ALGEBRA PRELIM EXAM
January, 2007

Ten points may be achieved on each of the following ten problems. Partial credit is possible on all of them. Very roughly speaking, 40 points will be considered a passing grade. No books or notes may be used during the exam.

1. Let F be a field. Let $M_5(F)$ denote the set of 5-by-5 matrices with entries in F, and let $\mathcal{N}$ denote the subset of nilpotent matrices,

$$\mathcal{N} := \{A \in M_5(F) \mid \text{there exists } N \text{ such that } A^N = 0\}.$$

Let X denote the subset of $M_5(F)$ consisting of those matrices which are squares of nilpotent matrices,

$$X := \{M \in M_5(F) \mid \text{there exists } A \in \mathcal{N} \text{ such that } M = A^2\}.$$

Observe that $G = \mathrm{GL}(5, F)$ acts on X by conjugation. Determine, with proof, the number of orbits for the action of G on X.

2. Let R be a commutative ring with identity. Prove or disprove: the tensor product (over R) of two torsion free R-modules is torsion-free.

3. Determine, with proof, the maximal order of an element in a Sylow 3-subgroup of the symmetric group on 9 letters.

4. Let G be a finite group, let H be a subgroup of G, and let $\mathbb{C}[G]$ (respectively, $\mathbb{C}[H]$) denote the complex group algebra of G (respectively, H). Let I_H^G denote the functor from the category of finite-dimensional $\mathbb{C}[H]$ modules to the category of finite-dimensional $\mathbb{C}[G]$ modules which, on the level of objects, is defined by

$$I_H^G(M) = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} M.$$

Prove or disprove: I_H^G is an exact functor.

5. Let F be a field with $\mathbb{Q} \subset F \subset \mathbb{C}$, where $F/\mathbb{Q}$ is a finite Galois extension. Let $\alpha \in F$ and let $f(X) \in \mathbb{Q}[X]$ be its minimal monic polynomial. Assume that the absolute value $|\alpha|$ of α is 1, and that $\mathrm{Gal}(F/\mathbb{Q})$ is abelian.

(a) Show that F is closed under complex conjugation.

(b) Prove that $|\beta| = 1$ for every complex root β of $f(X)$.

(c) Writing $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$, show that $|a_i| \leq 2^n$ for all i with $0 \leq i < n$.

(d) Prove that F contains only finitely many algebraic integers having absolute value 1 and deduce that each of these is a root of unity.

(over)

6. Let N be a normal subgroup of a finite group G. Assume that the center of N is trivial and that any automorphism of N is inner. Show that there exists a normal subgroup H of G such that G is isomorphic to the direct product of H and N.

7. Let R be a commutative ring with 1. Suppose I and J are two ideals of R such that $I + J = R$. Prove that $R/(IJ) \simeq R/I \oplus R/J$.

8. Let N be a positive integer. Consider the natural map from $\mathrm{SL}(2, \mathbb{Z})$ to $\mathrm{SL}(2, \mathbb{Z}/N)$ obtained by reducing each matrix entry modulo N. Prove or disprove: this map is surjective.

9. Let $\mathbb{Z}[x]$ be the ring of polynomials in x with integer coefficients. Let R denote the subring consisting of all polynomials having their coefficients of x and x^2 equal to 0.

(a) Show that $\mathbb{Q}(x)$ is the field of fractions of R.

(b) Compute the integral closure of R in $\mathbb{Q}(x)$.

(c) Does there exist a polynomial $g(x) \in R$ such that R is generated as a ring by 1 and $g(x)$?

10. Let R be a unique factorization domain with quotient field k. Let f be an element of $R[x]$. Show that if f can be factored as a product of lower-degree polynomials in $k[x]$, then f can be factored as a product of lower-degree polynomials in $R[x]$.

ALGEBRA PRELIM EXAM
August 22, 2006

Ten points may be achieved on each of the following ten problems. Partial credit is possible on all of them. Very roughly speaking, 40 points will be considered a passing grade. No books or notes may be used during the exam.

1. Let F be a field. Let X denote the set of 8-by-8 matrices A with entries in F such that $A^3 = 0$ but $A^2 \neq 0$. Observe that $G = \mathrm{GL}(8, F)$ acts on X by conjugation. Compute the number of orbits for the action of G on X.

2. Let R be a commutative integral domain with identity. Recall that an R-module is torsion free if for any $r \in R$ and $m \in M$, $rm = 0$ implies that either $r = 0$ or $m = 0$.

(a) Suppose R is a principal ideal domain and M and N are two R modules. Prove that if M and N are torsion free, so is $M \otimes_R N$.

(b) Given an example of an integral domain R and two torsion free R-modules M and N such that $M \otimes_R N$ is not torsion free.

3. Describe the structure of a Sylow 2-subgroup of the symmetric group S_n in terms of products and semidirect products of known groups for:

(a) $n = 4$; and

(b) $n = 8$.

4. Let G be a finite group and let $\mathbb{C}[G]$ denote the complex group algebra of G. Show that the invariants functor which (on the level of objects) takes a $\mathbb{C}[G]$ module M to

$$M^G := \{m \in M \mid f \cdot m = m \text{ for all } f \in \mathbb{C}[G]\}$$

is an exact functor.

5. Suppose R is a commutative ring with identity. Let J denote the intersection of all maximal ideals in R. Suppose M is a finitely-generated R-module such that

$$JM := \{jm \mid j \in J \text{ and } m \in M\} = \{0\}.$$

Prove that $M = \{0\}$.

(over)

6. Determine the Galois group of $x^4 - 2$ over each of the following fields: the complex numbers; the real numbers; the rational numbers; the integers modulo 5.

7. Determine the group of units in the ring $\mathbb{Q}[x]/\left((x^2+1)^2\right)$.

8. Determine (up to isomorphism) all groups G which satisfy the following condition: there exists a proper subgroup H of G which contains all proper subgroups of G.

9. Let V be a finite-dimensional vector space over a field of characteristic $p > 0$. Let $T : V \to V$ be a linear transformation with $T^{p^n} = \mathrm{Id}$ for some n. Prove that there is a nonzero vector $v \in V$ with $Tv = v$.

10. Prove that there are no simple groups of order 992.

ALGEBRA PRELIM EXAM
January 2006

Ten points may be achieved on each of the following ten problems. Partial credit for quality work, is possible on each problem. Very roughly speaking, 40 points will be considered a passing grade. No books or notes may be used during the exam.

(1) Let G be a finite non-abelian group of order p^3 where p is a prime number. If Z is the center of G, show that G/Z is isomorphic to $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$.

(2) Classify all groups of order $57 = 3 \cdot 19$.

(3) Show that

$$\mathbb{Q}/\mathbb{Z} \cong \bigoplus (\mathbb{Q}/\mathbb{Z})(p)$$

where p runs over all primes numbers and $(\mathbb{Q}/\mathbb{Z})(p)$ denotes the subgroup of elements of $\mathbb{Q}/\mathbb{Z}$ whose order is a power of p.

(4) Prove Gauss' Lemma over the integers: if $h(x) \in \mathbb{Z}[x]$ such that $h(x) = f(x)g(x)$ for nonconstant polynomials $f(x), g(x) \in \mathbb{Q}[x]$, prove that there exist rational numbers a and b so that $F(x) = af(x)$ and $G(x) = bg(x)$ are in $\mathbb{Z}[x]$ and so that $h(x) = F(x)G(x)$.

(5) Let R be a commutative ring with 1 and

$$0 \to K \to P \to M \to 0, \qquad 0 \to K' \to P' \to M \to 0$$

be short exact sequences of R-modules where P and P' are projective R-modules. Show that there is an isomorphism of R-modules $K \oplus P' \cong K' \oplus P$.

(6) Let G be the Galois group of the polynomial $x^4 - 5$ over the field F. Determine G in each of the following cases:
 (a) F is the real numbers
 (b) F is the rational numbers
 (c) F is the integers modulo 3.

(7) Let p be a prime number and F_p be the field with p elements. Let $f(x) = x^p - x - 1 \in F_p[x]$ and let K be the splitting field of $f(x)$. Show that:
 (a) if α is a root of $f(x)$, then $\alpha + 1$ is a root of $f(x)$,
 (b) $K = F_p(\alpha)$,
 (c) the Frobenius automorphism $\sigma(x) = x^p$ is a non-trivial element of $\mathrm{Gal}(K/F_p)$,
 (d) $f(x) \in F_p[x]$ is irreducible.

(8) Let K and F be fields such that K is a finite extension of F. Assume that there are only finitely many subfields L such that

$F \subset L \subset K$. Show that K is a primitive extension of F (i.e. $K = F(\theta)$ for some $\theta \in K$).

(9) Let V be a finite dimensional vector space over a field F and $T : V \to V$ be a linear transformation. Show that there is an integer $n > 0$ such that

$$V \cong \mathrm{Ker}(T^n) \oplus \mathrm{Im}(T^n).$$

(10) Let A be a square complex matrix such that $A^3 = A$. Prove that A is diagonalizable. Is this true over any algebraically closed field?

ALGEBRA PRELIM EXAM
August 2005

Ten points may be achieved on each of the following ten problems. Partial credit for quality work, is possible on each problem. Very roughly speaking, 45 points will be considered a passing grade. No books or notes may be used during the exam.

(1) Let G be a finite group and p a prime number such that $|G| = p^n$ for some $n > 0$. Prove that G has nontrivial center and that for all integers $0 < m < n$, G has at least one normal subgroup of order p^m.

(2) Let G be a group of order 63. Show that G contains an element of order 21.

(3) Let R be a commutative ring with 1, I, J ideals such that $I+J = R$. Show that $R/(IJ) \cong R/I \oplus R/J$.

(4) Let S be a multiplicative subset of commutative ring A not containing 0. Let P be a maximal element in the set of ideals of A whose intersection with S is empty. Show that P is prime.

(5) Let (R, m) be a local ring (i.e. a commutative ring with a unique maximal ideal m). Let M be a finitely generated R-module with $mM = M$. Show from first principles that $M = \{0\}$.

(6) ~~Find a galois extension K of $\mathbb{Q}$ with Galois group $\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$.~~

(7) Let α be a primitive fifth root of 1 and $f(x) = x^5 + 5x^2 + 4$. Show that $f(x)$ is irreducible in $\mathbb{Q}[\alpha]$.

(8) Let K be a finite field.
   (a) Show that $|K| = p^n$ for some prime p.
   (b) Show that $K^* = K - \{0\}$ is a cyclic group with respect to multiplication.

(9) Let A be an $n \times n$ matrix with complex coefficients. Show that $A^n = 0$ if and only if $tr(A^i) = 0$ for all $i > 0$.

(10) Suppose F is a field and T is an invertible $n \times n$ matrix with entries in F. Show that there exist constants $a_1, \ldots, a_k$ in F so that

$$T^{-1} = \sum_{i=1}^{k} a_i T^i.$$

ALGEBRA PRELIM EXAM
January 5, 2005

Ten points may be achieved on each of the following ten problems. Partial credit is possible on all of them. Very roughly speaking, 40 points will be considered a passing grade. No books or notes may be used during the exam.

1. Let $\mathcal{N}$ denote the set of two-by-two real matrices with determinant zero, let $\mathrm{SL}(2,\mathbb{R})$ denote the set of two-by-two matrices of determinant one, and finally let $\mathrm{GL}(2,\mathbb{R})$ denote the set of two-by-two matrices with nonzero determinant.
   (a) Show that both $\mathrm{GL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{R})$ act on $\mathcal{N}$ by conjugation.
   (b) Describe the orbits of $\mathrm{GL}(2,\mathbb{R})$ on $\mathcal{N}$.
   (c) Describe the orbits of $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{N}$.

2. Let R be a ring with identity. Suppose

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow K' \longrightarrow P' \longrightarrow M \longrightarrow 0$$

are two short exact sequences of R-modules. Suppose further that P and P' are projective. Prove that $K' \oplus P \simeq K \oplus P'$ as R-modules.

3. Let A be an n-by-n diagonal matrix with coefficients in a field F. Suppose the characteristic polynomial of A has the form

$$(x-c_1)^{d_1}(x-c_2)^{d_2}\cdots(x-c_k)^{d_k}$$

with $c_1,\ldots,c_k$ distinct. Let V denote the space of all n-by-n matrices over F which commute with A. Compute the dimension of V as a vector space over F.

4. Let R be a finite ring with identity. Let a be a nonzero element of R which is not invertible. Prove that a is a zero divisor.

5. Suppose R is a commutative Noetherian ring. Let $R[x]$ denote the ring of polynomial with coefficients in R. Prove that $R[x]$ is Noetherian.

(over)

6. Determine the number of subgroups of order 16 in the symmetric group S_6.

7. Let F be a field such that the multiplicative group of F is cyclic. Prove that F is a finite field.

8. Prove or disprove that $f(x) = x^5 - x + 1$ is irreducible over the rationals.

9. Let G_F denote the Galois group of $x^3 - 3x^2 - 3$ over the field F. Determine G_F when:

(a) F is the complex numbers;
(b) F is the real numbers;
(c) F is the rational numbers; and
(d) F is the integers modulo 2.

10. Prove that there are no simple groups of order 1365.

ALGEBRA PRELIM EXAM
August 16, 2004

Ten points may be achieved on each of the following ten problems. Partial credit is possible on all of them. Very roughly speaking, 40 points will be considered a passing grade. No books or notes may be used during the exam.

1. Set

$$J = \begin{pmatrix} 0 & 1 & & & & & & & & \\ & 0 & 1 & & & & & & & \\ & & 0 & & & & & & & \\ & & & 0 & 1 & & & & & \\ & & & & 0 & & & & & \\ & & & & & 0 & 1 & & & \\ & & & & & & 0 & & & \\ & & & & & & & 0 & & \\ & & & & & & & & 0 & \\ & & & & & & & & & 0 \end{pmatrix} \in M_{10}(\mathbb{C});$$

here it is understood that unless otherwise noted, the entries of J are 0. Let

$$S = \{X \in M_{10}(\mathbb{C}) \mid X^2 = gJg^{-1} \text{ for some } g \in \mathrm{GL}(10, \mathbb{C})\}.$$

Observe that $\mathrm{GL}(10, \mathbb{C})$ acts on S by conjugation. Compute the number of orbits for this action.

2. Let R be a commutative ring with identity. Given an R-module M, recall the R-module

$$\bigwedge M = \bigoplus_{j \geq 0} \bigwedge^j M,$$

where $\bigwedge^j M$ is the jth exterior power of M. Define a natural operation on morphisms so that the assignment $M \mapsto \bigwedge M$ becomes a functor and determine whether this functor is exact.

3. Let V be a finite-dimensional vector space over a field k, and let A and B be diagonalizable linear operators on V such that $AB = BA$. Prove that A and B are simultaneously diagonalizable; i.e. that there is a basis for V consisting of eigenvectors of both A and B.

4. Let R be a commutative ring with identity. Let G be a finite subgroup of $R^\times$, the group of units of R. Prove that if R is an integral domain, then G is cyclic.

5. Suppose R is a commutative Noetherian ring. Let $R[[x]]$ denote the ring of formal power series with coefficients in R. Prove that $R[[x]]$ is Noetherian.

(over)

6. Prove that there are no simple groups of order 616.

7. Let F be a finite field, and suppose that the subfield of F generated by $\{x^3 \mid x \in F\}$ is different from F. Show that the cardinality of F is 4.

8. Prove or disprove that $f(x) = x^4 + x^3 + x^2 + 6x + 1$ is irreducible over $\mathbb{Q}$.

9. Prove that the Galois group of $x^3 + x - 1$ over $\mathbb{Q}$ is the symmetric group on three letters.

10. Let G be a finite group with identity e. Suppose that for all $a, b \in G$ distinct from e, there exists an automorphism τ of G such that $\tau(a) = b$. Prove that G is abelian.

ALGEBRA PRELIM

January, 2004

INSTRUCTIONS: Work on as many problems as you wish. A passing grade is 40 points or more where each problem is worth 10 points. SHOW ALL WORK. You are not to use any books or notes while doing this test.

1. Suppose G is a finite group of order a power of a prime. If H is a nonidentity normal subgroup of G, prove that H contains a nonidentity element of Z(G). [$Z(G) = \{x \in G \mid xy = yx$ for all $y \in G\}$.]

2. Prove or disprove that $x^5 - x + 1$ is irreducible over the rationals.

3. Let V be a finite dimensional vector space of the field of rational numbers and let T be a linear transformation of V into itself such that T^5 is the identity mapping. If there is no nonzero vector in V which T maps to itself, prove that the dimension of V is a multiple of 4.

4. Suppose H is a subgroup of the finite group G and the index of H in G is the smallest prime dividing the order of G. Prove that H is a normal subgroup of G.

5. Compute the Galois group of the polynomial $x^4 - 2$ over each of these fields: (a) the complex numbets (b) the reals, (c) the rationals, and (d) the integers modulo 5.

6. Let R be a ring and let M be an irreducible R-module. Prove that one of the following statements is true about M considered as an abelian group:
   (1) M is the direct sum of (perhaps infinitely many) copies of the additive group of all rational numbers
   (2) M is the direct sum of (perhaps infinitely many) copies of a cyclic group of order p for some prime p.

7. Let R be a ring and let L be the set of all nilpotent elements of R.
   (a) If R is commutative, prove that L is an ideal.
   (b) Give an example of a ring R in which L is not an ideal.

8. Prove that a group of order 112 cannot be simple.

9. Let R be a semisimple right Artinian ring (in particular, this implies that R has an identiy 1). Assume $x \in R$ and define the sets A_x and B_x by

$$A_x = \{y \in R \mid y \neq 0 \text{ and } xy = 0\}$$

$$B_x = \{y \in R \mid xy = 1\}.$$

Prove that exactly one of these sets is empty.

10. Let E be an extension of the field F and let α be an element of E which is algebraic over F of degree n.

(a) If n is prime to 2, prove that $F(\alpha^2) = F(\alpha)$.

(b) Give an example in which n is prime to 3 but $F(\alpha^3) \neq F(\alpha)$.

ALGEBRA PRELIM

August, 2003

INSTRUCTIONS: Work on as many problems as you wish. A passing grade is 40 points or more where each problem is worth 10 points. SHOW ALL WORK. You are not to use any books or notes while doing this test.

1. Suppose H is a subgroup of the group G such that $x^2 \in H$ for all $x \in G$. Prove that H is a normal subgroup of G.

2. Suppose A is an $n \times n$ complex matrix with $n > 1$. If A has rank 1, prove that A is similar to a diagonal matrix if, and only if, trace(A) $\neq 0$. [the trace of a matrix is the sum of the entries on the main diagonal.]

3. Let R be the ring of polynomials in one variable over a field F. Prove that R has infinitely many maximal ideals.

4. (a) Suppose K is a field of prime order and E is the algebraic closure of K. If F is any subfield of E, prove that the multiplicative group of F is periodic, i.e., every element of the multiplicative group has finite order.

   (b) Suppose F is a field whose multiplicative group is periodic. Prove that for some prime p, F is isomorphic to a subfield of the algebraic closure of the field of order p.

5. Let G be the Galois group of the polynomial $x^3 - 3x^2 - 3$ over the field F. Determine G in each of the following cases:

   (a) F is the complex numbers.

   (b) F is the real numbers.

   (c) F is the rational numbers.

   (d) F is the integers modulo 2.

6. If G is a simple group of order 60, prove that G is isomorphic to A_5, the alternating group of degree 5.

7. Suppose G is a group of order p^n where p is a prime and $n > 1$. If A is the group of all automorphisms of G, prove that p divides the order of A.

8. Determine, with proof, all rings R such that R contains at most 11 elements and $x^{11} = x$ for all $x \in R$.

9. Prove that there is a rational polynomial whose Galois group over the rationals is the direct product of two groups of order 3.

PRELIMINARY EXAMINATION IN ALGEBRA

August 13, 2002

Instructions: Answer as many questions or parts of questions that you wish. A passing score consists of five complete answers or a reasonable equivalent.

1. Let G be a group of order p^n, where p is a prime number and $n > 0$ is an integer. Show that the center of G is not trivial.

2. Let $E \subseteq F$ be an extension of finite fields. Let q be the number of elements of E, and let $\phi : F \to F$ be defined by $\phi(a) = a^q$. Prove that F is a Galois extension of E and that ϕ generates its Galois group.

3. Describe the 2-Sylow subroup of S_6 (that is, show that it is isomorphic to a group described in terms of known groups).

4. Prove that a finite abelian group is determined up to isomorphism by the number of elements of each order.

5. Show that there are no simple groups of order 616.

6. Let $E \subseteq F$ be an extension of finite fields. Prove that the trace map from F to E is surjective.

7. Let G be a finite group, and let $\mathcal{C}$ be the category of finitely generated $\mathbf{Q}[G]$-modules. Show the the functor which takes a $\mathbf{Q}[G]$module to its submodule of invariants (that is, the set of $m \in M$ such that $gm = m$ for all $g \in G$) is exact.

8. Determine for which fields the polynomial $X^3 - X + 1$ has a multiple root.

9. Let E be a finite separable extension of a field K and let F be any finite field extension of K. Show that $E \otimes_K F$ is a finite product of fields.

10. Prove that a principal ideal domin is a unique factorization domain.

11. Let R be a an algebra over a field k of dimension n. Let I be the intersection of all maximal left ideals of R. Show that I is a 2-sided ideal and that $I^n = 0$.
Give an example for which $I^{n-1} \neq 0$.

PRELIMINARY EXAMINATION IN ALGEBRA

August 13, 2001

Instructions: Answer as many questions or parts of questions that you wish. A passing score consists of five complete answers or a reasonable equivalent.

1. Let G be a finite group, and let H be a subgroup of index p, where p is the smallest prime number dividing the order of G. Show that H is a normal subgroup of G.

2. Show that there are no simple groups of order 132.

3. Show that a polynomial $f(X)$ of degree n over the field $\mathbb{Z}/p\mathbb{Z}$ is irreducible if and only if $f(X)$ is relatively prime to the polynomial $X^{p^m} - X$ for all m less than or equal to $n/2$.

4. Show that there exists a Galois extension of the rational numbers with Galois group $\mathbb{Z}/7\mathbb{Z}$.

5. Ler R be a unique factorization domain with quotient field K. Let $f(X)$ be a polynomial in $R[X]$. Show that if $f(X)$ can be factored into polynomials of lower degree in $K[X]$, then $f(X)$ can be factored into polynomials of lower degree in $R[X]$.

6. Let G be a finite group. Prove the following version of Schur's Lemma: if $\rho : G \to M_n(\mathbb{C})$ is a finite dimensional irreducible representation of G over the complex numbers, then every element of $M_n(\mathbb{C})$ that commutes with $\rho(g)$ for all $g \in G$ is a scalar matrix.

7. Let N be a positive integer. Prove that the natural map from $SL_2(\mathbb{Z})$ to $SL_2(\mathbb{Z}/N\mathbb{Z})$ is surjective.

8. Let G be a nonabelian group of order p^3 for a prime p and and let Z be its center. Show that $G/Z \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$.

9. Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$, and that $\mathbb{Q} \otimes \mathbb{Z}/N\mathbb{Z} = 0$ for every positive integer N, while $\mathbb{Q} \otimes (\prod \mathbb{Z}/N\mathbb{Z}) \neq 0$, where the product is taken over all positive integers N.

10. Show that if a 3 by 3 matrix with entries in $\mathbb{Q}$ satisfies $A^8 = I$, then it satisfies $A^4 = I$.

11. Let G be a finite abelian group and let n be an integer. Show that the map which sends g to ng is a group homomorphism and that it is an isomorphism if and only if n is relatively prime to the order of G.

ALGEBRA PRELIM

August, 2000

INSTRUCTIONS: Work on as many problems as you wish. A passing grade is 40 points or more where each problem is worth 10 points. SHOW ALL WORK. You are not to use any books or notes while doing this test.

1. If G is a group of order 105, prove that G has a normal cyclic subgroup of order 35.

2. Let $f(x)$ be a polynomial of degree $n > 2$ over $\mathbb{Q}$, the field of rational numbers. Let K be the splitting field of $f(x)$ over $\mathbb{Q}$ and suppose that the Galois group of $f(x)$ over $\mathbb{Q}$ is S_n, the group of all permutations of n objects. Prove the following:
   (a) $f(x)$ is irreducible over $\mathbb{Q}$.
   (b) Let α be a root of $f(x)$ in K and let σ be an automorphism of the field $\mathbb{Q}(\alpha)$. Prove that $\sigma = 1$.

3. Find an integer n such that the abelian group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ is a quotient of the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^*$ of integers relatively prime to n modulo n.

4. Let G be the Galois group of the polynomial $x^3 + 2x - 2$ over the field F. Determine G in each of the following cases:
   (a) F is the complex numbers.
   (b) F is the real numbers.
   (c) F is the rational numbers.
   (d) F is the integers modulo 3.

5. Let H be a proper subgroup of the finite group G. Prove that there is an element x in G such that x is not conjugate in G to any element of H.

6. Let A and B be commuting $n \times n$ matrices over a field F. If A and B are both diagonalizable over F, prove that A and B are simultaneously diagonalizable (i.e., there is an $n \times n$ nonsingular matrix S over F such that both $S^{-1}AS$ and $S^{-1}BS$ are diagonal).

7. Let R be an infinite simple ring (i.e., a ring with infinitely many elements and with no 2-sided ideals except $\{0\}$ and R). Show that exactly one of the following holds.

   (a) R is a division ring.

   (b) R has an infinite number of right ideals.

8. Let G be a nonabelian group of order p^3 where p is a prime. Prove that Z(G) and G' are equal and have order p. [Here Z(G) is the center of G and G' is the commutator subgroup of G, i.e., G' is generated by all elements of the form $x^{-1}y^{-1}xy$ with $x, y \in G$.]

9. Determine all rings R with an identity such that R contains at most 9 elements and $x^9 = x$ for all $x \in R$.

10. Suppose F is a field and $K = F(\alpha)$ is a finite Galois extension of F. Assume that the Galois group of K over F is cyclic generated by an automorphism σ where $\alpha^\sigma = \alpha + 1$. Prove that F has characteristic $p \neq 0$ and that $\alpha^p - \alpha \in F$.

PRELIMINARY EXAMINATION IN ALGEBRA

August 16, 1999

Instructions: Answer as many questions or parts of questions that you wish. A passing score consists of six complete answers or a reasonable equivalent.

1. Show that there are no simple groups of order 1365.

2. Prove that the symmetric group S_n on n elements is generated by the transpositions $(12), (23), \ldots, (n-1\ n)$.

3. Let G be a group of order 55 acting on a set X consisting of 24 elements. Show that there is at least one element fixed by every element of G.

4. Let F be a field, and let $f(X)$ be a polynomial in $F[X]$. Show that $f(X)$ has a multiple root in some extension of F if and only if $f(X)$ and its derivative $f'(X)$ are not relatively prime.

5. Show that if α and β are elements of a field E which are algebraic over a subfield F, then $\alpha + \beta$ and $\alpha\beta$ are algebraic over F.

6. Let E be a normal extension of $\mathbb{Q}$ contained in $\overline{\mathbb{Q}}$ (the algebraic closure of $\mathbb{Q}$), and let K be an extension of $\mathbb{Q}$ contained in E. Let K' be a subfield of $\overline{\mathbb{Q}}$ which is isomorphic to K. Show that K' is contained in E and the Galois groups of E/K and E/K' are conjugate in the Galois group of $E/\mathbb{Q}$.

7. Prove that if A is a commutative Noetherian ring, then $A[X_1, \ldots, X_n]$ (the polynomial ring in n variables over A) is Noetherian.

8. Let G be a finite group. Prove that every short exact sequence of $\mathbb{C}[G]$-modules splits, where $\mathbb{C}[G]$ is the group algebra of G over the complex numbers.

9. Let K be an arbitrary field, and let L be an extension of K obtained by adjoining a root of the polynomial $X^n - 1$ for some integer $n \geq 1$. Show that L is Galois over K with abelian Galois group.

10. Show that a free group F is torsion-free; that is, that if $g \neq 1$ is an element of F, then $g^n \neq 1$ for all $n \geq 1$.

11. Let M be an n by n matrix with entries in $\mathbb{C}$. Assume that there is a e polynomial $f(X)$ in $\mathbb{Q}[X]$ which is irreducible over $\mathbb{Q}$ such that $f(M) = 0$ (i.e. it is the zero matrix). Show that M is diagonalizable over $\mathbb{C}$.

12. Show that the endomorphism ring of a simple module is a division ring.

PRELIMINARY EXAMINATION IN ALGEBRA

August 20, 1998

Instructions: Answer as many questions or parts of questions that you wish. A passing score consists of five complete answers or a reasonable equivalent.

1. Let q be a prime number. Show that the number of non-isomorphic abelian q-groups of order q^n is $p(n)$, where $p(n)$ denotes the number of partitions of n.

2. Let M be a square matrix over $\mathbb{C}$ such that M^k is the identity matrix for some integer $k > 0$. Show that M is diagonalizable.

3. Let k be a field containing a primitive n^{th} root of unity. Show that for any $a \in k$ the Galois group of $X^n - a$ over k is abelian.

4. Show that there are no simple groups of order 245.

5. State and prove the Hilbert Basis Theorem.

6. Prove Wedderburn's theorem: if the ring R has a faithful simple module that is finite-dimensional over its commutant ring, then $R \cong M_n(D)$ for some $n \geq 1$, where D is a division ring.

7. Prove that a free group has no nonidentity elements of finite order.

8. Let K/k be a Galois extension with Galois group G. Show that every 1-cycle on G with values in K^* is a 1-coboundary.

9. If H is a finite group, prove that there exists a field k and a Galois extension of k with Galois group H.

10. Prove that the Galois group of $X^3 - X - 1$ over $\mathbb{Q}$ is S_3, the symmetric group on three elements.

11. Let p be a prime number, and let G be a finite group of order p^n acting on a set X consisting of r elements. Show that the number of points fixed by all elements of G is congruent to r modulo p.

ALGEBRA PRELIM

September, 1997

Instructions: Work on as many problems as you wish. A passing grade is 40 points or more with each problem worth 10 points. SHOW ALL WORK. You are not to use any books or notes while doing this test.

1. Let G be a finite group with the property that for each positive integer n, there are at most 2 subgroups of order n in G. Prove that G is cyclic.

2. Let V be a finite dimensional vector space over an algebraically closed field F. Let T be a linear transformation of V into itself, let λ be an eigenvalue of multiplicity m of T (i.e., m is the multiplicity of λ as a root of the characteristic polynomial of T), and let U be the subspace $U = \{v \in V \mid T(v) = \lambda v\}$. Prove that the dimension of U is $\leq m$.

3. Let R be a ring and let L be the set of all nilpotent elements of R.
   (a) If R is commutative, prove that L is an ideal.
   (b) Give an example of a ring R in which L is not an ideal.

4. Prove that there is a polynomial p(x) with rational coefficients such that the Galois group of p(x) over the field of rational numbers is cyclic of order 5.

5. Let H be a proper subgroup of the finite group G. Prove that G has a conjugacy class C such that C contains no element of H.

6. Let A and B be finite rings with the same number of elements. Assume that n is an integer > 1 and $x^n = x$ for all $x \in A$ and $y^n = y$ for all $y \in B$.
   (a) If n = 6, prove that A and B are isomorphic.
   (b) If n = 4, prove by an example that A and B need not be isomorphic.

7. Let E be an extension of the field F and let α be an element of E which is algebraic over F of degree n.
   (a) If n is prime to 2, prove that $F(\alpha^2) = F(\alpha)$.
   (b) Give an example in which n is prime to 3 but $F(\alpha^3) \neq F(\alpha)$.

8. Let A_n be the alternating group of degree n (i.e., n is the number of objects permuted by A_n). If $n \geq 5$ and H is a subgroup of A_n such that the order of H is $\frac{(n-1)!}{2}$, then prove that H is isomorphic to A_{n-1}.

9. Let R be a semisimple right Artinian ring (in particular, this implies that R has an identity 1). Assume $x \in R$ and define the sets A_x and B_x by

$$A_x = \{y \in R \mid y \neq 0 \text{ and } xy = 0\}$$
$$B_x = \{y \in R \mid xy = 1\}.$$

Prove that exactly one of these sets is empty.

10. Let F be a field such that the multiplicative group of F is cyclic. Prove that F is a finite field.

PRELIMINARY EXAMINATION IN ALGEBRA

September 9, 1996

Instructions: Answer as many questions or parts of questions that you wish. A passing score consists of five complete answers or a reasonable equivalent.

1. Let G be a finite group of order p^n acting on a set X consisting of k elements. Show that the number of points fixed by all elements of G is congruent to k modulo p.

2. Find all rings (with 1), up to isomorphism, containing exactly 25 elements.

3. Show that there are no simple groups of order 520.

4. Let $K \subseteq L$ be finite fields. Prove that the extension L of K is Galois with cyclic Galois group.

5. Let E be a finite extension of the rationals. Show that the number of roots of unity in E is finite.

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6. Let $P_\bullet$ be a complex of projective modules with $P_n = 0$ for $n < 0$. Let $E_\bullet$ be an exact complex. Show that any map of complexes from $P_\bullet$ to $E_\bullet$ is homotopic to zero.

7. Prove the Cayley-Hamilton Theorem (the theorem that states that for every matrix A we have $f(A) = 0$, where $f(X)$ is the characteristic polynomial of A).

8. Let $f : \mathbb{Z}^n \to \mathbb{Z}^n$ be a map of free Abelian groups of rank n defined by a matrix A with non-zero determinant. Show that the cokernel of f is finite and that its order is equal to the absolute value of the determinant of A.

9. Let K be a field of characteristic zero. Let L be the splitting field of the polynomial $X^n - a$ for some $a \in K$ and some integer $n \geq 1$. Show that the Galois group of L over K is solvable.

10. Let I and J be ideals of a commutative ring R. Show that $(R/I) \otimes_R (R/J)$ is isomorphic to $R/(I + J)$.

11. Let G be a finite subgroup of the group $GL_n(\mathbb{C})$ of invertible n by n matrices with entries in $\mathbb{C}$.

a. Show that every element of G is diagonalizable over $\mathbb{C}$.

b. Give an example of a finite subgroup of $GL_n(\mathbb{Q})$ which contains elements which are not diagonalizable over $\mathbb{Q}$.

12. Show that if a module M is a sum of simple modules, then it is a direct sum of simple modules.

ALGEBRA PRELIM

September, 1995

INSTRUCTIONS: Work on as many problems as you wish. A passing grade is 40 points or more where each problem is worth 10 points. SHOW ALL WORK. You are not to use any books or notes while doing this test.

1. If A is a square matrix over the field of complex numbers, prove that A is similar to its transpose.

2. Determine, with proof, all rings R such that R contains at most 10 elements and $x^{10} = x$ for all $x \in R$.

3. Suppose G is a finite simple group such that G has exactly 5 distinct Sylow p-subgroups for some prime p. Prove that G is isomorphic to A_5, the alternating group of order 60.

4. Suppose F is a field of characteristic 0 with the property that every polynomial of odd degree over F has at least one root in F. Assume that E is an extension of degree 2 over F with the property that for each $\alpha \in E$, the polynomial $x^2 - \alpha$ has a root in E. Prove that E is algebraically closed.

5. If R is a simple ring and n is a positive integer, prove that R_n, the ring of all $n \times n$ matrices over R, is simple.

6. (a) Let H be a maximal subgroup of the finite group G and let M be a minimal normal subgroup of G. Assume that M is abelian but H does not contain M. Prove that G = HM, $H \cap M = \{1\}$ (the identity subgroup), and $|G:H|$ (the index of H in G) is a power of a prime.

   (b) Use the result in (a) to show that that if G is a finite solvable group and H is a maximal subgroup of G, then $|G:H|$ is a power of a prime.

7. Let F be the field of integers modulo 7. Determine which finite extensions of F contain exactly four 4th roots of 1.

8. Let R be a commutative ring with an identity and assume that I and J are ideals in R such that $I + J = R$. Prove that $IJ = I \cap J$ and that R/IJ is isomorphic to the direct product $(R/I) \times (R/J)$.

9. Assume that A and B are solvable subgroups of the group G and $G = AB$. Here $AB = \{ab \mid a \in A, b \in B\}$.

   (a) Give an example showing that G does not have to be solvable.

   (b) Prove that if either A or B is a normal subgroup of G, then G is solvable.

10. Let R be a ring and let M be an irreducible R-module. Prove that one of the following statements is true about M considered as an abelian group:

    (1) M is the direct sum of (perhaps infinitely many) copies of the additive group of all rational numbers

    (2) M is the direct sum of (perhaps infinitely many) copies of a cyclic group of order p for some prime p.

Preliminary Examination in Algebra

September 1994

Instructions: A passing grade consists of eight problems fully worked, or partial credit for the equivalent of 8 problems. All rings are associative (but not necessarily commutative) and possess a unit element. A field is a commutative division ring.

1. a) No group of order 156 is simple.
   b) Every group of order 289 is abelian.

2. If a square matrix M of complex numbers satisfies the equation

$$M^{1994} = \text{Identity},$$

then M is diagonalizable.

3. a) Define the elementary divisors of a square matrix M over the principal ideal domain R.
   b) Calculate the elementary divisors of the matrix of integers $\begin{pmatrix} 4 & 6 \\ 6 & 12 \end{pmatrix}$.

4. a) Let R be a ring and let M be an R-module. Prove that if M is generated by simple submodules, then M is the direct sum of simple submodules.
   b) Show that every vector space over a field has a basis.

5. Show that if the field F contains a primitive n^{th}-root of unity, then the Galois group of the polynomial $x^n - a \in F[x]$, $a \in F$, is abelian.

6. Show that if

$$0 \to K \to P \to M \to 0$$

and

$$0 \to K' \to P' \to M \to 0$$

are short exact sequences of R-modules, where P and P' are projective, then

$$K \oplus P' \cong K' \oplus P.$$

7. If R is a ring and M is a simple R-module, then $\mathrm{End}_R(M)$ is a division ring.

8. If R is a left noetherian ring, then $R[x]$ is left noetherian.

9. If $p(x) \in F[x]$ where F is a finite field, then $p(x)$ is separable over $F[x]$.

10. A finite subgroup of the multiplicative group of nonzero elements of a field is cyclic.

11. Show that the elements $e_1 \wedge e_2 + e_3 \wedge e_4$, $e_1 \wedge e_3 + e_3 \wedge e_4$, and $e_5 \wedge e_6 + e_1 \wedge e_3$ are linearly independent in $\Lambda^2 \mathbf{R}^6$, where $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ is a basis for $\mathbf{R}^6$.

12. Explain what a free group is and prove that any two free bases for the free group F have the same cardinality.

13. Prove that the group given by the presentation

$$\langle x, y \mid x^2 = 1, y^2 = 1, xyx = yxy \rangle$$

has 6 elements.

14. a) Explain what it means for a *contravariant* functor $F : \mathcal{C} \to \mathrm{Set}$ to be representable ('Set' is the category of sets).
b) Show how the power set $P(S) = \{T \mid T \subseteq S\}$ defines a contravariant functor $P : \mathrm{Set} \to \mathrm{Set}$ and prove that P is representable.