UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Applied Mathematics

August 16th, 2017.

Instructions: This examination has two parts consisting of five problems in part A and five in part B. You are to work three problems from part A and three problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be detailed as possible. All problems are weighted equally.

Part A.

A1. Let H be a Hilbert space. Show that an isometric linear operator $T\colon H \to H$ satisfies $T^*T = I$, where I is the identity operator on H.

A2. Let (X, d) be a complete metric space and f a contraction on X with contraction constant $\alpha < 1$. Let x_f be the unique fixed point of f. Fix $x_0 \in X$ and consider the sequence of points defined by $x_n = f(x_{n-1})$ for $n \in \mathbb{N}$.

(a) Prove that $d(x_n, x_f) \leq \frac{\alpha^n}{1-\alpha}\, d(x_0, x_1)$. Hint: consider $d(x_m, x_n)$ for $m > n$.

(b) Let $S = [1, \infty)$ and $f\colon S \to S$ be defined by $f(x) = \frac{1}{2}(x + 2/x)$. Prove that f is a contraction mapping on S and compute a contraction constant. Determine all fixed points of f on S. Using part (a), explain why the sequence (x_n) might be useful.

A3. Let X be an inner product space with inner product $\langle \cdot, \cdot \rangle$. Let (a_j) be a sequence of complex numbers and let (ψ_j) and (ϕ_j) be two orthonormal sequences in X. Let $T\colon X \to X$ be an operator defined by

$$Tx = \sum_{j=1}^{\infty} a_j \langle \psi_j, x \rangle \phi_j.$$

State and prove a (nontrivial) sufficient condition on (a_j) so that T is a compact operator.

A4. The right-shift operator $R\colon \ell^2 \to \ell^2$ is defined by $R(\xi_1, \xi_2, \ldots) = (0, \xi_1, \ldots)$ and the left-shift operator $L\colon \ell^2 \to \ell^2$ is defined by $L(\xi_1, \xi_2, \ldots) = (\xi_2, \xi_3, \ldots)$.

(a) Show that L and R are adjoint to each other.

(b) Show that $\|R\| = \|L\| = 1$.

(c) Prove that the spectrum of L is given by

$$\sigma_p(L) = \{\lambda \in \mathbb{C} \colon |\lambda| < 1\}, \quad \sigma_c(L) = \{\lambda \in \mathbb{C} \colon |\lambda| = 1\}, \quad \sigma_r(L) = \emptyset.$$

(d) Prove that the spectrum of R is given by

$$\sigma_p(R) = \emptyset, \quad \sigma_c(R) = \{\lambda \in \mathbb{C} \colon |\lambda| = 1\}, \quad \sigma_r(R) = \{\lambda \in \mathbb{C} \colon |\lambda| < 1\}.$$

A5. Let X be a complex Banach space and $T \in B(X, X)$. Show that the resolvent, $R_\lambda(T)$, satisfies $\|R_\lambda(T)\| \to 0$ as $\lambda \to \infty$.

Part B.

B1. (a) State and prove Liouville's theorem.

(b) Show that every polynomial of degree $n \geq 1$ in the complex plane has at least one root.

B2. Find the real-valued function $\Phi(z)$ that solves Laplace's equation in the region between the two noncentric circles $|z| = 1$ and $|z - 1| = \frac{5}{2}$ subject to the boundary conditions $\Phi = 0$ on $|z| = 1$ and $\Phi = 1$ on $|z - 1| = \frac{5}{2}$.

B3. Consider the transformation on the extended complex plane $\bar{\mathbb{C}}$

$$f(z) = w = z + \frac{1}{z}.$$

(a) Find all possible fixed points.

(b) Show that the image of the points in the upper half z-plane that are exterior to the circle $|z| = 1$ corresponds to the entire upper half w-plane.

(c) Find the image of the unit circle.

(d) Show that $f(z)$ is a two-to-one function.

B4. Evaluate the integral

$$\int_0^\infty \frac{\sin(ax)}{x} dx \quad \text{for} \quad a > 0.$$

You must clearly show the contours of integration and treat integration over each contour separately.

B5. Consider the Dawson's integral

$$D(x) = e^{-x^2} \int_0^x e^{t^2} dt \quad \text{as} \quad x \to \infty.$$

(a) Find the first non-constant term of the asymptotic expansion of $D(x)$ and show that the relationship is asymptotic.

(b) Find the full asymptotic series of $D(x)$.

HINT: Introduce a cutoff parameter.