University of Utah, Department of Mathematics
August 2018, Algebra Qualifying Exam

There are ten problems on the exam. You may attempt as many problems as you wish; five correct solutions count as a pass. Show all your work, and provide reasonable justification for your answers.

1. Determine, up to isomorphism, the groups of order 30.

2. Let G be a finite simple group with identity e. Suppose that A and B are distinct maximal proper subgroups of G. If A and B are abelian, prove that $A \cap B = \{e\}$.

   Hint: Prove that $A \cap B$ is normal in G.

3. Suppose that p is the smallest prime dividing the order of a group G, and that P is a Sylow p-subgroup of G. If P is cyclic, show that
$$N_G(P) = C_G(P),$$
   i.e., that the normalizer of P in G agrees with the centralizer of P in G.

4. Find all solutions of $x^2 = 1$ in the ring $\mathbb{Z}/91$.

5. Consider the ideal $I = (2,\ 1+\sqrt{-5})$ in the ring $R = \mathbb{Z}[\sqrt{-5}]$. Is I a prime ideal? Is I a projective R-module?

6. Let $A = \mathbb{F}_3[x]$, i.e., A is a polynomial ring in one variable over the field with 3 elements. Suppose M and N are finitely generated A-modules such that
$$M \oplus \frac{A}{x^3+1} \;\cong\; N \oplus \frac{A}{x+1} \oplus \frac{A}{x+1} \oplus \frac{A}{x+1}.$$
   Are M and N isomorphic?

7. Let $R = \mathbb{Z}[i]$ be the ring of Gaussian integers. Consider the R-module M generated by two elements x and y, subject to the relations $ix + 2y = 0$ and $2x - iy = 0$. How many elements does M have?

8. Let R be the ring $\mathbb{Q}[x,y]$, and let I be the ideal $I = (x,y)$. What is the $\mathbb{Q}$-vector space rank of $\operatorname{Tor}_1^R(R/I,\ I)$?

9. Determine the extension degree of the splitting field of $x^7 - 1$ over $\mathbb{F}_{11}$.

10. Determine the Galois group of $x^6 + 3$ over $\mathbb{Q}$.