence of odd natural numbers, starting with −3, as

$$f(n) = 2n - 5,\ n = 1,\ 2,\ 3, \cdots$$
$$a_n = 2n - 5,\ n = 1,\ 2,\ 3, \cdots$$
$$\{2n - 5\}_{n=1}^{\infty}$$
$$\{2n - 5\}$$

Example 7.1.2. Find the first four terms of each sequence.

1. $f(n) = -3n + 11,\ n = 1,\ 2,\ 3, \cdots$

2. $a_n = a_{n-1}\left(\frac{1}{2}\right)^{n-2},\ a_1 = 8,\ n = 2,\ 3,\ 4, \cdots$

Solution.

1. For $f(n) = -3n + 11$, we find

$$f(1) = -3(1) + 11 = 8$$
$$f(2) = -3(2) + 11 = 5$$
$$f(3) = -3(3) + 11 = 2$$
$$f(4) = -3(4) + 11 = -1$$

The sequence is $8,\ 5,\ 2, -1, \cdots$

[2] The domain may also be a subset of consecutive integers; for example, the whole numbers. The starting n value could be 0 or any integer.

2. For the recursively defined sequence $a_n = a_{n-1}\left(\frac{1}{2}\right)^{n-2}$, we begin with $a_1 = 8$, then find the second term by substituting $n = 2$ in the formula.

$$a_1 = 8$$

$$a_2 = a_1\left(\frac{1}{2}\right)^{2-2} = 8(1) = 8 \quad \text{substitute } n = 2 \text{ in formula: } a_n = a_{n-1}\left(\frac{1}{2}\right)^{n-2}$$

$$a_3 = a_2\left(\frac{1}{2}\right)^{3-2} = 8\left(\frac{1}{2}\right) = 4 \quad \text{substitute } n = 3 \text{ in formula}$$

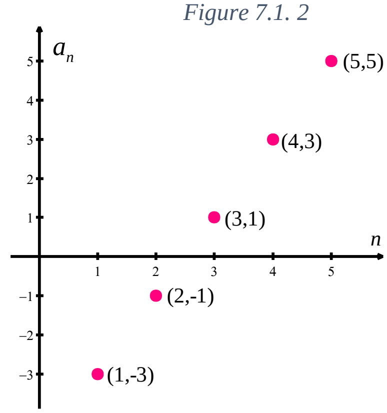
$$a_4 = a_3\left(\frac{1}{2}\right)^{4-2} = 4\left(\frac{1}{4}\right) = 1 \quad \text{substitute } n = 4 \text{ in formula}$$

The sequence is $8,\ 8,\ 4,\ 1, \cdots$.

□

Because a sequence is a function, we can graph a sequence. However, it will only consist of isolated points as in the following graph of $a_n = 2n - 5$, for $n = 1$, 2, 3, 4, 5.

Figure 7.1. 2



Note that we can only graph a finite number of terms, although the domain in this example is all natural numbers. As a word of caution, we should not give in to the desire to connect the points together by lines or curves. It is worth observing however that, were we able to connect the points, we would have a line with slope 2. Since we do not have a line, the most we can say is that we have a constant rate of change of 2, which is consistent with the nature of a linear function, and indicative of an **arithmetic sequence**.

Example 7.1.3. Find the first four terms of each sequence and graph each sequence.

1. $\left\{(-1)^{n+1}\frac{n}{n+1}\right\}_{n=1}^{\infty}$

2. $\left\{\frac{5}{3^{n-1}}\right\}_{n=1}^{\infty}$

Solution.

1. The first four terms of the explicit formula $\left\{(-1)^{n+1}\frac{n}{n+1}\right\}_{n=1}^{\infty}$ are found as follows.

n	1	2	3	4
$(-1)^{n+1}\dfrac{n}{n+1}$	$(-1)^{1+1}\dfrac{1}{1+1}=\dfrac{1}{2}$	$(-1)^{2+1}\dfrac{2}{2+1}=-\dfrac{2}{3}$	$(-1)^{3+1}\dfrac{3}{3+1}=\dfrac{3}{4}$	$(-1)^{4+1}\dfrac{4}{4+1}=-\dfrac{4}{5}$

This sequence, with its first four terms listed, is $\frac{1}{2},-\frac{2}{3},\frac{3}{4},-\frac{4}{5},\cdots$

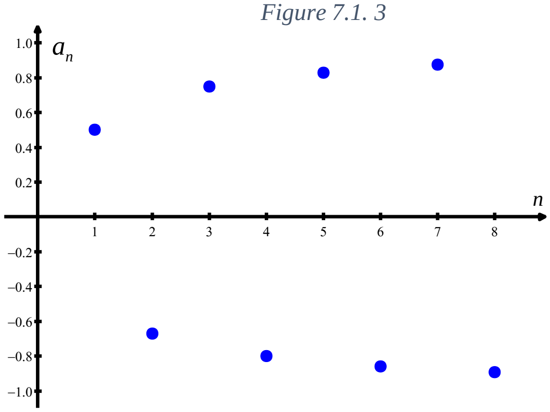
2. For $\left\{\dfrac{5}{3^{n-1}}\right\}_{n=1}^{\infty}$, we find the first four terms with n values of 1, 2, 3 and 4.

n	1	2	3	4
$\dfrac{5}{3^{n-1}}$	$\dfrac{5}{3^{1-1}}=\dfrac{5}{1}=5$	$\dfrac{5}{3^{2-1}}=\dfrac{5}{3}$	$\dfrac{5}{3^{3-1}}=\dfrac{5}{9}$	$\dfrac{5}{3^{4-1}}=\dfrac{5}{27}$

We have the sequence $5,\frac{5}{3},\frac{5}{9},\frac{5}{27},\cdots$

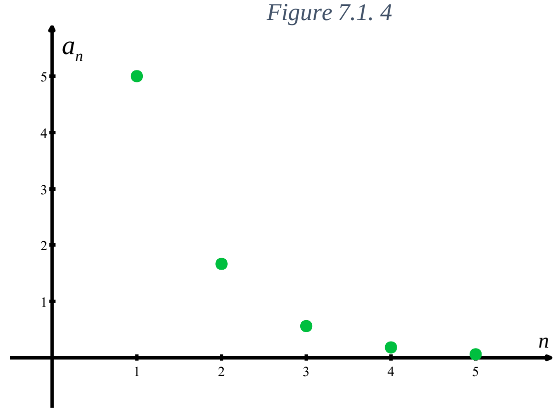
The graphs of several terms of these sequences are shown below.

Figure 7.1. 3



$$\left\{(-1)^{n+1}\frac{n}{n+1}\right\}_{n=1}^{\infty}=\left\{\frac{1}{2},-\frac{2}{3},\frac{3}{4},-\frac{4}{5},\ldots\right\}$$

Figure 7.1. 4



$$\left\{\frac{5}{3^{n-1}}\right\}_{n=1}^{\infty}=\left\{5,\frac{5}{3},\frac{5}{9},\frac{5}{27},\cdots\right\}$$

□

As seen in the blue graph on the left, the sequence values alternate between positive and negative. Any sequence with alternating positive and negative values is called an **alternating sequence**. The green graph, to the right, has the general shape of an exponential function. This is consistent with the nature of a **geometric sequence**, which we will define later.

Arithmetic and Geometric Sequences

As you may have noticed, sequences can vary greatly. We will focus in this chapter on two types: arithmetic and geometric. Arithmetic sequences are sequences in which each pair of consecutive terms differ by a fixed amount. Thus, the example of consecutive odd integers starting with −3 is an arithmetic sequence, having a fixed amount of 2 between terms. Geometric sequences are sequences in which each term is a constant multiple of the term before it. For example, the sequence in **Example 7.1.3** with terms $5, \frac{5}{3}, \frac{5}{9}, \frac{5}{27}, \cdots$ is geometric since we multiply each term by $\frac{1}{3}$ to arrive at the next term.

Definition 7.2. The sequence $\{a_n\}$ is called

- an **arithmetic sequence** if $a_n = a_{n-1} + d$ for some constant d and all values of n and $n-1$ in its domain. The constant d is called the **common difference**.
- a **geometric sequence** if $a_n = ra_{n-1}$ for some nonzero constant r and all values of n and $n-1$ in its domain. The constant r is called the **common ratio**.

Note that $a_n = a_{n-1} + d$ is equivalent to $a_n - a_{n-1} = d$ and $a_n = ra_{n-1}$ is equivalent to $\frac{a_n}{a_{n-1}} = r$.

Example 7.1.4. Identify the sequences from **Example 7.1.1** as arithmetic, geometric or neither and explain your reasoning.

Solution.

Sequence	Type	Reasoning
a) $1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots$	Neither	No common difference and no common multiple.
b) $1, 3, 6, 10, 15, \cdots$	Neither	No common difference and no common multiple.
c) $1, 4, 9, 16, 25, \cdots$	Neither	No common difference and no common multiple.
d) $-7, -4, -1, 2, 5, 8, \cdots$	Arithmetic	Each term is 3 more that the previous term.
e) $64, 32, 16, 8, 4, 2, \cdots$	Geometric	Each term is $\frac{1}{2}$ of the previous term.
f) $37, 32, 27, 22, 17, 12, \cdots$	Arithmetic	Each term is the previous term plus −5.
g) $81, 54, 36, 24, 16, \cdots$	Geometric	Each term is $\frac{2}{3}$ of the previous term.
h) $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \cdots$	Neither	No common difference and no common multiple.
i) $-5, \frac{15}{2}, -\frac{45}{4}, \cdots$	Geometric	Each term is $-\frac{3}{2}$ of the previous term.

□

Example 7.1.5. Determine if the following sequences are arithmetic, geometric or neither. If arithmetic, find the common difference d; if geometric, find the common ratio r.

1. $a_n = 2n+1,\ n=1,2,3,\cdots$ 2. $a_n = (-1)^{n+1}\dfrac{n}{n+1},\ n=1,2,3,\cdots$ 3. $a_n = \dfrac{5}{3^{n-1}},\ n=1,2,3,\cdots$

Solution. A good rule of thumb to keep in mind when working with sequences is 'When in doubt, write it out!' Writing out the first several terms can help you identify the pattern of the sequence.

1. The sequence $a_n = 2n+1$ generates the odd numbers $3, 5, 7, 9, \cdots$. Computing the first few differences, we have $a_2 - a_1 = 2$, $a_3 - a_2 = 2$ and $a_4 - a_3 = 2$. This suggests the sequence is arithmetic. To verify this, we find

$$\begin{aligned} a_n - a_{n-1} &= [2n+1] - [2(n-1)+1] \\ &= 2n+1-2n+2-1 \\ &= 2 \end{aligned}$$

This establishes that the sequence is arithmetic with common difference $d = 2$.

2. We write out the first several terms: $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \cdots$. We find $a_2 - a_1 = -\frac{2}{3} - \frac{1}{2} = -\frac{7}{6}$ and $a_3 - a_2 = \frac{3}{4} + \frac{2}{3} = \frac{17}{12}$. Hence, the sequence is not arithmetic. To see if it is geometric, we compute $\frac{a_2}{a_1} = -\frac{4}{3}$ and $\frac{a_3}{a_2} = -\frac{9}{8}$. Since there is no 'common ratio', the sequence is not geometric either.

3. The sequence has terms $5, \frac{5}{3}, \frac{5}{9}, \frac{5}{27}, \cdots$ and we see right off that $\frac{5}{3} - 5 \neq \frac{5}{9} - \frac{5}{3}$ so the sequence is not arithmetic. Computing the first few ratios gives us $\frac{a_2}{a_1} = \frac{1}{3}$, $\frac{a_3}{a_2} = \frac{1}{3}$ and $\frac{a_4}{a_3} = \frac{1}{3}$ and suggests that the sequence is geometric. To verify this, we show $\frac{a_n}{a_{n-1}} = r$ for all n.

$$\frac{a_n}{a_{n-1}} = \frac{\frac{5}{3^{n-1}}}{\frac{5}{3^{(n-1)-1}}} = \frac{5}{3^{n-1}} \cdot \frac{3^{n-2}}{5} = \frac{3^{n-2}}{3^{n-1}} = \frac{3^n 3^{-2}}{3^n 3^{-1}}$$

After a bit more manipulation, we have $\frac{a^n}{a^{n-1}} = \frac{3^1}{3^2} = \frac{1}{3}$ so the sequence is geometric with common ratio $r = \frac{1}{3}$.

□

As stated earlier, we will focus on writing arithmetic and geometric sequences in this chapter. Let's proceed by looking more closely at the arithmetic sequence $-7, -3, 1, 5, 9, 13, 17, \cdots$ to see how we might write a formula. Noting that the common difference is 4, we could easily come up with the recursive formula $a_n = a_{n-1} + 4$, but let's go a step further and look for an explicit formula.

$\mathbf{a_1 = -7}$ ↘ $+4$ ↗ $\mathbf{a_2 = -3}$ ↘ $+4$ ↗ $\mathbf{a_3 = 1}$ ↘ $+4$ ↗ $\mathbf{a_4 = 5}$ ↘ $+4$ ↗ $\mathbf{a_5 = 9}$ ↘ $+4$ ↗ $\mathbf{a_6 = 13}$ ↘ $+4$ ↗ $\mathbf{a_7 = 17}$

To get to the ***second*** term, we add ***one*** 4 to the first term; to get the the ***third*** term, we add ***two*** 4's to the first term; to get to the ***fourth*** term, we add ***three*** 4's to the first term; and so on until the ***seventh*** term where we add ***six*** 4's to the first term.

$$a_1 = -7$$
$$a_2 = -7 \underbrace{+\ 4}_{\mathbf{1}\text{ addend}}$$
$$a_3 = -7 \underbrace{+\ 4\ +\ 4}_{\mathbf{2}\text{ addends}}$$
$$a_4 = -7 \underbrace{+\ 4\ +\ 4\ +\ 4}_{\mathbf{3}\text{ addends}}$$

In other words, to get to ***any*** term, we add the common difference to the starting term ***one less*** time than the position of our term.

This same logic may be seen in a geometric sequence such as $8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \cdots$

$\mathbf{a_1 = 8}$ ↘ $\times\frac{1}{2}$ ↗ $\mathbf{a_2 = 4}$ ↘ $\times\frac{1}{2}$ ↗ $\mathbf{a_3 = 2}$ ↘ $\times\frac{1}{2}$ ↗ $\mathbf{a_4 = 1}$ ↘ $\times\frac{1}{2}$ ↗ $\mathbf{a_5 = \frac{1}{2}}$ ↘ $\times\frac{1}{2}$ ↗ $\mathbf{a_6 = \frac{1}{4}}$ ↘ $\times\frac{1}{2}$ ↗ $\mathbf{a_7 = \frac{1}{8}}$

Thus, in a geometric sequence, rather than add to get to the next term, we multiply. The basic logic is the same. To get to the ***second*** term, we multiply the first term by $\frac{1}{2}$ ***one*** time; to get the the ***third*** term, we multiply the first term by $\frac{1}{2}$ ***two*** times; to get to the ***fourth*** term, we multiply the first term by $\frac{1}{2}$ ***three*** times; and so on until the ***seventh*** term where we multiply the first term by $\frac{1}{2}$ ***six*** times.

$$a_1 = 8$$
$$a_2 = 8 \times \underbrace{\frac{1}{2}}_{1\text{ factor}}$$
$$a_3 = 8 \times \underbrace{\frac{1}{2} \times \frac{1}{2}}_{2\text{ factors}}$$
$$a_4 = 8 \times \underbrace{\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}}_{3\text{ factors}}$$

In other words, to get to ***any*** term, we multiply the first term by the common ratio ***one less*** time than the position of our term. We have the following result.

Theorem 7.1. Formulas for Arithmetic and Geometric Sequences

- An arithmetic sequence with the first term a_1 and common difference d is given by

$$a_n = a_1 + (n-1)d\,,\ n = 1,\ 2,\ 3, \cdots$$

- A geometric sequence with the first term a_1 and common ratio r is given by

$$a_n = a_1 r^{n-1},\ n = 1,\ 2,\ 3, \cdots$$

Note: A sequence in the form $a_n = (\text{an expression of } n)$ is arithmetic if the expression is linear and is geometric if the expression is exponential.

Example 7.1.6. Find an explicit formula for the nth term of the following sequences. Assume each sequence is arithmetic or geometric.

1. $-6, -1, 4, 9, \cdots$
2. $-5, \frac{10}{3}, -\frac{20}{9}, \frac{40}{27}, \cdots$

Solution.

1. Since each term is 5 units larger than the previous term, the common difference is 5. Noting that the first term is −6, the formula for any term is

$$a_n = -6 + 5(n-1)$$
$$a_n = 5n - 11$$

We can check by evaluating the formula to confirm any term. For example, using the formula, we find $a_3 = 5(3) - 11 = 4$ and this agrees with the third term in the given sequence.

2. There is not a common difference, so we check to see if there is a common multiple.

$$-5r=\frac{10}{3}\Rightarrow r=-\frac{2}{3}$$

After verifying that $\left(\frac{10}{3}\right)\left(-\frac{2}{3}\right)=-\frac{20}{9}$ and $\left(-\frac{20}{9}\right)\left(-\frac{2}{3}\right)=\frac{40}{27}$, we use $r=-\frac{2}{3}$ and $a_1=-5$ to write a formula for this geometric sequence.

$$a_n=-5\left(-\frac{2}{3}\right)^{n-1}$$

□

Example 7.1.7. The first term of an arithmetic sequence is 15 and its 10th term is −12. Find the common difference and an explicit formula.

Solution. We are given that $a_1=15$ and $a_{10}=-12$. Using the formula $a_n=a_1+(n-1)d$ for $n=10$, we find the common difference as follows.

$$\begin{aligned}a_{10}&=a_1+(10-1)d\\-12&=15+9d\\9d&=-27\\d&=-3\end{aligned}$$

The common difference $d=-3$ and the first term $a_1=15$ give us the formula.

$$\begin{aligned}a_n&=a_1+(n-1)d\\a_n&=15+(n-1)(-3)\\a_n&=-3n+18\end{aligned}$$

□

Example 7.1.8. The first term of an arithmetic sequence is −12 and its common difference is 5. Which term of this sequence has value 58?

Solution. We are given that $a_1=-12$ and $d=5$. We want to find the value of n for which $a_n=58$. Using the formula from **Theorem 7.1**, we can solve for n.

$$\begin{aligned}a_n&=a_1+(n-1)d\\58&=-12+(n-1)(5)\\58&=5n-17\\5n&=75\\n&=15\end{aligned}$$

The number 58 is the 15th term of this arithmetic sequence.

□

Example 7.1.9. Find a_{13} in the arithmetic sequence with $a_{10}=-14$ and $a_{30}=-54$.

Solution. We are given that $a_{10} = -14$ and $a_{30} = -54$, and we'd like to find both the first term and the common difference in our search for a_{13}. Using the formula $a_n = a_1 + (n-1)d$, we get the system of equations:

$$\begin{cases} a_{10} = a_1 + (10-1)d \\ a_{30} = a_1 + (30-1)d \end{cases} \Rightarrow \begin{cases} -14 = a_1 + 9d \\ -54 = a_1 + 29d \end{cases}$$

Using elimination, we find $d = -2$ and $a_1 = 4$. So $a_{13} = 4 + (13-1)(-2) = -20$.

□

Example 7.1.10. The fifth term of a geometric sequence is $\frac{4}{5}$ and its common ratio is $\frac{2}{3}$. Find the first term of this sequence. Find an explicit formula.

Solution. We are given that $a_5 = \frac{4}{5}$ and $r = \frac{2}{3}$. Using the formula $a_n = a_1 r^{n-1}$ for $n = 5$, we can solve for a_1.

$$\begin{aligned} a_5 &= a_1 r^{5-1} \\ \frac{4}{5} &= a_1 \left(\frac{2}{3}\right)^4 \\ \frac{4}{5} &= \frac{16}{81} a_1 \\ a_1 &= \frac{4}{5} \cdot \frac{81}{16} = \frac{81}{20} \end{aligned}$$

The first term is $a_1 = \frac{81}{20}$ and the formula is $a_n = \left(\frac{81}{20}\right)\left(\frac{2}{3}\right)^{n-1}$.

□

7.1 Exercises

1. What are the main differences between using a recursive formula and using an explicit formula to describe an arithmetic sequence?
2. Describe the similarities between exponential functions and geometric sequences. How are they different?

In Exercises 3 – 12, describe how subsequent terms may be determined and state the following two terms.

3. $3,\ 5,\ 7,\ 9,\cdots$
4. $\frac{1}{16},-\frac{1}{8},\ \frac{1}{4},-\frac{1}{2},\cdots$
5. $1,\ \frac{2}{3},\ \frac{4}{5},\ \frac{8}{7},\cdots$
6. $1,\ \frac{2}{3},\ \frac{1}{3},\ \frac{4}{27},\cdots$
7. $1,\ \frac{1}{4},\ \frac{1}{9},\ \frac{1}{16},\cdots$
8. $4,\ 7,\ 12,\ 19,\ 28,\cdots$
9. $-4,\ 2,-10,\ 14,-34,\cdots$
10. $1,\ 1,\ \frac{4}{3},\ 2,\ \frac{16}{5},\cdots$
11. $-\frac{1}{2},\ \frac{1}{4},-\frac{1}{8},\ \frac{1}{16},\cdots$
12. $1,\ 2,\ 6,\ 24,\ 120,\cdots$

In Exercises 13 – 24, find the first four terms of each sequence. Assume n is a natural number, $n \geq 2$.

13. $a_1 = 9$, $a_n = a_{n-1} + n$
14. $a_1 = 3$, $a_n = (-3)a_{n-1}$
15. $a_1 = -4$, $a_n = \frac{a_{n-1}+2n}{a_{n-1}-1}$
16. $a_1 = -1$, $a_n = \frac{(-3)^{n-1}}{a_{n-1}-2}$
17. $a_1 = -30$, $a_n = (2+a_{n-1})\left(\frac{1}{2}\right)^n$
18. $a_1 = 3$, $a_n = a_{n-1} - 1$
19. $a_1 = 12$, $a_n = \frac{a_{n-1}}{100}$
20. $a_1 = 2$, $a_n = 3a_{n-1} + 1$
21. $a_1 = -2$, $a_n = \frac{a_{n-1}}{(n+1)(n+2)}$
22. $a_1 = 117$, $a_n = \frac{1}{a_{n-1}}$
23. $a_1 = \frac{1}{24}$, $a_2 = 1$, $a_n = (2a_{n-2})(3a_{n-1})$
24. $a_1 = -1$, $a_2 = 5$, $a_n = a_{n-2}(3 - a_{n-1})$

In Exercises 25 – 36, determine if the sequence is arithmetic, geometric or neither. If it is arithmetic, find the common difference d; if it is geometric, find the common ratio r. Plot the points of the sequence on a graph.

25. $-6, -12, -24, -48, -96, \cdots$

26. $11.4,\ 9.3,\ 7.2,\ 5.1,\ 3, \cdots$

27. $\frac{1}{3},\ \frac{1}{6},\ \frac{1}{12},\ \frac{1}{24}, \cdots$

28. $-1,\ \frac{1}{2}, -\frac{1}{4},\ \frac{1}{8}, -\frac{1}{16}, \cdots$

29. $17,\ 5, -7, -19, \cdots$

30. $4,\ 16,\ 64,\ 256,\ 1024, \cdots$

31. $6,\ 8,\ 11,\ 15,\ 20, \cdots$

32. $2,\ 22,\ 222,\ 2222, \cdots$

33. $0.9,\ 9,\ 90,\ 900, \cdots$

34. $\{3n-5\}_{n=1}^{\infty}$

35. $a_n = n^2 + 3n + 2$, $n \geq 1$

36. $\left\{3\left(\frac{1}{5}\right)^{n-1}\right\}_{n=1}^{\infty}$

In Exercises 37 – 45, find an explicit formula for the nth term of the sequence.

37. $3,\ 5,\ 7,\ 9, \cdots$

38. $32,\ 24,\ 16,\ 8, \cdots$

39. $-2, -4, -8, -16, \cdots$

40. $1,\ 3,\ 9,\ 27, \cdots$

41. $-5,\ 95,\ 195,\ 295, \cdots$

42. $-17, -217, -417, -617, \cdots$

43. $-1, -\frac{4}{5}, -\frac{16}{25}, -\frac{64}{125}, \cdots$

44. $2,\ \frac{1}{3},\ \frac{1}{18},\ \frac{1}{108}, \cdots$

45. $3, -1,\ \frac{1}{3}, -\frac{1}{9}, \cdots$

In Exercises 46 – 49, use the given information to write the first five terms of the arithmetic sequence.

46. $a_1 = -25$, $d = -9$

47. $a_1 = 0$, $d = \frac{2}{3}$

48. $a_1 = 17$, $a_7 = -31$

49. $a_{13} = -60$, $a_{33} = -160$

In Exercises 50 – 53, use the given information to write the first five terms of the geometric sequence.

50. $a_1 = 8$, $r = 0.3$

51. $a_1 = 5$, $r = \frac{1}{5}$

52. $a_7 = 64$, $a_{10} = 512$

53. $a_6 = 25$, $a_8 = 6.25$

In Exercises 54 – 59, use the given information to find the specified term of the arithmetic sequence.

54. Find a_5 if $a_1 = 3$ and $d = 4$

55. Find a_6 if $a_1 = 6$ and $d = 7$

56. Find a_1 if $a_6 = 12$ and $a_{14} = 28$

57. Find a_1 if $a_7 = 21$ and $a_{15} = 42$

58. Find a_4 if $a_1 = 33$ and $a_7 = -15$

59. Find a_{21} if $a_3 = -17.1$ and $a_{10} = -15.7$

In Exercises 60 – 65, use the given information to find the specified term of the geometric sequence.

60. Find a_5 if $a_1 = 2$ and $r = 3$

61. Find a_4 if $a_1 = 16$ and $r = -\frac{1}{3}$

62. Find a_{12} in the sequence $-1, 2, -4, 8, \cdots$

63. Find a_7 in the sequence $-2, \frac{2}{3}, -\frac{2}{9}, \frac{2}{27}, \cdots$

64. Find a_8 if $a_1 = 4$ and $a_n = -3a_{n-1}$

65. Find a_{12} if $a_n = -\left(-\frac{1}{3}\right)^{n-1}$

7.2 Series

Learning Objectives

- Use summation notation.
- Find the sum of a finite arithmetic sequence.
- Solve applications of arithmetic series.
- Find the value of an infinite geometric series with a finite sum.
- Find the sum of a finite geometric sequence.
- Solve applications of geometric series.

As we get older, we benefit or suffer from our accumulated behavior; we may enjoy retirement if we have saved yearly or we may suffer from diseases related to being overweight if we have consumed too many calories daily. If a life is a sequence of events, then the future is influenced by the accumulation of those events. In mathematics, the sum of numbers in a sequence is called a **series**. Adding finitely many of those numbers results in a **finite series**, while adding infinitely many of those numbers is an **infinite series**. We express a series, or the sum of a sequence, using the following notation.

Summation Notation

Definition 7.3. Summation Notation: Consider the sequence $\{a_n\}$.

- The sum of the terms a_n, from $n = j$ to $n = p$, can be written as $\sum_{n=j}^{p} a_n = a_j + a_{j+1} + \cdots + a_p$.
- The sum $a_j + a_{j+1} + a_{j+2} + \cdots$ is written as $\sum_{n=j}^{\infty} a_n = a_j + a_{j+1} + a_{j+2} + \cdots$.

The symbol Σ is the Greek capital letter sigma and is read as 'sigma' or 'summation'. The letter n is the **index**. The value j is called the **lower limit** and p is called the **upper limit**.

Note that in the upper limit of the summation $\sum_{n=j}^{p} a_n$ we did not write $n = p$, since n being the index is clear from the lower limit of the summation. It is, hower, acceptable to write the upper limit at $n = p$ if you prefer. Also note that we will refer to the value of a series, if it exists, as a sum. The word **sum** may refer to the value of a finite or infinite summation.

Example 7.2.1. Find the following sums.

1. $\sum_{n=2}^{6}(3n-5)$
2. $\sum_{n=1}^{4}\frac{52}{100^n}$
3. $\sum_{k=0}^{4}\frac{(-1)^{k+1}}{k+1}(x-2)^k$

Solution.

1. To evaluate the series $\sum_{n=2}^{6}(3n-5)$, we must find the sum of the sequence generated by the formula $a_n=3n-5$, where the first term has $n=2$ and the last term has $n=6$. We can do this by finding each term and adding them together.

$$\sum_{n=2}^{6}(3n-5)=\underbrace{(3\cdot2-5)}_{a_2}+\underbrace{(3\cdot3-5)}_{a_3}+\underbrace{(3\cdot4-5)}_{a_4}+\underbrace{(3\cdot5-5)}_{a_5}+\underbrace{(3\cdot6-5)}_{a_6}$$
$$=1+4+7+10+13$$
$$=35$$

2. For $\sum_{n=1}^{4}\frac{52}{100^n}$, we add the sequence generated by $a_n=\frac{52}{100^n}$, starting with $n=1$ and going to $n=4$.

$$\sum_{n=1}^{4}\frac{52}{100^n}=\underbrace{\left(\frac{52}{100^1}\right)}_{a_1}+\underbrace{\left(\frac{52}{100^2}\right)}_{a_2}+\underbrace{\left(\frac{52}{100^3}\right)}_{a_3}+\underbrace{\left(\frac{52}{100^4}\right)}_{a_4}=\frac{52}{100}+\frac{52}{10000}+\frac{52}{1000000}+\frac{52}{100000000}$$

After obtaining a common denominator and simplifying, we have $\frac{52525252}{100000000}=0.52525252$.

3. The series $\sum_{k=0}^{4}\frac{(-1)^{k+1}}{k+1}(x-2)^k$ is asking us to add the sequence $a_k=\frac{(-1)^{k+1}}{k+1}(x-2)^k$, where we start with $k=0$ and end with $k=4$, resulting in the following.

$$\underbrace{\left(\frac{(-1)^{0+1}}{0+1}(x-2)^0\right)}_{a_0}+\underbrace{\left(\frac{(-1)^{1+1}}{1+1}(x-2)^1\right)}_{a_1}+\underbrace{\left(\frac{(-1)^{2+1}}{2+1}(x-2)^2\right)}_{a_2}+\underbrace{\left(\frac{(-1)^{3+1}}{3+1}(x-2)^3\right)}_{a_3}+\underbrace{\left(\frac{(-1)^{4+1}}{4+1}(x-2)^4\right)}_{a_4}$$

After simplifying, we have $\sum_{k=0}^{4}\frac{(-1)^{k+1}}{k+1}(x-2)^k=-1+\frac{1}{2}(x-2)-\frac{1}{3}(x-2)^2+\frac{1}{4}(x-2)^3-\frac{1}{5}(x-2)^4$

□

In each of the problems in the previous example, we were given the formula for a_n, along with index values for the first and last term, and we evaluated the first and last terms along with the terms in between the two. Now we will go in the opposite direction. In other words, we start with the terms and generate the summation notation. We make use of our knowledge of arithmetic and geometric sequences, looking for a common difference or common ratio as we go from term to term to determine the type of sequence. Once we know what type of sequence we are dealing with, we can refer to the formulas from **Section 7.1**.

Example 7.2.2. Write the following sums using summation notation. Assume the terms in each sum result from an arithmetic or a geometric sequence.

1. $-5-2+1+\cdots+22$
2. $5-\frac{5}{2}+\frac{5}{4}-\frac{5}{8}+\cdots$
3. $5-\frac{5}{2}+\frac{5}{4}-\frac{5}{8}+\cdots+\frac{5}{1024}$

Solution.

1. For $-5-2+1+\cdots+22$, we first determine a general formula for its terms. Since the difference between the first and second term is 3, and the difference between the second and third term is 3, we will assume the pattern is arithmetic. We still need to find a formula for the terms and the position number for the last term, 22. For the formula, we use the formula for an arithmetic sequence, $a_n = a_1 + (n-1)d$, with $a_1 = -5$ and $d = 3$, to get

$$\begin{aligned} a_n &= -5 + (n-1)(3) \\ a_n &= 3n - 8 \end{aligned}$$

   Now, to determine the value of n for the term 22, we set $a_n = 22$ in the formula we just found.

$$\begin{aligned} a_n &= 3n-8 \\ 22 &= 3n-8 \\ n &= 10 \end{aligned}$$

   In summation notation, $-5-2+1+\cdots+22 = \sum_{n=1}^{10}(3n-8)$.

2. Again, we need to determine if the pattern of $5-\frac{5}{2}+\frac{5}{4}-\frac{5}{8}+\cdots$ is arithmetic or geometric. We quickly recognize that it is not arithmetic since there is no common difference. There is, however, a common ratio of $-\frac{1}{2}$ so it is geometric. The lack of a last term, indicated by '$\cdots$', indicates we are adding infinitely many terms. We simply need to find the formula for a_n. Using the formula for a geometric sequence, $a_n = a_1 r^{n-1}$, along with $a_1 = 5$ and $r = -\frac{1}{2}$, we find $a_n = 5\left(-\frac{1}{2}\right)^{n-1}$.

   Finally, using summation notation,

$$5-\frac{5}{2}+\frac{5}{4}-\frac{5}{8}+\cdots = \sum_{n=1}^{\infty} 5\left(-\frac{1}{2}\right)^{n-1}$$

3. The terms in the series $5-\frac{5}{2}+\frac{5}{4}-\frac{5}{8}+\cdots+\frac{5}{1024}$ are the same as in part 2 of this example.

However, this time there is a last term, $\frac{5}{1024}$. From part 2, we have $\sum_{n=1}^{?}5\left(-\frac{1}{2}\right)^{n-1}$, so we need only to determine the position of $\frac{5}{1024}$. Checking different n values, we see that $\frac{5}{1024}=5\left(-\frac{1}{2}\right)^{n-1}$ holds for $n=11$.

$$5-\frac{5}{2}+\frac{5}{4}-\frac{5}{8}+\cdots+\frac{5}{1024}=\sum_{n=1}^{11}5\left(-\frac{1}{2}\right)^{n-1}$$

□

Although the summation notation is new, it is just a way of writing addition for many terms that are each generated in the same way. Because it is just addition, the usual properties of addition hold. For example, the order of addition does not matter. Below, we list properties of summation, using mathematical notation, and noting that p may represent an integer or infinity. In the case where $p=\infty$, these properties hold if each infinite sum is defined. We will discuss certain infinite sums later. They will be discussed extensively in Calculus.

Properties of Summation

- $\sum_{n=j}^{p}(a_n \pm b_n)=\sum_{n=j}^{p}a_n \pm \sum_{n=j}^{p}b_n$
- $\sum_{n=j}^{p}a_n=\sum_{n=j}^{h}a_n+\sum_{n=h+1}^{p}a_n$, for any integer $j \le h < p$
- $\sum_{n=j}^{p}ca_n=c\sum_{n=j}^{p}a_n$, for any constant c
- $\sum_{n=j}^{p}a_n=\sum_{n=j+h}^{p+h}a_{n-h}$, for any integer h (if $p=\infty$, replace $p+h$ with ∞)

We now turn our attention to finding sums, limiting our sums to arithmetic and geometric series. We begin with arithmetic series.

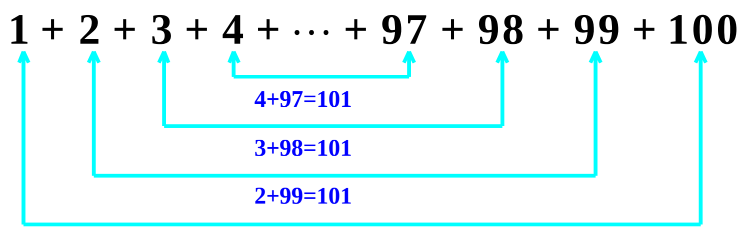
Arithmetic Series

There is a story often told in math classes about Carl Friedrich Gauss (1777 – 1855), one of the world's most famous mathematicians, when he was about 9 years old. The legend goes like this: Gauss's teacher often gave Gauss and his classmates tedious arithmetic tasks. On one such occasion, the teacher asked students to add all of the integers from 1 to 100:

$$1+2+3+4+\cdots+97+98+99+100$$

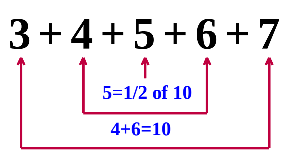
All of the students, except Gauss, added the numbers one at a time. When the teacher asked him why he wasn't working on the problem, Gauss replied that he already knew the answer, 5050. The teacher was astonished and asked how he knew the answer without doing the computation. Gauss explained that if you add the first term and the last term you get 101; then add the second term and the second to the last term, you again get 101; and then the third and third to last term, again 101. As a matter of fact, you always get 101 if you continue in this manner. There are a total of 50 sums of 101 (a sum is made up of a pair of terms), so the sum is $50(101)=5050$.

Figure 7.2. 1



In other words, we are taking the first term plus the last term and then multiplying that sum by half the number of terms in the series. If we have n terms, this is equivalent to $\frac{n}{2}(a_1+a_n)$. Before moving on, let's quickly check that this works for an odd number of terms, say $3+4+5+6+7$.

Figure 7.2. 2



With these five terms, we find $\frac{5}{2}(3+7)=25$, since the middle term is half the sum of the first and last terms, so the formula $\frac{n}{2}(a_1+a_n)$ appears to work for an odd number of terms as well, bringing us to the following theorem.

Theorem 7.2. Arithmetic Sum:

Consider the arithmetic sequence $\{a_k\}$ with first term a_1 and common difference d. Then $a_k = a_1 + (k-1)d$, $k = 1, 2, 3, \cdots$, and the sum of the first n terms, $S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$, is

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(2a_1 + (n-1)d),\ n \geq 2$$

As verification of this theorem, consider the finite arithmetic sequence $a_k = a_1 + (k-1)d$, $k = 1, 2, \cdots, n$ and its sum $S_n = \sum_{k=1}^{n}\left[a_1 + (k-1)d\right] = a_1 + (a_1 + d) + \cdots + (a_1 + (n-2)d) + (a_1 + (n-1)d)$. To determine the value of the sum, we write the sum twice, the second time in reverse order, and add the two.

$$\begin{aligned}
S_n &= a_1 + (a_1 + d) + \cdots (a_1 + (n-2)d) + (a_1 + (n-1)d) \\
S_n &= (a_1 + (n-1)d) + (a_1 + (n-2)d) + \cdots (a_1 + d) + a_1 \\
2S_n &= (2a_1 + (n-1)d) + (2a_1 + (n-1)d) + \cdots (2a_1 + (n-1)d) + (2a_1 + (n-1)d)
\end{aligned}$$

The right side is the sum of n copies of $(2a_1 + (n-1)d)$, so

$$2S_n = n(2a_1 + (n-1)d)$$

$$S_n = \frac{n}{2}(2a_1 + (n-1)d)$$

Using the fact that the last, or *n*th term in the sequence is $a_n = a_1 + (n-1)d$, we can also write this sum as $S_n = \frac{n}{2}(a_1 + a_n)$.

Example 7.2.3. Find the value of the arithmetic[3] sum $\sum_{k=1}^{12}(4k-5)$.

Solution. We could write out each of the terms and then add them all up, but this would be tedious, so we choose to use the formula. We can see that there are 12 terms, so $n = 12$. We proceed with finding the value of the first and the last term by setting $k = 1$ and $k = 12$, respectively.

$$k = 1:\quad a_1 = 4(1) - 5 = -1$$

$$k = 12:\ a_{12} = 4(12) - 5 = 43$$

The sum is $S_{12} = \frac{12}{2}(-1 + 43) = 252$.

□

[3] To verify that this sum is arithmetic, we find that we have a common difference, i.e. $a_{k+1} - a_k = 4$.

Example 7.2.4. The first term of an arithmetic sequence is 15 and its 10[th] term is −12. Find the sum of its first 10 terms. Find the sum of its first 20 terms.

Solution. We are given $a_1 = 15$ and $a_{10} = -12$. The sum of the first 10 terms is

$$\begin{aligned} S_{10} &= \frac{10}{2}\left(15 + (-12)\right) \\ &= 15 \end{aligned}$$

To find the sum of the first 20 terms, we need to find the value of a_{20}. We start by finding the value of d. Using $a_n = a_1 + (n-1)d$ with $a_1 = 15$, $a_{10} = -12$ and $n = 10$, we have

$$\begin{aligned} -12 &= 15 + (10-1)d \\ -27 &= 9d \\ d &= -3 \end{aligned}$$

Then $a_{20} = 15 + (-3)(20-1) = -42$. With this and the sum formula, we find

$$\begin{aligned} S_{20} &= \frac{20}{2}\left(15 + (-42)\right) \\ &= -270 \end{aligned}$$

□

Example 7.2.5. The first term of an arithmetic sequence is −12 and its common difference is 5. How many terms of this sequence must we add to get a sum of 345?

Solution. We are given $a_1 = -12$ and $d = 5$, and we are looking for n so that $S_n = 345$ where

$S_n = \frac{n}{2}(a_1 + a_n)$. By substituting $a_n = a_1 + (n-1)d$, we have

$$\begin{aligned} S_n &= \frac{n}{2}\left(a_1 + \left(a_1 + (n-1)d\right)\right) \\ &= \frac{n}{2}\left(2a_1 + (n-1)d\right) \end{aligned}$$

Now we set $S_n = 345$, $a_1 = -12$ and $d = 5$ to solve for n.

$$\begin{aligned} \frac{n}{2}\left[2(-12) + (n-1)(5)\right] &= 345 \\ n(-24 + 5n - 5) &= 690 \\ 5n^2 - 29n - 690 &= 0 \end{aligned}$$

$$n = \frac{29 \pm \sqrt{(-29)^2 - 4(5)(-690)}}{2(5)}$$
$$n = \frac{29 \pm \sqrt{14641}}{10}$$
$$n = \frac{29 \pm 121}{10}$$

We get $n = -9.2$ or $n = 15$. Since n must be a whole number, we find $n = 15$ and conclude that adding 15 terms in the sequence will result in 345.

□

Applications of Arithmetic Series

Example 7.2.6. On the Sunday after a minor surgery, Margaret is able to walk a half-mile. Each Sunday, she walks an additional quarter mile. After 8 weeks, what will be the total number of miles she has walked?

Solution. This problem can be modeled as an arithmetic series with $a_1 = \frac{1}{2}$ and $d = \frac{1}{4}$. We are looking for the total number of miles walked after 8 weeks, so we know that $n = 8$ and that we are looking for S_8. Using the formula for S_n, with $n = 8$, we have everything we need to find the sum of miles walked except the value of a_8. We find a_8 using the formula $a_n = a_1 + (n-1)d$.

$$a_8 = \frac{1}{2} + (8-1)\left(\frac{1}{4}\right) = \frac{9}{4}$$

We can now use the formula for an arithmetic sum to determine the number of miles Margaret walked.

$$S_8 = \frac{8}{2}\left(\frac{1}{2} + \frac{9}{4}\right) = 11$$

Margaret walked a total of 11 miles.

□

Before moving on to geometric series, we ponder the results of summing infinitely many arithmetic terms. Suppose Gauss had been asked to add up all of the natural numbers.

$$1 + 2 + 3 + 4 + 5 + \cdots$$

This would mean adding progressively larger and larger numbers or, in other words, bigger and bigger terms. The sum just keeps getting larger. This sum is said to 'go to infinity' or **diverge**. This always happens for arithmetic sequences where $a_1 \neq 0$, even if $d = 0$. For example, if $a_1 = 1$ and $d = 0$ we have $1 + 1 + 1 + \cdots$. We leave it to the reader to think about whether summing infinitely many arithmetic terms can ever result in a finite number.

Geometric Series

We begin by adding a finite number of terms of the form $a_k = a_1 r^{k-1}$, where $k = 1, 2, \cdots, n$. The sum of these terms is $S_n = \sum_{k=1}^{n} a_1 r^{k-1} = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-2} + a_1 r^{n-1}$. To find the value of S_n, we use a bit of arithmetic; we multiply the terms of the series by r, subtract the result from S_n to give us $S_n - rS_n$, then proceed to solve for S_n.

$$\begin{array}{rcccccccc} S_n = & a_1 + & a_1 r + & a_1 r^2 + & \cdots & + & a_1 r^{n-2} + & a_1 r^{n-1} & \\ rS_n = & & a_1 r + & a_1 r^2 + & \cdots & + & a_1 r^{n-2} + & a_1 r^{n-1} + & a_1 r^n \\ \hline S_n - rS_n = & a_1 & & & & & & & -a_1 r^n \end{array}$$

Thus,

$$\begin{aligned} S_n - rS_n &= a_1 - a_1 r^n \\ S_n (1 - r) &= a_1 - a_1 r^n \\ S_n &= \frac{a_1 - a_1 r^n}{1 - r} = \frac{a_1 (1 - r^n)}{1 - r} \end{aligned}$$

Theorem 7.3. Geometric Sum (finite number of terms):

Consider the geometric sequence $\{a_k\}$ with first term a_1 and common ratio r, $r \neq 1$. Then $a_k = a_1 r^{k-1}$, $k = 1, 2, 3, \cdots$, and the sum of the first n terms, $S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$, is

$$S_n = \frac{a_1 (1 - r^n)}{1 - r}$$

In the case when $r = 1$, $a_k = a_1$ and $S_n = \underbrace{a_1 + a_1 + a_1 + \cdots + a_1}_{n \text{ times}} = na_1$.

Example 7.2.7. Find the value of the sum $5 - \frac{5}{2} + \frac{5}{4} - \frac{5}{8} + \cdots + \frac{5}{1024}$.

Solution. This is a finite geometric sum with $a_1 = 5$ and $r = -\frac{1}{2}$, since $\left(-\frac{5}{2}\right) \div (5) = -\frac{1}{2}$ and $\left(\frac{5}{4}\right) \div \left(-\frac{5}{2}\right) = -\frac{1}{2}$. We need only to know the number of terms, n, to find the sum with the formula in Theorem 7.3. To determine the value of n, we use the formula $a_n = a_1 r^{n-1}$ with $a_1 = 5$ and $a_n = \frac{5}{1024}$.

$$\frac{5}{1024} = 5\left(-\frac{1}{2}\right)^{n-1}$$

$$\frac{1}{1024} = \left(-\frac{1}{2}\right)^{n-1}$$

$$\left(-\frac{1}{2}\right)^{10} = \left(-\frac{1}{2}\right)^{n-1} \quad \text{since } \left(-\frac{1}{2}\right)^{10} = \frac{1}{1024}$$

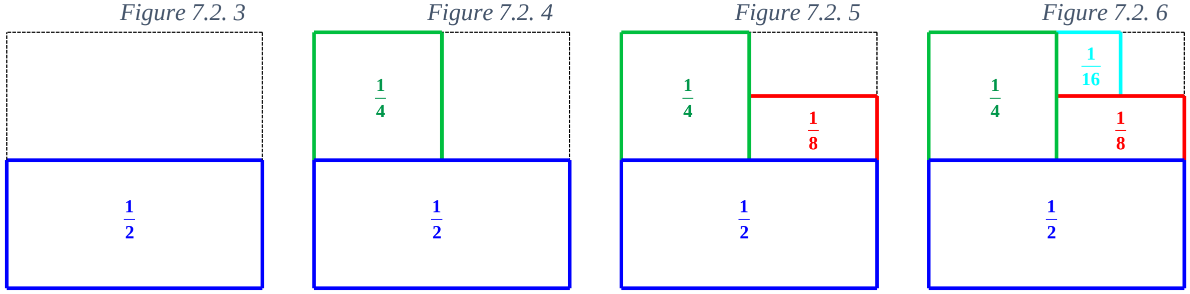
We have $n-1=10 \Rightarrow n=11$. We may now use the formula $S_n = \frac{a_1\left(1-r^n\right)}{1-r}$ with $n=11$, $a_1 = 5$ and $r = -\frac{1}{2}$ to find the sum.

$$S_{11} = \frac{5\left(1-\left(-\frac{1}{2}\right)^{11}\right)}{1-\left(-\frac{1}{2}\right)} = \frac{5\left(1+\frac{1}{2048}\right)}{\frac{3}{2}} = \frac{10245}{2048}\cdot\frac{2}{3} = \frac{3415}{1024}$$

□

An interesting question, to be explored further in Calculus, is 'when is it possible to add an infinite number of numbers and get a finite sum?' We talked about adding an infinite number of terms in an arithmetic sequence. Now we examine the sum of infinitely many geometric terms. Suppose you add all of the terms starting with $\frac{1}{2}$ where each subsequent term is half of the previous term: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$. As an illustration, in the following diagram the indicated square has a side length of 1, and we sum areas within the square as we move from left to right.

Figure 7.2. 3 *Figure 7.2. 4* *Figure 7.2. 5* *Figure 7.2. 6*



$\frac{1}{2}$ $\quad\frac{1}{2}+\frac{1}{4}$ $\quad\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$ $\quad\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$

Notice that we keep adding smaller and smaller pieces and that the 'pieces' seem to be filling in a 'missing piece'. The sum appears to be getting closer and closer to 1, and indeed the sum is 1.

While the geometric series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1$, this is obviously not the case for all geometric series. For example, the geometric series $1+2+4+8+\cdots$ diverges, much like the arithmetic series we discussed earlier. As it turns out, the series $\sum_{k=1}^{\infty} a_1 r^{k-1}$ has a finite value when $-1<r<1$; we say the series **converges**. For all other values of r, the infinite geometric series will not have a finite value. As to what happens and why, we leave that discussion to your Calculus course.

> **Theorem 7.4. Geometric Series (infinite number of terms):**
> Consider the geometric sequence $\{a_n\}$ with first term a_1 and common ratio r, $-1<r<1$. Then $a_k = a_1 r^{k-1}$, $k=1, 2, 3, \cdots$, and the sum of this infinite geometric series, $S=\sum_{k=1}^{\infty} a_k = a_1+a_2+a_3+\cdots$, is
> $$S=\frac{a_1}{1-r}$$

The sum $\sum_{k=1}^{\infty} a_1 r^{k-1}$ is not defined for $|r|\geq 1$, assuming $a_1 \neq 0$.

Example 7.2.8. Find the value of the series, if it exists.

1. $5-\frac{5}{2}+\frac{5}{4}-\frac{5}{8}+\cdots$
2. $\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)\left(\frac{3}{2}\right)^{k-1}$
3. $\sum_{k=1}^{\infty}\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)^{k-1}$

Solution.

1. The series $5-\frac{5}{2}+\frac{5}{4}-\frac{5}{8}+\cdots$ is geometric with $a_1=5$ and $r=-\frac{1}{2}$. Because $|r|<1$, the series will converge to a value. We use the formula in **Theorem 7.4** to find that value.
$$5-\frac{5}{2}+\frac{5}{4}-\frac{5}{8}+\cdots=\frac{5}{1-\left(-\frac{1}{2}\right)}=\frac{5}{\left(\frac{3}{2}\right)}=\frac{10}{3}$$

2. The series $\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)\left(\frac{3}{2}\right)^{k-1}$ is written in summation notation. Since $\frac{3}{2}$ is the number being raised to a power, this means we are multiplying each term by $\frac{3}{2}$ to get to the next term. This tells us that the ratio is $\frac{3}{2}$, which is larger than the absolute value of 1, so the series will diverge.

3. Again, the series is written in summation notation. However, for $\sum_{k=1}^{\infty}\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)^{k-1}$, we have $r=\frac{2}{3}$, which has an absolute value less than 1, so the series will converge to a value. We find $a_1=\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)^{1-1}=\frac{3}{2}$ and use the formula from **Theorem 7.4** to find the sum.

$$\sum_{k=1}^{\infty}\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)^{k-1}=\frac{\frac{3}{2}}{1-\frac{2}{3}}=\left(\frac{3}{2}\right)\cdot\left(\frac{3}{1}\right)=\frac{9}{2}$$

□

Applications of Geometric Series

A rational number is a real-valued number that is the ratio of two integers or, in decimal notation, has a decimal expansion that ends or repeats. The following application of geometric series allows us to write a repeating decimal as a ratio of two integers.

Example 7.2.9. Write the rational number $4.\overline{52}=4.525252\cdots$ as a ratio of two integers.

Solution. We begin by noting that the decimal portion of this number can be written as a geometric series, to which we may apply the geometric series formula.

$$\begin{aligned}4.\overline{52}&=4.525252\cdots\\&=4+0.525252\cdots\\&=4+\underbrace{0.52+0.0052+0.000052+\cdots}_{\text{geometric series}} \quad a_1=0.52=\frac{52}{100}\text{ and } r=0.01=\frac{1}{100}\\&=4+\frac{\frac{52}{100}}{1-\frac{1}{100}}\end{aligned}$$

We find $4.\overline{52}=4+\frac{52}{99}=\frac{448}{99}$.

□

An interesting fact that can be shown with the method used in **Example 7.2.9** is that $0.\overline{9}=0.999\cdots=1$. Try it on your own!

An important applicaton of the geometric sum formula is the investment plan called an **annuity**. Annuities differ from the kind of investments we studied in **Section 4.5** in that payments are deposited into the account on an on-going basis. In the following example, we look at an **ordinary annuity**, which requires equal payments made at the end of consecutive periods, each having the same length. To find the value of an annuity, we must sum all of the payments and the interest earned.

Example 7.2.10. A deposit of $50 is made at the end of each month in a savings account that offers 6% annual interest, compounded monthly. Find the value of the savings account after 30 years.

Solution.

We begin by assessing the values, at the end of 30 years, of the individual monthly deposits of $50. In Section 4.5, we used the compound interest formula $A = P\left(1+\frac{r}{n}\right)^{nt}$ for principal P, annual interest rate r, and n compoundings per year over a period of t years. In this example, we have $r = 0.06$, $n = 12$, and $P = 50$, so $A = 50\left(1+\frac{0.06}{12}\right)^{12t} = 50(1.005)^{12t}$. Since $12t$ is the number of months that compounding takes place, and payments are made at the end of each month, our initial deposit of $50 will gain interest over $12\times 30-1=359$ months; the second deposit of $50 will gain interest over 358 months, the third over 357 months, etc.

Deposit #	Months Compounding	Resulting Value
1	359	$50(1.005)^{359}$
2	358	$50(1.005)^{358}$
3	357	$50(1.005)^{357}$
⋮	⋮	⋮
360	0	$50(1.005)^{0}$

The sum of these values that result from monthly deposits is

$$\begin{aligned}&50(1.005)^{359}+50(1.005)^{358}+50(1.005)^{357}+\cdots+50(1.005)^{0}\\ &=50(1.005)^{0}+50(1.005)^{1}+50(1.005)^{2}+\cdots+50(1.005)^{359}\end{aligned}$$

This is the sum of a geometric sequence with $a_1 = 50$, $r = 1.005$ and $nt = 360$.

$$\begin{aligned}S_{360} &= \frac{50\left(1-1.005^{360}\right)}{1-1.005} \quad \text{using formula } S_{nt} = \frac{a_1\left(1-r^{nt}\right)}{1-r}\\ &\approx 50225.75\end{aligned}$$

Thus, the account contains $50,225.75 after 30 years.

□

7.2 Exercises

1. What is the difference between an arithmetic sequence and an arithmetic series?
2. Describe the criteria for determining if an infinite geometric series has a finite sum. Give an example of an infinite geometric series that has a finite sum, and another that does not.

In Exercises 3 – 11, find the value of the sum.

3. $\sum_{a=1}^{14} a$
4. $\sum_{n=1}^{6} n(n-2)$
5. $\sum_{k=1}^{17} k^2$
6. $\sum_{g=4}^{9} (5g+3)$
7. $\sum_{k=3}^{8} \frac{1}{k}$
8. $\sum_{j=0}^{5} 2^j$
9. $\sum_{k=0}^{2} (3k-5)x^k$
10. $\sum_{i=1}^{4} \frac{1}{4}(i^2+1)$
11. $\sum_{n=1}^{100} (-1)^n$

In Exercises 12 – 19, rewrite the sum using summation notation.

12. $8+11+14+17+20$
13. $1-2+3-4+5-6+7-8$
14. $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}$
15. $1+2+4+\cdots+2^{29}$
16. $2+\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\frac{6}{5}$
17. $-\ln(3)+\ln(4)-\ln(5)+\cdots+\ln(20)$
18. $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}$
19. $\frac{1}{2}(x-5)+\frac{1}{4}(x-5)^2+\frac{1}{6}(x-5)^3+\frac{1}{8}(x-5)^4$

In Exercises 20 – 37, use formulas from this section to find the sum, if possible.

20. $\sum_{n=1}^{10} (5n+3)$
21. $\sum_{n=1}^{20} (2n-1)$
22. $\sum_{k=0}^{15} (3-k)$
23. $\sum_{n=1}^{10} \left(\frac{1}{2}\right)^n$
24. $\sum_{n=1}^{5} \left(\frac{3}{2}\right)^n$
25. $\sum_{k=0}^{5} 2\left(\frac{1}{4}\right)^k$
26. $1+4+7+\cdots+295$
27. $4+2+0-2-\cdots-146$
28. $1+3+9+\cdots+2187$
29. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{256}$
30. $3-\frac{3}{2}+\frac{3}{4}-\frac{3}{8}+\cdots+\frac{3}{256}$
31. $4+2+1+\frac{1}{2}+\cdots$

32. $\frac{1}{2}+1+2+4+\cdots$

33. $-1-\frac{1}{4}-\frac{1}{16}-\frac{1}{64}-\cdots$

34. $\sum_{k=1}^{\infty} 3\left(\frac{1}{4}\right)^{k-1}$

35. $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^{n}$

36. $1-\frac{3}{4}+\frac{9}{16}-\frac{27}{64}+\cdots$

37. $\sum_{n=1}^{\infty} 4\left(-\frac{1}{2}\right)^{n-1}$

38. The sum of terms $50-k^2$ from $k=x$ through $k=7$ is 115. What is x?

39. Write an explicit formula for a_k such that $\sum_{k=0}^{6} a_k = 189$. Assume this is an arithmetic series.

40. Find the smallest value of n such that $\sum_{k=1}^{n}(3k-5) > 100$.

41. How many terms must be added before the series $-1-3-5-7-\cdots$ has a sum less than -75?

42. Cindy devised a week-long study plan to prepare for finals. On the first day, she plans to study for 1 hour, and each successive day she will increase her study time by 30 minutes. How many hours will Cindy have studied after one week?

43. A testing center is designed with 10 seats in the first row, 12 seats in the second row, 14 seats in the third row, and so forth. The testing center has 15 rows of seating. What is the maximum number of students who may be testing at any one time?

44. A brick wall is built with 300 bricks in the first row, 299 bricks in the second row, and each successive row contains one less brick. If the top row contains 177 bricks, what is the total number of bricks required to build the wall?

45. Find the sum $1+2+3+\cdots+1000$.

In Exercises 46 – 51, express the repeating decimal as a fraction of integers.

46. $0.\overline{7}$

47. $0.\overline{13}$

48. $2.\overline{3}$

49. $4.\overline{17}$

50. $10.\overline{159}$

51. $-5.8\overline{67}$

In Exercises 52 – 57, compute the future value of the annuity with the given terms. In all cases, assume the payment is made at the end of each month, the interest rate given is the annual rate, and interest is compounded at the end of each month.

52. payments are \$300, interest rate is 2.5%, term is 17 years.

53. payments are \$50, interest rate is 1.0%, term is 30 years.

54. payments are $100, interest rate is 2.0%, term is 20 years.

55. payments are $100, interest rate is 2.0%, term is 25 years.

56. payments are $100, interest rate is 2.0%, term is 30 years.

57. payments are $100, interest rate is 2.0%, term is 35 years.

58. Discuss with your classmates what goes wrong when trying to find the following sums.[4]

(a) $\sum_{k=1}^{\infty} 2^{k-1}$ (b) $\sum_{k=1}^{\infty} (1.0001)^{k-1}$ (c) $\sum_{k=1}^{\infty} (-1)^{k-1}$

[4] When in doubt, write them out!

7.3 Binomial Expansion

Learning Objectives

- Expand binomial powers using
  - Pascal's Triangle
  - Binomial Theorem
- Find an indicated term in the expansion of a binomial

A binomial is simply a polynomial with two terms. In this section, we are interested in powers of binomials of the form $(a+b)^n$, for $n=0,\ 1,\ 2,\ 3,\cdots$. The expressions $(a+b)^2$ and $(a+b)^3$ occur frequently in simplifying, factoring and solving equations.

Identity	In sentence form
$(a+b)^2 = a^2+2ab+b^2$	*The sum of two terms squared is the first squared, plus twice the first times the second, plus the second squared.*
$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$	*The sum of two terms cubed is the first cubed, plus three times the first squared times the second, plus three times the first times the second squared, plus the second cubed.*

As demonstrated below, these identities can be applied in general and there is no need to memorize rules for $(a-b)^2$ or $(a-b)^3$.

$$\begin{aligned}(2x+3y)^2 &= ((2x)+(3y))^2\\ &= (2x)^2+2(2x)(3y)+(3y)^2\\ &= 4x^2+12xy+9y^2\end{aligned}$$

$$\begin{aligned}(a-b)^3 &= (a+(-b))^3\\ &= a^3+3a^2(-b)+3a(-b)^2+(-b)^3\\ &= a^3-3a^2b+3ab^2-b^3\end{aligned}$$

Example 7.3.1. Simplify $4(a-3)^3-a(2a-9)^2$.

Solution.

$$\begin{aligned}4(a-3)^3-a(2a-9)^2 &= 4\left(a^3+3a^2(-3)+3a(-3)^2+(-3)^3\right)-a\left((2a)^2+2(2a)(-9)+(-9)^2\right)\\ &= 4\left(a^3-9a^2+27a-27\right)-a\left(4a^2-36a+81\right)\\ &= 4a^3-36a^2+108a-108-4a^3+36a^2-81a\\ &= 27a-108\end{aligned}$$

□

Example 7.3.2. Solve $x^3+6x^2+12x+8=125$.

Solution. The left side of this equation is the expansion of $(a+b)^3$ with $a=x$ and $b=2$.

$$x^3+6x^2+12x+8=125$$
$$x^3+3(x^2)(2)+3(x)(2^2)+2^3=125$$
$$(x+2)^3=5^3$$

Now, $x+2=5$, from which we find $x=3$.

□

Consider $(a+b)^n$ in general. This is the product of n copies of $(a+b)$, which we can expand by multiplying out term by term.

$$(a+b)^n=\overbrace{(a+b)\times(a+b)\times\cdots\times(a+b)}^{n\text{ copies}}$$
$$=a^n+\square a^{n-1}b+\square a^{n-2}b^2+\cdots+\square a^2b^{n-2}+\square ab^{n-1}+b^n$$

Multiplying out the a terms in each of the n copies, we get a^n, and there is only one way of obtaining it, so the first term of the expansion is a^n. Now, multiplying the a terms from the $n-1$ copies and the b from the remaining copy, we get $a^{n-1}b$. However, we can obtain this term more than one way. Reducing the number of a terms and replacing them with b terms, we get the expression $a^{n-k}b^k$ for $k=2,\ 3,\ 4,\cdots,n-1$. Again, we can obtain each of these in more than one way. Finally, multiplying the b terms from each copy, we get b^n, and there is only one way of obtaining it, so the last term of the expansion is b^n.

Pascal's Triangle

Except for the first and last terms, at this time, we don't know the coefficients of the other terms. In order to discover these, we list expansions of several binomial powers and try to recognize the pattern of the terms and coefficients.

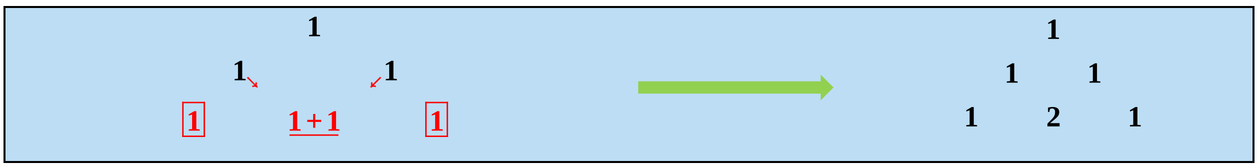
$$(a+b)^0=1$$
$$(a+b)^1=a+b$$
$$(a+b)^2=(a+b)(a+b)=a^2+2ab+b^2$$
$$(a+b)^3=(a+b)(a+b)^2=a^3+3a^2b+3ab^2+b^3$$
$$(a+b)^4=(a+b)(a+b)^3=a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

Here, the expansion is written in descending order of the powers of the first term. As discussed above, in the expansion of $(a+b)^n$ for $n=1,\ 2,\ 3,\cdots$, the first term is a^n. For the middle terms we decrease the power of a by one and increase the power of b by one until we reach the last term b^n. To discover what's going on with the middle terms, let's just look at the coefficients.

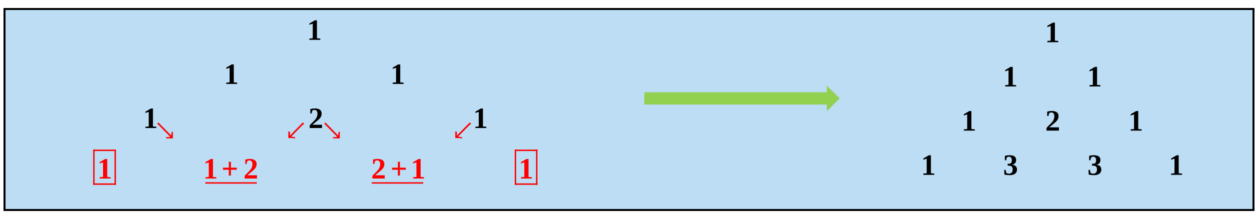
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Notice that each new row can be formed from the previous one by adding a 1 at the beginning and another 1 at the end. Then each term in the middle is the sum of the two above it, as shown below.

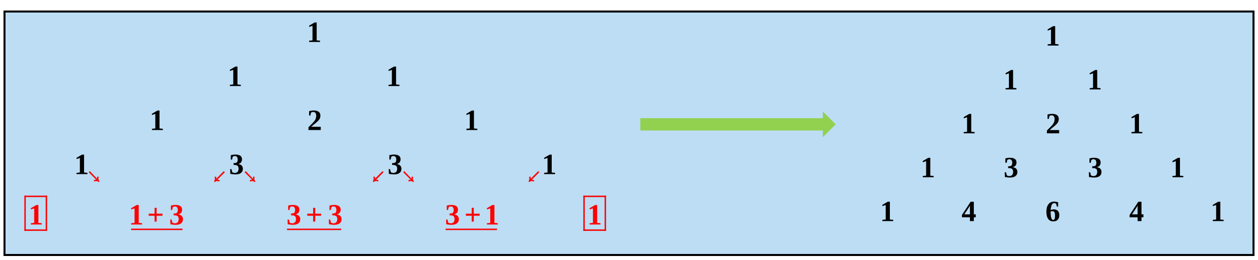
3rd row:



4th row:



5th row:



Following this pattern, using the last row, we get a new row.

1 4 6 4 1
1 5 10 10 5 1

This row is precisely the coefficients of the expansion $(a+b)^5$.

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Of course, you can check this by performing the operation $(a+b)^5 = (a+b)(a+b)^4$.

The table formed by these rows of coefficients is called **Pascal's Triangle**. Blaise Pascal was a French mathematician and physicist who lived in the 17th century. However, this result was well known both in Iran and China in the 10th century and might even go as far back as the 2nd century BCE in India. Notice that there is symmetry in each row, making it easy to check our work.

Example 7.3.3. Use Pascal's Triangle to expand the following.

1. $(2x-3)^4$ 2. $(a-b)^5$ 3. $(a+b)^6$

Solution. We write one more row of Pascal's Triangle.

$n=0$							**1**						
$n=1$						**1**		**1**					
$n=2$					**1**		**2**		**1**				
$n=3$				**1**		**3**		**3**		**1**			
$n=4$			**1**		**4**		**6**		**4**		**1**		
$n=5$		**1**		**5**		**10**		**10**		**5**		**1**	
$n=6$	**1**		**6**		**15**		**20**		**15**		**6**		**1**

1. For $(2x-3)^4$, we have $n=4$.

$$\begin{aligned}(2x-3)^4 &= ((2x)+(-3))^4 \\ &= (2x)^4+4(2x)^3(-3)+6(2x)^2(-3)^2+4(2x)(-3)^3+(-3)^4 \\ &= 16x^4-96x^3+216x^2-216x+81\end{aligned}$$

2. We use the row for $n=5$ to expand $(a-b)^5$.

$$\begin{aligned}(a-b)^5 &= (a+(-b))^5 \\ &= a^5+5a^4(-b)+10a^3(-b)^2+10a^2(-b)^3+5a(-b)^4+(-b)^5 \\ &= a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5\end{aligned}$$

3. The row for $n=6$ gives us the coefficients for the expansion of $(a+b)^6$.

$$(a+b)^6 = a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$$

□

The Binomial Theorem

Recall our discussion regarding the 'missing coefficients' in the expansion of $(a+b)^n$.

$$\begin{aligned}(a+b)^n &= \overbrace{(a+b)\times(a+b)\times\cdots\times(a+b)}^{n\text{ copies}} \\ &= a^n+\square a^{n-1}b+\square a^{n-2}b^2+\cdots+\square a^2b^{n-2}+\square ab^{n-1}+b^n\end{aligned}$$

The Binomial Theorem provides us with a way of filling in these 'missing coefficents' one at a time. There may be occasions when we require only one term, say the fifth term, in an expansion and the Binomial Theorem allows us to find that single term. Before proceeding, we introduce the **factorial**, and note that the notation for the factorial is $n!$.

Definition 7.4. Factorial: The factorial of a nonnegative integer is defined as follows.

$$0! = 1$$
$$n! = 1 \times 2 \times 3 \times \cdots \times n \text{ for } n = 1,\ 2,\ 3, \cdots$$

Example 7.3.4. Evaluate $n!$ for $n = 0,\ 1,\ 2, \cdots,\ 5$.

Solution. Using **Definition 7.4**, we have the following.

$$\begin{aligned}
0! &= 1 \\
1! &= 1 \\
2! &= 1 \times 2 = 2 \\
3! &= 1 \times 2 \times 3 = 6 \\
4! &= 1 \times 2 \times 3 \times 4 = 24 \\
5! &= 1 \times 2 \times 3 \times 4 \times 5 = 120
\end{aligned}$$

□

Note that $0! = 1$ does not follow the pattern of the definition $n! = 1 \times 2 \times 3 \times \cdots \times n$, and also note that $1! = 1$. In the next example, we evaluate some expressions involving factorials.

Example 7.3.5. Evaluate the following.

1. $(9-3)!$
2. $\frac{5!}{2!}$
3. $\frac{10!}{8!}$
4. $\frac{100!}{97!3!}$

Solution.

1. $$\begin{aligned}(9-3)! &= 6! \\ &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \\ &= 720\end{aligned}$$

2. $$\begin{aligned}\frac{5!}{2!} &= \frac{1 \times 2 \times 3 \times 4 \times 5}{1 \times 2} \\ &= 3 \times 4 \times 5 \\ &= 60\end{aligned}$$

3. $$\begin{aligned}\frac{10!}{8!} &= \frac{\overbrace{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8}^{8!} \times 9 \times 10}{8!} \\ &= 9 \times 10 \\ &= 90\end{aligned}$$

4. $$\begin{aligned}\frac{100!}{97!3!} &= \frac{\overbrace{1 \times 2 \times 3 \times \cdots \times 97}^{97!} \times 98 \times 99 \times 100}{(97!)(1 \times 2 \times 3)} \\ &= \frac{98 \times 99 \times 100}{1 \times 2 \times 3} \\ &= 161,700\end{aligned}$$

□

Note: Although many calculators have a button for factorials, a calculator cannot evaluate $100!$. So, to evaluate part 4 of the above example, it is essential to recognize that $100! = (97!) \times 98 \times 99 \times 100$ and simplify the fraction.

Let's take another look at our earlier expansion $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$ and, as we work toward the Binomial Theorem, rewrite the coefficients in terms of factorials. Our goal is that each coefficient is a fraction with a numerator of $5!$.

$$\begin{aligned}(a+b)^5 &= 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5 \\ &= \frac{5!}{5!}a^5 + \frac{1\times2\times3\times4\times5}{1\times2\times3\times4\times1}a^4b + \frac{1\times2\times3\times4\times5}{1\times2\times3\times2}a^3b^2 + \frac{1\times2\times3\times4\times5}{1\times2\times2\times3}a^2b^3 + \frac{1\times2\times3\times4\times5}{1\times2\times3\times4}ab^4 + \frac{5!}{5!}b^5 \\ &= \frac{5!}{5!}a^5 + \frac{5!}{4!1!}a^4b + \frac{5!}{3!2!}a^3b^2 + \frac{5!}{2!3!}a^2b^3 + \frac{5!}{4!}ab^4 + \frac{5!}{5!}b^5\end{aligned}$$

Seeing a pattern beginning to emerge, we work with the equation a bit more to get

$$(a+b)^5 = \frac{5!}{0!5!}a^5 + \frac{5!}{1!4!}a^4b + \frac{5!}{2!3!}a^3b^2 + \frac{5!}{3!2!}a^2b^3 + \frac{5!}{4!1!}ab^4 + \frac{5!}{5!0!}b^5$$

Now each coefficient is of the form $\dfrac{5!}{(\text{power of } b)!(\text{power of } a)!}$. This hold in general and we call this type of fraction a **binomial coefficient**.

Definition 7.5. The Binomial Coefficient: For nonnegative integers n and k with $n \geq k$, the binomial coefficient $\binom{n}{k}$, read as 'n choose k', is defined by $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

Although their definition involves a fraction, the binomial coefficients are always integers.

Example 7.3.6. Evaluate the following binomial coefficients.

1. $\binom{6}{0}$ 2. $\binom{10}{7}$ 3. $\binom{4}{4}$ 4. $\binom{7}{3}$ 5. $\binom{7}{4}$

Solution.

1. $\binom{6}{0} = \dfrac{6!}{0!(6-0)!} = \dfrac{6!}{1\times 6!} = 1$

2. $\binom{10}{7} = \dfrac{10!}{7!(10-7)!} = \dfrac{10\times9\times8\times7!}{7!3!} = \dfrac{10\times9\times8}{3\times2\times1} = 120$

3. $\binom{4}{4} = \dfrac{4!}{4!(4-4)!} = \dfrac{4!}{4!0!} = \dfrac{4!}{4!(1)} = 1$

4. $\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} = \frac{7\times6\times5\times4!}{3\times2\times1\times4!} = \frac{7\times6\times5}{3\times2\times1} = 35$

5. $\binom{7}{4} = \frac{7!}{4!(7-4)!} = \frac{7!}{4!3!} = \frac{7\times6\times5\times4!}{4!\times3\times2\times1} = \frac{7\times6\times5}{3\times2\times1} = 35$

□

We return to the expansion of $(a+b)^5$, noting that $\binom{5}{0} = \frac{5!}{0!(5-0)!} = \frac{5!}{0!5!}$, $\binom{5}{1} = \frac{5!}{1!(5-1)!} = \frac{5!}{1!4!}$, and so forth. This gives us

$$(a+b)^5 = \binom{5}{0}a^5 + \binom{5}{1}a^4b + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}ab^4 + \binom{5}{5}b^5$$

The name 'binomial coefficient' now makes more sense. Additionally, if we think of the coefficient $\binom{5}{2}$ as the number of ways we can choose the 2 b's occurring in a^3b^2 out of the 5 possible b's in $(a+b)^5$, the phrase '5 choose 2' makes better sense. We are now ready to state our last result.

Theorem 7.5. The Binomial Theorem: For any positive integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

or $$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$

While we have not proved these results, they follow from pattern recognition in our expansion of $(a+b)^5$. Notice that we have not included $\binom{n}{0}$ and $\binom{n}{n}$ in the second equation in **Theorem 7.5**. Each of these coefficients is equivalent to 1 so will not change any values in the equation.

Example 7.3.7. Use the Binomial Theorem to expand $(a+b)^6$.

Solution. By the Binomial Theorem, we have

$$
\begin{aligned}
(a+b)^6 &= \sum_{k=0}^{6}\binom{6}{k}a^{6-k}b^k \\
&= a^6+\binom{6}{1}a^{6-1}b+\binom{6}{2}a^{6-2}b^2+\binom{6}{3}a^{6-3}b^3+\binom{6}{4}a^{6-4}b^4+\binom{6}{5}a^{6-5}b^5+b^6 \\
&= a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6
\end{aligned}
$$

□

Example 7.3.8. Use the Binomial Theorem to expand $(2x-3)^4$.

Solution. We begin by writing $(2x-3)^4=((2x)+(-3))^4$. Then, by the Binomial Theorem, we have

$$
\begin{aligned}
(2x-3)^4 &= ((2x)+(-3))^4 \\
&= \sum_{k=0}^{4}\binom{4}{k}(2x)^{4-k}(-3)^k \\
&= (2x)^4+\binom{4}{1}(2x)^{4-1}(-3)^1+\binom{4}{2}(2x)^{4-2}(-3)^2+\binom{4}{3}(2x)^{4-3}(-3)^3+(-3)^4 \\
&= 16x^4+(4)(8x^3)(-3)+(6)(4x^2)(9)+(4)(2x)(-27)+81 \\
&= 16x^4-96x^3+216x^2-216x+81
\end{aligned}
$$

□

The expansions in the previous two examples first showed up in **Example 7.3.3**, and referring back to that example we observe that the Binomial Theorem does not appear to be a time saver. However, the following examples demonstrate the usefulness of the Binomial Theorem.

Example 7.3.9. Find the seventh term in the expansion of $(a+b)^{10}$.

Solution. By the Binonial Theorem, the expansion of $(a+b)^{10}$ is

$$(a+b)^{10}=\sum_{k=0}^{10}\binom{10}{k}a^{10-k}b^k$$

The seventh term corresponds to $k=6$, so the seventh term is

$$\binom{10}{6}a^{10-6}b^6=\frac{10!}{6!(10-6)!}a^{10-6}b^6=\frac{(6!)(7\times8\times9\times10)}{(6!)(1\times2\times3\times4)}a^4b^6$$

After simplifying, we find the seventh term is $210a^4b^6$.

□

Example 7.3.10. Consider $(2x-y)^7$. Find the term in its expansion that contains x^3 and the term that contains y^5.

Solution. By the Binomial Theorem,

$$(2x-y)^7 = \sum_{k=0}^{7} \binom{7}{k} (2x)^{7-k} (-y)^k$$

The term that contains x^3 corresponds to $k=4$ since we want $7-k=3$. The term containing x^3 is

$$\binom{7}{4} (2x)^{7-4} (-y)^4 = \frac{7!}{4!(7-4)!} (2x)^3 y^4 = \frac{5\times 6\times 7}{1\times 2\times 3} (8x^3) y^4$$

After simplifying, we find the term containing x^3 is $280x^3y^4$. Since the term that contains y^5 corresponds to $k=5$, it is

$$\binom{7}{5} (2x)^{7-5} (-y)^5 = \frac{7!}{5!(7-5)!} (2x)^2 (-y)^5 = -\frac{6\times 7}{1\times 2} (4x^2) y^5$$

The term containing y^5 is $-84x^2y^5$.

□

7.3 Exercises

1. What is a binomial coefficient and how is it calculated?

2. When is it an advantage to use the Binomial Theorem? Explain.

In Exercises 3 – 8, evaluate the expression.

3. $6!$

4. $\dfrac{10!}{7!}$

5. $\dbinom{12}{6}!$

6. $\dfrac{100!}{99!}$

7. $\dfrac{7!}{2^3 3!}$

8. $\dfrac{9!}{4!3!2!}$

In Exercises 9 – 17, evaluate the binomial coefficient.

9. $\dbinom{6}{2}$

10. $\dbinom{5}{3}$

11. $\dbinom{7}{4}$

12. $\dbinom{8}{3}$

13. $\dbinom{9}{7}$

14. $\dbinom{10}{9}$

15. $\dbinom{117}{0}$

16. $\dbinom{25}{11}$

17. $\dbinom{200}{199}$

In Exercises 18 – 29, expand the given binomial.

18. $(4a-b)^3$

19. $(x+2)^5$

20. $(5a+2)^3$

21. $(3a+2b)^3$

22. $(2x+3y)^4$

23. $(2x-1)^4$

24. $(4x+2y)^5$

25. $(3x-2y)^4$

26. $(4x-3y)^5$

27. $\left(\dfrac{1}{3}x+y^2\right)^3$

28. $\left(\dfrac{1}{x}+3y\right)^5$

29. $\left(x-x^{-1}\right)^4$

In Exercises 30 – 37, use the Binomial Theorem to find the indicated term.

30. The fourth term of $(2x-3y)^4$

31. The fourth term of $(3x-2y)^5$

32. The third term of $(6x-3y)^7$

33. The eighth term of $(7+5y)^{14}$

34. The seventh term of $(a+b)^{11}$

35. The fifth term of $(x-y)^7$

36. The term containing x^3 in the expansion $(2x-y)^5$

37. The term containing x^{117} in the expansion $(x+2)^{118}$

38. You've just won three tickets to see the new film, '$8.\overline{9}$.' Five of your friends, Brenda, Cindy, Michael, Rachel and Sadie, are interested in seeing it with you. With the help of your classmates, list all the possible ways to distribute your two extra tickets among your five friends. Now suppose you've come down with the flu. List all the different ways you can distribute the three tickets among these five friends. How does this compare with the first list you made? What does this have to do with the fact that $\binom{5}{2}=\binom{5}{3}$?

Key Equations

Recursive Formula for an Arithmetic Sequence: $a_n = a_{n-1} + d$

***n*th Term of an Arithmetic Sequence:**
$a_n = a_1 + (n-1)d$

Recursive Formula for a Geometric Sequence: $a_n = ra_{n-1}$

***n*th Term of a Geometric Sequence:**
$a_n = a_1 r^{n-1}$

Summation Notation:

$$\sum_{n=1}^{j} a_n = a_1 + a_2 + a_3 + \cdots + a_j$$

Properties of Summation:

- $\sum_{n=m}^{p} (a_n \pm b_n) = \sum_{n=m}^{p} a_n \pm \sum_{n=m}^{p} b_n$
- $\sum_{n=m}^{p} a_n = \sum_{n=m}^{j} a_n + \sum_{n=j+1}^{p} a_n$
  for any integer $m \le j < p$
- $\sum_{n=m}^{p} ca_n = c\sum_{n=m}^{p} a_n$ for any constant c
- $\sum_{n=m}^{p} a_n = \sum_{n=m+j}^{p+j} a_{n-j}$ for any integer j

Sum of an Arithmetic Series:

$$S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

Sum of a Geometric Series:

$$S_n = \frac{a_1\left(1-r^n\right)}{1-r}$$

Sum of an Infinite Geometric Series:

$$S_\infty = \frac{a_1}{1-r}, \text{ if } -1 < r < 1$$

Pascal's Triangle: (Triangle continues indefinitely following the same pattern**)**

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Binomial Coefficient: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Binomial Theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$= a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{n-1} ab^{n-1} + b^n$$

Key Terms

Alternating Sequence: A sequence in which the sign of the terms alternates between positive and negative

Annuity: An investment account where payments are deposited into the account on an ongoing basis

Annuity-Due: Annuity with deposits made at the beginning of each compounding period

Arithmetic Sequence: A sequence in which each pair of consecutive terms differ by a fixed amount

Common Difference: The difference between consecutive terms in an arithmetic sequence

Common Ratio: The ratio of one term to the previous term in a geometric sequence

Convergent Series: The infinite sum approaches some finite number

Divergent Series: A series where the sum keeps getting larger

Explicit Formula: Formula used to find the *n*th term of a sequence

Finite Series: Sum of a finite number of terms in a sequence

Geometric Sequence: A sequence in which each term is a constant multiple of the previous term

Infinite Sequence: The sequence continues indefinitely

Infinite Series: Sum of an infinite number of terms in a sequence

Ordinary Annuity: Annuity with equal payments made at the end of each compounding period

Pascal's Triangle: Table formed by rows of coefficients of a binomial expansion

Recursive Formula: Defines each new term of a sequence by one or more of the previous terms

Sequence: A function with domain all natural numbers; an ordered collection

Series: Sum of the terms in a sequence

Terms of a Sequence: Numbers in the sequence