PhD Preliminary Qualifying Examination: Applied Mathematics

August 13, 2007

Instructions: Answer three questions from part A and three questions from part B. Indicate clearly which questions you wish to have graded.

Part A.

1. The dilation equation is

$$\phi(x) = \sum_k c_k \phi(2x - k)$$

and the associated wavelet is

$$W(x) = \sum_k (-1)^k c_{1-k} \phi(2x - k).$$

(a) Show that the function

$$N_2(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

satisfies the dilation equation and determine the values of c_k. Sketch the associated wavelet. Briefly explain whether or not $N_2(x)$ can act as a scaling function for a multiresolution analysis?

(b) Prove that if the functions $\{\phi(x - k)\}_k$ form an orthonormal set then

$$\int_{-\infty}^{\infty} W(x)W(x - m)dx = \delta_{0,m} \int_{-\infty}^{\infty} \phi^2(x)dx.$$

(c) Prove that

$$\int_{-\infty}^{\infty} W(x)\phi(x - m)dx = 0$$

for all m.

2. (a) Show that a test function $\psi(x)$ is of the form $\psi = (x\phi)'$ where ϕ is a test function if and only if

$$\int_{-\infty}^{\infty} \psi(x)dx = 0 \text{ and } \int_{0}^{\infty} \psi(x)dx = 0.$$

(b) Solve the following equation in the sense of distribution:

$$x^2 \frac{d\phi}{dx} = 0.$$

3. Consider the integral equation

$$\phi(x) - \lambda \int_0^\pi \sin(x+t)\phi(t)dt = \cos(x) + \sin(x), \quad 0 \le x \le \pi.$$

(a) Find the unique solution when $\lambda \neq \pm 2/\pi$.

(b) Use the Fredholm alternative theorem to show that there is no solution when $\lambda = 2/\pi$.

(c) Use the Fredholm alternative theorem to show that when $\lambda = -2/\pi$ there is a one–parameter family of solutions of the form

$$\phi(x) = \frac{\cos(x) + \sin(x)}{2} + \alpha(\cos(x) - \sin(x)).$$

4. Suppose that L is a bounded linear operator in a Hilbert space H with closed range.

(a) Prove the Fredholm alternative theorem for solutions of the inhomogeneous equation $Lu = f$ with $u, f \in H$.

(b) Suppose that L is invertible and has a complete orthonormal set of eigenfunctions ϕ_n, integer n. Solve the inhomogeneous equation of part (a) using an eigenfunction expansion.

(c) Suppose that the adjoint operator L^* has a nontrivial nullspace $\mathcal{N}(L^*)$ spanned by the functions $\psi_i, i = 1, \ldots m$ and f does not lie in the orthogonal complement of $\mathcal{N}(L^*)$. Let L have a nullspace spanned by the functions $\phi_i, i = 1, \ldots, m$. Explain the construction of the smallest least squares solution.

5. (a) By constructing the one–dimensional Green's function solve the inhomogeneous equation

$$\frac{d^2u}{dx^2} = f(x), \quad u(0) = \alpha, \quad u(1) = \beta.$$

(b) Use Green's functions to solve

$$\frac{d^2u}{dx^2} - \alpha^2 u = f(x) \quad \text{on } L^2(-\infty, \infty)$$

with α real.

Part B.

1. Formulate and prove the Cauchy integral formula.

2. Solve Laplace's equation $\Delta\phi = 0$ in the domain between the two nonconcentric circles $|z| = 1$ and $|z-1| = 5/2$, subject to the boundary conditions $\phi = a$ on $|z| = 1$ and $\phi = b$ on $|z-1| = 5/2$.

3. Evaluate the integral

$$I = \int_0^\infty \frac{x^\alpha dx}{(x+1)^2}$$

(where α is a constant parameter, so that the integral converges).

4. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\sin \alpha x}{x} dx$$

(α is a constant parameter). Explain your steps.

5. Find the two-term asymptotic expansion of the integral

$$I(s) = \int_{-\infty}^{\infty} \frac{e^{s(ix+1)^2}}{(x+i)^2} dx, \qquad s \text{ is real and } s \to +\infty.$$