ALGEBRA EXAM - AUGUST 2007

1) Let $\mathbb{F}_q$ be a finite field of order q. Compute the order of $GL_n(\mathbb{F}_q)$, the group of invertible $n \times n$ matrices with coefficients in $\mathbb{F}_q$.

2) Let $I \subseteq \mathbb{Z}[x]$ be an ideal consisting of all polynomials $a_0 + a_1x + a_2x^2 + \cdots$ such that 8 divides a_0, 4 divides a_1 and 2 divides all other coefficients. Find a finite set of generators of the ideal I.

3) Let M be a $\mathbb{Z}$-module consisting of 3×1-matrics with integer entries. Let $N \subseteq M$ be a $\mathbb{Z}$-submodule generated by the columns of the matrix

$$\begin{pmatrix} 2 & 7 & 3 & 4 \\ 3 & 8 & 7 & 11 \\ 4 & 10 & 10 & 16 \end{pmatrix}.$$

Find a minimal presentation for the quotient M/N.

4) Show that the ring $\mathbb{Z}[\sqrt{-1}]$ is a euclidean domain.

5) Let T be a complex square matrix such that $T^{2007} = I$, where I is the identity matrix. Show that T can be diagonalized.

6) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$.

7) Let $\zeta_8 = e^{\frac{\pi}{4}i}$. Determine the Galois group of $K = \mathbb{Q}(\zeta_8)$ and then determine all quadratic extensions of $\mathbb{Q}$ contained in K.

8) Compute the Galois group $G(E/F)$ where E is the splitting field of the polynomial $x^3 - 2$ and F is

(1) $\mathbb{Q}$
(2) $\mathbb{R}$
(3) $\mathbb{F}_5$
(4) $\mathbb{F}_7$

9) Let m be a maximal ideal in $\mathbb{R}[x, y]$ containing x^2+y^2+1. What is the quotient $\mathbb{R}[x, y]/m$? Justify your answer.

10) Let D_8 be the group of symmetries of a square. Show that the 2 dimensional representation (as symmetries of the square) is irreducible by calculating the character table.