University of Utah, Department of Mathematics
January 2013, Algebra Qualifying Exam

Show all your work and provide reasonable proofs/justification. You may attempt as many problems as you wish. **Five** *correct solutions count as a pass; ten half-correct solutions may not!*

(1) Determine the number of Sylow p-subgroups of $GL_2(\mathbb{F}_p)$.

(2) Show that $(\mathbb{Q}/\mathbb{Z}, +)$ has one and only one subgroup of order n, for each integer $n \geq 1$, and that this subgroup is cyclic.

(3) Determine representatives for the conjugacy classes in $GL_3(\mathbb{F}_2)$.

(4) Let R be a commutative ring with $1 \neq 0$. Recall that the nilradical of R is the ideal $\mathfrak{N}(R) = \{x \in R : x^n = 0 \text{ for some positive integer } n\}$. Prove that the following are equivalent:
(i) R has exactly one prime ideal;
(ii) $R/\mathfrak{N}(R)$ is a field.

(5) If I, J are ideals in the commutative ring R, prove that

$$R/I \otimes_R R/J \cong R/(I + J),$$

as R-modules.

(6) In the category of $\mathbb{Z}$-modules:
(a) Is $\mathbb{Z}$ injective?
(b) Is $\mathbb{Z}/8\mathbb{Z}$ projective?

(7) Show that if p is an odd prime, the polynomial $x^{p^n} - x + 1$ is irreducible over $\mathbb{F}_p$ only when $n = 1$.

(8) Show that $f(x) = x^4 + 4x^2 + 2$ is irreducible over $\mathbb{Q}$, and find its Galois group over $\mathbb{Q}$.

(9) Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree $n \geq 4$ and let K be a splitting field of f over $\mathbb{Q}$. Suppose that $\mathrm{Gal}(K/\mathbb{Q})$ is the symmetric group S_n. If $\alpha \in K$ is a root of $f(x)$, show that $\alpha^n \notin \mathbb{Q}$.

(10) Let A be a real $n \times n$ matrix. We say that A is *a difference of two squares* if there exist real $n \times n$ matrices B and C with $BC = CB = 0$ and $A = B^2 - C^2$.
(a) If A is a diagonal matrix, show that it is a difference of two squares.
(b) If A is a symmetric matrix that is not necessarily diagonal, again show that it is a difference of two squares.
(c) Suppose A is a difference of two squares, with corresponding matrices B and C as above. If B has a nonzero real eigenvalue, prove that A has a positive real eigenvalue.