PhD Preliminary Qualifying Examination

Applied Mathematics

Tuesday, August 16, 2016

Instructions: Answer three questions from Part A and three questions from Part B. Indicate clearly which questions you wish to have graded.

Part A.

1. Consider the operator $T : \ell^2 \to \ell^2$ defined for $x = (\xi_j) \in \ell^2$ by

$$Tx = (0, \xi_1\alpha_1, \xi_2\alpha_2, \ldots),$$

for some sequence (α_j) with $\alpha_j \to 0$. Show that T is compact.

2. Let X be a Banach space and consider the *non-linear* mapping $F : X \to X$ defined by

$$F(x) = y - \alpha\|x\|x,$$

where $y \in X$ is fixed and α is a scalar. Show that there is a constant $C > 0$ such that for any $|\alpha| < C$, F is a contraction on the open ball of radius 1 centered at y.

3. Let M be a subset of a Hilbert space H. Assume M is such that for any $v, w \in H$ for which the equality

$$\langle v, x\rangle = \langle w, x\rangle$$

holds for all $x \in M$, we must have $v = w$. Show that $M^\perp = \{0\}$.

4. Let X be a real Banach space. Assume $f \in X^*$ has a closed nullspace $\mathcal{N}(f)$. The goal of this problem is to show that f must be a bounded linear functional.

   (a) Let $x_0 \in X$ be such that $f(x_0) = 1$. Explain why there is an $\epsilon > 0$ for which the ball

   $$B(x_0, \epsilon) \equiv \{x \in X \mid |x - x_0| < \epsilon\}$$

   satisfies $B(x_0, \epsilon) \subset X - \mathcal{N}(f)$.

   (b) Prove that $f(x) > 0$ for all $x \in B(x_0, \epsilon)$, where ϵ is as in part (a).
   To prove this, you may assume for contradiction that there is some $y \in B(x_0, \epsilon)$ with $f(y) < 0$.

   (c) Any $x \in B(x_0, \epsilon)$ can be written as $x = x_0 + \epsilon u$ for some u with $\|u\| < 1$.
   Use the result of part (b) to show that $|f(u)| < 1/\epsilon$.

   (d) To conclude, give an upper bound for $\|f\|$.

5. Let $T : X \to X$ be a bounded linear operator on a complex Banach space X.
   Prove that $\sigma(T)$ lies in the complex plane disk: $\{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\|\}$.

Part B.

1. The following function $f(x)$ [x is a real variable] can be represented by a series (Taylor or Laurent) in powers of $(x-7)$. Find the radius of convergence of the series in each case

$$\begin{aligned}
&\text{(a)} \qquad f(x) &=& \; e^{(x-7)^{10}} \, , \\
&\text{(b)} \qquad f(x) &=& \; \left(\frac{\sin x - 3}{\sin x - 2}\right)^2 \, , \\
&\text{(c)} \qquad f(x) &=& \; \frac{\sin x - 2}{x^2 - 49}
\end{aligned}$$

[You do not need to find the series themselves.]

2. (a) Prove: If a function is analytic then its real and imaginary parts are harmonic.

   (b) Prove: If a function $u(x, y)$ is harmonic in a domain D, then $u(x, y)$ cannot attain a strict local maximum in D. [Hint: If $f(z) = u + iv$, then function $\phi(z) = e^{f(z)}$ has absolute value $|\phi(z)| = e^u$.]

3. Integrate

$$\begin{aligned}
&\text{(a)} \qquad \int_0^\infty \frac{\sin \alpha\, x}{x}\, dx \, , \\
&\text{(b)} \qquad \int_0^\infty \frac{x^\alpha}{1+x}\, dx \, , \\
&\text{(c)} \qquad \int_0^\infty \sin x^2 \, dx
\end{aligned}$$

[Explain the logic (in particular, the choice of contour), but do not worry about getting the exact numbers; α is a real parameter.]

4. (a) Consider a 2-dimensional map $(x, y) \to (u, v)$.
   Explain: If the map is given by an analytic function [$u + iv = f(x + iy)$] and at some point $z_0 = x_0 + iy_0$ the derivative $f'(z_0) \neq 0$, then this map preserves small shapes in the vicinity of z_0.

   (b) Is a square conformally equivalent to a circle?

   (c) Explain: A map by analytic function preserves Laplace's equation.

5. Solve Laplace's equation

$$u_{xx} + u_{yy} = 0 \quad \text{in the domain} \quad D = \{(x, y) \in \mathbb{R}^2 : \quad (x-1)^2 + y^2 > 1 \quad \& \quad (x-2)^2 + y^2 < 4\}$$

subject to the boundary condition

$$u(x, y) = a \text{ when } (x-1)^2 + y^2 = 1 \quad \text{and} \quad u(x, y) = b \text{ when } (x-2)^2 + y^2 = 4$$

[a and b are real parameters; the two circles touch each other; express your answer in terms of the original real variables x, y].