**University of Utah, Department of Mathematics**
**Fall 2008, Algebra Preliminary Exam**

*Four correct solutions count as a pass; eight half-correct solutions may not!*

1. Determine the groups of order $1225 = 5^2 \cdot 7^2$ up to isomorphism.

2. Let $p$ be a prime integer. Let $G$ be a finite $p$-group, with identity $e$, and center $Z$.

   If $N \neq \{e\}$ is a normal subgroup of $G$, prove that $N \cap Z \neq \{e\}$.

3. Let $K$ be a field and let $f(x) \in K[x]$ be a monic polynomial of degree $n$ with $f(0) \neq 0$. Suppose $f(x)$ has $n$ distinct roots in its splitting field, and that the set of roots is closed under multiplication. Determine $f(x)$.

4. Let $M$ be an element of $GL_n(\mathbb{C})$ that has finite order. Is $M$ necessarily diagonalizable? Prove or disprove.

5. Let $n$ be an integer. Let $M$ be a square matrix with integer entries, such that the sum of the entries of each row equals $n$. Prove that $n$ divides the determinant of $M$.

6. A square matrix $M$ is *idempotent* if $M^2 = M$. Prove that two idempotent $n \times n$ matrices over a field are similar if and only if they have the same rank.

7. Is every ideal of the ring $\mathbb{Z} \times \mathbb{Z}$ a principal ideal? Prove or disprove.

8. (a) Determine the rank and signature of the real quadratic form

   $$x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 .$$

   (There are different definitions of signature, so state the one you use!)

   (b) Up to isometry, what is the number of distinct quadratic forms on $\mathbb{R}^4$?

9. Take a regular $n$-sided polygon inscribed in a circle of radius 1. Label the vertices $P_1, \ldots, P_n$, and let $\lambda_k$ be the length of the line joining $P_k$ and $P_n$ for $1 \leqslant k \leqslant n-1$. Prove that

   $$\lambda_1 \cdots \lambda_{n-1} = n .$$

10. Let $\omega = e^{2\pi i/3}$. Let $\sigma$ and $\tau$ be automorphisms of $\mathbb{C}(x)$ which fix $\mathbb{C}$ and satisfy

    $$\sigma(x) = \omega x, \qquad \tau(x) = 1/x.$$

    Prove that the group $\langle \sigma, \tau \rangle$ is isomorphic to the symmetric group $S_3$, and determine the subfield of $\mathbb{C}(x)$ that is fixed by this group.