University of Utah, Department of Mathematics
Spring 2011, Algebra Qualifying Exam

Show all your work and provide reasonable proofs/justification. You may attempt as many problems as you wish. Four correct solutions count as a pass; eight half-correct solutions may not!

(1) Let p be a prime. Show that an element in the symmetric group S_n has order p if and only if it is a product of commuting p-cycles. Show by an explicit example that this need not be the case if p is not prime.

(2) Prove that the number of Sylow p-subgroups of $GL_2(\mathbb{F}_p)$ is $p+1$.

(3) Let G be a p-group with $|G| > p$. Show that
  (a) G has nontrivial center;
  (b) G has a normal subgroup of every order $p^m < |G|$.

(4) For the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}$, find:
  (a) the rational canonical form over $\mathbb{Q}$;
  (b) the Jordan canonical form over $\mathbb{C}$.

(5) Prove that the ring $\mathbb{Z}[i]$ is a Euclidean domain.

(6) Let G be a finite abelian group and H a subgroup of G. Show that G has a subgroup isomorphic with G/H.

(7) Let m, n be positive integers and d their greatest common divisor. Show that

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}.$$

(8) For a ring R define its nilradical $\mathfrak{n}(R) = \{x \in R : x^n = 0 \text{ for some n} \in \mathbb{Z}\}$.
  (a) If R is commutative, prove that $\mathfrak{n}(R)$ is an ideal of R.
  (b) Is the nilradical an ideal even if R is noncommutative?

(9) What is the Galois group of $x^4 - 5$ over:
  (a) $\mathbb{Q}$;
  (b) $\mathbb{Q}(\sqrt{5})$;
  (c) $\mathbb{Q}(i)$.

(10) Show that the extension $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is Galois over $\mathbb{Q}$, and determine the Galois group.

(11) Show that the polynomial $x^2 + y^2 - 1$ is irreducible in $\mathbb{Q}[x, y]$. Is it irreducible in $\mathbb{C}[x, y]$?