University of Utah, Department of Mathematics
August 2015, Algebra Qualifying Exam

Show all your work, and provide reasonable justification for your answers. You may attempt as many problems as you wish; five correct solutions count as a pass.

1. Determine, up to isomorphism, the groups of order 44.

2. Prove that there is no simple group of order 192.

3. Let $R = \mathbb{Q}[x,y]$ and let $I = \langle x,y\rangle$. Compute the vector space rank of $\mathrm{Ext}^1_R(I,R)$ over $\mathbb{Q}$.

4. Let M be a 3×3 complex matrix with $M^6 = M^4$ and $M^4 + M^2 = 2M^3$. Determine possible Jordan forms of M.

5. (i) List the prime ideals of the ring $R = \mathbb{Z}[x,y]/\langle 3+x,\ y(y-1)+x^2,\ x\rangle$.

   (ii) Give an example of an integral domain with exactly 2 prime ideals.

6. Suppose that $R = \mathbb{Z}[i]$ is the Gaussian integers (here i is a square root of -1). Let M and N be finitely generated R-modules such that
$$M \oplus R^{\oplus 2} \oplus R/\langle 2\rangle \ \cong\ N \oplus R \oplus R/\langle 1+i\rangle \oplus R \oplus R/\langle 1-i\rangle.$$
   Is it true that $M \cong N$?

7. Prove that $\mathbb{F} = \mathbb{Z}[t]/\langle 3,\ t^3 - t^2 + 1\rangle$ is a field. Find the number of solutions of $x^{13}+1 = 0$ in $\mathbb{F}$, and also the number of solutions of $x^{13}-1=0$ in $\mathbb{F}$.

8. Compute the Galois group of $x^4 - 2$ over $\mathbb{Q}$.

9. Let E be the splitting field of $f(x) = x^{14}+1$ over $\mathbb{F}_2$, and let K be the splitting field $g(x) = x^{21}+1$ over $\mathbb{F}_2$. Prove that K contains an isomorphic copy of E, and compute the extension degree of K over E.

10. Let $f(x)$ be a degree 5 polynomial in $\mathbb{Q}[x]$ that is not solvable by radicals. Let L be its splitting field over $\mathbb{Q}$.

    (i) Prove that there is at most one field K with $\mathbb{Q}\subset K\subset L$ and $[K:\mathbb{Q}] = 2$.

    (ii) If α,β are irrational elements in L such that α^2 and β^2 are rational, prove that $\alpha\beta$ is rational.