University of Utah, Department of Mathematics
August 2016, Algebra Qualifying Exam

Show all your work, and provide reasonable justification for your answers. You may attempt as many problems as you wish; five correct solutions count as a pass.

1. Determine, up to isomorphism, the groups of order 75.
2. Let G be a nonabelian simple group. Prove that $\operatorname{Aut} G$ contains a normal subgroup isomorphic to G.
3. Is the group of positive rational numbers under multiplication torsion-free? Is it a free abelian group?
4. Determine, up to conjugacy, all real 5×5 matrices with minimal polynomial $(x-1)^2(x+1)$.
5. Let A be a square matrix with coefficients in $\mathbb{C}$. Prove that there exist polynomials $s(t)$ and $n(t)$ with coefficients in $\mathbb{C}$, such that $s(A)$ is diagonalizable, $n(A)$ is nilpotent, and $A = s(A) + n(A)$.
6. Find all the maximal ideals of $\mathbb{Z}[\sqrt{-13}]$ that contain 7, and prove that their product is precisely the ideal (7).
7. Let p be a prime number, n a positive integer, and V the field $\mathbb{F}_{p^n}$ viewed as a vector space over $\mathbb{F}_p$. Prove that the Frobenius map $\varphi\colon V \longrightarrow V$, assigning v to v^p, is diagonalizable over $\mathbb{F}_p$ if and only if n divides $p-1$.
8. Determine the minimal polynomial of $\sin(2\pi/5)$ over $\mathbb{Q}$.
9. Determine the Galois group of $x^4 - 16x^2 + 4$ over $\mathbb{Q}$.
10. Let L/K be a Galois extension of fields of characteristic zero. Suppose $f(x)$ is an irreducible polynomial of degree 5 in $K[x]$. If $f(x)$ has no roots in L, prove that it is irreducible in $L[x]$.