UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Applied Mathematics
August 15, 2018

Instructions: This examination has two parts consisting of five problems in part A and five in part B. You are to work three problems from part A and three problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three from each part will be graded.

All problems are worth 10 points and a passing score is 40.

Part A.

1. Consider the nonlinear integral equation

$$f(x) - \lambda \int_a^b k\,(x, y; f(y))\,dy = \phi(x)$$

where $\lambda \in \mathbb{R}$, ϕ is continuous on $[a, b]$, k is continuous on $C = [a, b] \times [a, b] \times \mathbb{R}$, and k satisfies a Lipschitz condition of the form

$$|k(x, y; z_1) - k(x, y; z_2)| \leq L|z_1 - z_2|$$

in its 'functional argument'. Show that the nonlinear integral equation has a unique solution, f, for any λ such that $|\lambda| < \frac{1}{L(b-a)}$.

2. Let $T\colon X \to X$ be a compact linear operator on a normed space X. Show that for every $\lambda \neq 0$, the null space of $T_\lambda = T - \lambda I$ is finite dimensional.

3. Let $a \in \ell^2 \setminus \{0\}$. Consider the bounded linear operator $T\colon \ell^2 \to \ell^2$ defined by

$$Tu = \langle u, a\rangle a.$$

   (a) Show that the operator T is compact and self-adjoint.

   (b) Use the Fredholm Alternative Theorem to find a simple condition on a guaranteeing that the equation

$$(T - I)u = v$$

   has a unique solution u for all $v \in \ell^2$. What is the solution?

4. Consider the operator $T\colon L^2[0, 1] \to L^2[0, 1]$ defined by $(Tx)(t) = tx(t)$.

   (a) Show that T is self-adjoint and satisfies $\langle x, Tx\rangle \geq 0$ for every $x \in L^2[0, 1]$.

   (b) Find the spectrum $\sigma(T)$, point spectrum $\sigma_p(T)$, continuum spectrum $\sigma_c(T)$, residual spectrum $\sigma_r(T)$ and resolvent $\rho(T)$.

5. Let $\delta \in \mathscr{D}'(\mathbb{R})$ denote the 'delta function'. Find $u \in \mathscr{D}'(\mathbb{R})$ that solves the following linear partial differential equation:

$$\partial^2 u = \delta.$$

Is the solution unique?

Hint: You can solve this equation by guessing the distribution and then verifying that is solves the equation.

Part B.

1. Function $f(x)$ (see below) is expanded in powers of $x - x_0$, i.e. in the Taylor or Laurent series with center x_0 (x is a real variable, x_0 is a real number). Find the radius of convergence of this series.

$$(a) \qquad f(x) = \frac{\sin x^2}{(x-3)^2(x^2+1)}, \qquad x_0 = 3,$$

$$(b) \qquad f(x) = \frac{\sin x}{x}, \qquad x_0 = 5,$$

$$(c) \qquad f(x) = \frac{\sin x}{\sin x + 3}, \qquad x_0 = 8.$$

2. Solve Laplace's equation $\phi_{xx} + \phi_{yy} = 0$
in the domain $D = \{(x,y) \in \mathbb{R}^2 : \ x^2 + y^2 > 1, \ (x+2)^2 + y^2 < 9\}$
with boundary conditions: $\phi(x,y) = a$ when $x^2+y^2 = 1$ and $\phi(x,y) = b$ when $(x+2)^2+y^2 = 9$.

3. Calculate the following integrals (a and b are positive parameters)

$$(a) \qquad I(a) = \int_{-\infty}^{\infty} \frac{\sin ax}{x}\, dx,$$

$$(b) \qquad I(a,b) = \int_{-\infty}^{\infty} \frac{\cos ax \ - \ \cos bx}{x^2}\, dx.$$

4. $F(\omega)$ is the Fourier transform of $f(x)$ (which is continuous and not identical zero), x and ω are real variables. Explain why it is impossible that *both* functions $f(x)$ and $F(\omega)$ have finite support.

5. Consider integral

$$I(s) = \int_C \frac{e^{sz^2}}{z^2 - 1}\, dz, \quad C \text{ is the vertical line } \operatorname{Re} z = 3, \quad \text{from } z = 3 - i\infty \text{ to } z = 3 + i\infty,$$

with large (real) parameter s. Find the three-term asymtotic expansion (approximation with three non-zero terms) of this integral as $s \to +\infty$.