PhD Preliminary Qualifying Examination

Applied Mathematics

Aug 19 2014

Instructions: Answer three questions from Part A and three questions from Part B. Indicate clearly which questions you wish to have graded.

Part A.

1. Let $T : L^1[0,1] \to L^2[0,1]$ be a bounded linear operator. Let $a \in L^\infty[0,1]$ and $g \in L^1[0,1]$.

   (a) Prove that there exists a constant $C > 0$ such that if $\|a\|_{L^\infty} < C$ then the equation
   $$f + a(Tf)^2 = g \tag{1}$$
   has a unique solution f in the set
   $$B = \{u \in L^1[0,1] : \|u - g\|_{L^1} \le 1\}.$$
   Note: the constant C may depend on T and/or g.

   (b) Suggest an iterative procedure for approximating the unique solution f to equation (1).

2. Let X be a separable Hilbert space and (e_n) be a total orthonormal sequence of X. Let $T : X \to X$ be a bounded linear operator satisfying
   $$\sum_{n=1}^{\infty} \|Te_n\|^2 < \infty. \tag{2}$$
   Define the operators $T_k : X \to X$, $k = 1, 2, \ldots$, that applied to an arbitrary
   $$x = \sum_{n=1}^{\infty} x_n e_n \in X \quad \text{give} \quad T_k x = \sum_{n=1}^{k} x_n T e_n.$$

   (a) Are the operators T_k compact? Explain why or why not.

   (b) Using equation (2), show that the sequence (T_k) converges uniformly to T, i.e. show that $\|T - T_k\| \to 0$ as $k \to \infty$.

   (c) Is T compact? Explain why or why not.

3. Assume a compact self-adjoint operator $T : X \to X$, where X is a separable Hilbert space, satisfies $\langle Tx, x\rangle \ge 0$ for all $x \in X$.

   (a) State the spectral theorem for compact self-adjoint operators.

   (b) Show that all the eigenvalues of T are non-negative.

   (c) Construct an operator $S : X \to X$ such that $T = S^2$.

4. Let $l^{1\prime}$ be the dual space of l^1. The objective here is to show that $l^{1\prime}$ is isomorphic to l^∞. You may use that a Schauder basis for l^1 (and l^∞) is (e_k), where $e_k = (\delta_{kj})$, i.e. the sequence with zeroes everywhere except for a 1 in the $k-$th position.

   (a) Show that any $f \in l^{1\prime}$ corresponds to a sequence $g = (\gamma_k) \in l^\infty$.

   (b) Show that any sequence $g = (\gamma_k) \in l^\infty$ corresponds to some $f \in l^{1\prime}$.

   (c) Parts (a) and (b) define a bijective linear mapping $G : l^{1\prime} \to l^\infty$. Prove that G is an isomorphism by showing $\|f\|_{l^{1\prime}} = \|G(f)\|_{l^\infty}$ for any $f \in l^{1\prime}$.

5. Let $K : L^2[0,1] \to L^2[0,1]$ be the linear operator defined by

$$(Kx)(t) = \int_0^1 k(t,\tau)x(\tau)d\tau$$

   where $k : [0,1] \times [0,1] \to \mathbb{R}$ is continuous.

   (a) Show that K is bounded and give an upper bound for $\|K\|$ (in terms of k).

   (b) Show that there is a constant C such that if $|\mu| < C$, the operator $I + \mu K$ is invertible.

   (c) Assuming $|\mu| < C$, where C is as in part (b), express $(I + \mu K)^{-1}$ as a series. Specify whether the series converges and in what sense.

Part B.

1. Find the radius of convergence of the Taylor series with the center at $x_0 = 1$ for the following function

   (a)
$$f(x) = x^{1/3}$$

   (b)
$$f(z) = \frac{(\sin x + 2)^2}{(\sin x - 2)^2}$$

   (x is a real variable; you do not need to find the series themselves).

2. (a) Show that any analytic function (not identically equal to zero) can have only isolated zeros inside its analyticity domain.

   (b) Prove the Uniqueness Theorem: If two functions are analytic in a domain D and equal on some set of points that has a limiting point inside D, then these functions are identically equal in D.

3. Evaluate the integrals:

   (a) $I = \displaystyle\int_0^\infty \frac{\sin x}{x}\, dx,$

   (b) $I = \displaystyle\int_0^\infty \frac{x^\alpha}{2+x}\, dx \qquad (-1 < \alpha < 0).$

4. (a) Calculate the Fourier transform of a Gaussian $f(x) = e^{-x^2}$.

   (b) Formulate and prove Heisenberg's uncertainty principle.

5. Consider integral
$$I(s) = \int_C \frac{e^{s(z^2-1)}}{z-1}\, dz$$
   with large parameter $s \to +\infty$; C is the vertical line $\operatorname{Im} z = 3$, from $z = 3 - i\infty$ to $z = 3 + i\infty$.

   (a) What is the saddle point z_0?
   What is the path C_0 of steepest descent from z_0?
   What is the path of steepest ascent from z_0?

   (b) Find the three-term asymtotic expansion of the integral.