UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Applied Mathematics
January 3, 2019

Instructions: This examination has two parts consisting of five problems in part A and five in part B. You are to work three problems from part A and three problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three from each part will be graded.

All problems are worth 10 points and a passing score is 40.

Part A.

1. Let (e_n) be an orthonormal sequence of a Hilbert space H. Show that $e_n \to 0$ weakly.

2. Let $T\colon X \to X$ be a bounded linear operator on a complex Banach space X. Prove that the spectrum $\sigma(T)$ lies in the complex plane disk: $\{\lambda \in \mathbb{C}\colon |\lambda| \leq \|T\|\}$.

3. Let T be a bounded linear operator on a complex Hilbert space H. Show that the operator $I + T^*T$ is injective.

4. Let H be a Hilbert space. Recall that an operator $T \in B(H)$ is *Hilbert-Schmidt* if there exits a complete orthonormal sequence, $(e_n) \in H$ such that $\sum_{n=1}^{\infty} \|Te_n\|^2 < \infty$. Show that every Hilbert-Schmidt operator is compact.

5. Consider the distribution $u \in \mathscr{D}'(\mathbb{R})$ defined by

$$\langle u, \phi \rangle = \int_{-\infty}^{\infty} |x|\phi(x)\ dx, \qquad \phi \in \mathscr{D}(\mathbb{R}).$$

   Find its derivative, ∂u.

Part B.

1. Prove the fundamental theorem of algebra: *Every polynomial of degree $n(\geq 1)$ has exactly n zeros (each zero being counted according to its multiplicity).*

2. The following function $f(x)$ [x is a real variable] is expanded in a series in powers of $(x+3)$. Find the radius of convergence of the series in each case.

$$
\begin{aligned}
&\text{(a)} \qquad f(x) = e^{x^2}, \\
&\text{(b)} \qquad f(x) = \left(\frac{\sin x - 3}{\sin x - 2}\right)^2, \\
&\text{(c)} \qquad f(x) = \frac{\sin(x+1)}{(x+1)(x+3)}
\end{aligned}
$$

3. Evaluate the integrals

$$
\begin{aligned}
&(a) \qquad \int_{-\infty}^{\infty} \frac{\sin ax}{x} dx \qquad (a \text{ is a real parameter}), \\
&(b) \qquad \int_{0}^{\infty} \frac{x^\alpha}{x+1} dx \qquad (\alpha \text{ is a real parameter}, \ -1 < \alpha < 0).
\end{aligned}
$$

4. (a) Consider a 2-dimensional map $(x, y) \to (u, v)$.
   Explain: If the map is given by an analytic function $[u + iv = f(x + iy)]$ and at some point $z_0 = x_0 + iy_0$ the derivative $f'(z_0) \neq 0$, then this map preserves small shapes in the vicinity of z_0.

   (b) Does there exist a conformal transformation $u + iv = f(x + iy)$ that maps triangle $\Delta = \{(x, y) : 0 < x < 1, \ 0 < y < 1, \ x + y < 1\}$ to circle $D = \{(u, v) : u^2 + v^2 < 1\}$?

5. Consider integral

$$
I(s) = \int_C \frac{e^{sz^2}}{z^2 - 1} \, dz, \quad C \text{ is the vertical line Re } z = 2, \ \ \text{from } z = 2 - i\infty \text{ to } z = 2 + i\infty,
$$

with large positive parameter s. Find the three-term asymptotic expansion (approximation with three non-zero terms) of this integral as $s \to +\infty$.