PhD Preliminary Qualifying Examination: Applied Mathematics

January, 2009

Instructions: Answer three questions from part A and three questions from part B. Indicate clearly which questions you wish to have graded.

Part A.

1. (a) The least squares psuedo-inverse of A is a matrix B which satisfies what two properties?

   (b) Find the pseudoinverse of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

2. The Haar function $\phi(x)$ is defined by

$$\begin{aligned} \phi(x) &= 1 \text{ when } 0 < x < 1 \\ &= 0 \text{ elsewhere} \end{aligned}$$

   and the mother Haar wavelet $W(x)$ is defined by

$$\begin{aligned} W(x) &= 1 \text{ when } 0 < x < 1/2 \\ &= -1 \text{ when } 1/2 < x < 1 \\ &= 0 \text{ elsewhere.} \end{aligned}$$

   Express the function

$$\begin{aligned} f(x) &= 2 \text{ when } 0 < x < 1/4 \\ &= 0 \text{ when } 1/4 < x < 1/2 \\ &= -1 \text{ when } 1/2 < x < 3/4 \\ &= 1 \text{ when } 3/4 < x < 1 \\ &= 0 \text{ elsewhere} \end{aligned}$$

   as a linear combination of 4 orthogonal functions: the Haar function and 3 Haar wavelets $W_{mn}(x) = 2^{m/2}W(2^m x - n)$, where m and n are appropriate integers.

3. (a) Define what is meant by a precompact (or sequentially compact) set and what is meant by a compact linear operator, assuming linear operators and bounded sets have been already been defined.

   (b) If K is a compact linear operator and if $\{\phi_n\}$ is an infinite set of orthonormal functions then (using Bessel's inequality, if needed) show that

$$\lim_{n\to\infty} K\phi_n = 0.$$

4. For the differential operator $Lu = u'' + (x^2 - 1)u'$ on the interval $[0, 1]$ with boundary conditions $u(0) = u'(1)$ and $u(1) = u'(0)$ find the adjoint operator and its domain.

5. (a) Write down the set of linear equations which if solved give the Green's function for the operator $Lu = d^3u/dx^3$ on the interval $[0, 1]$ with boundary conditions $u(0) = u'(0) = u(1) = 0$. (There is no need to explicitly solve these equations).

   (b)Letting $G(x, y)$ denote the Green's function which solves (a), and using a particular solution of $d^3u/dx^3 = 0$, find the general solution to $d^3u/dx^3 = f(x)$ on the interval $[0, 1]$ with boundary conditions $u(0) = u'(0) = 0$ and $u(1) = 1$.

Part B.

1. (a) Verify that the real and imaginery parts of a complex analytic function satisfy Laplace's equation.

   (b) Give an interpretation of the complex function $w(z) = z + \frac{a^2}{z}$ as a flow past some object. That is, determine the streamlines for this function. Where are the stagnation points?

2. (a) Use contour integration to explicitly evaluate $f(a) = \int_0^\infty \frac{x \sin x dx}{x^2 - a}$ for complex a. For what values of complex a is this possible?

   (b) Is the function $f(a)$ an analytic function of a? Identify all singularities, branch points, branch cuts, etc.

3. The windowed Fourier transform of a function $f(x) \in L^2(-\infty, \infty)$ is defined by

$$Gf(\omega, \mu) = \int_{-\infty}^{\infty} g(x - \mu) f(x) e^{-\omega x} d\mu dx.$$

   Suppose $g(x)$ is a real valued function. State and verify the formula for the reconstruction of $f(x)$ from Gf, including any additional conditions on $g(x)$. You may assume the validity of the Fourier transform.

4. (a) Find the spherically symmetric eigenfunctions and corresponding eigenvalues for the Laplacian on a spherical domain of radius R with Dirichlet boundary conditions $u(R) = 0$.

   (b) Use these eigenfunctions to solve the heat equation on a spherical domain of radius R, with $u(R, t) = U_0$ and $u(r, 0) = 0$.

   (c) Estimate the time t at which $u(0, t)$ is $\frac{U_0}{2}$. How does this time depend on the radius R?

5. (a) Find the first term of the asymptotic representation of

$$n! = \int_0^\infty \exp(-t) t^{a-1} dt$$

   for large n. (This approximation is called Stirling's formula.)

   (b) What is the order of the error term and how can this be rigorously justified? (Invoke the appropriate theorem.)