UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Differential Equations

August 15th, 2016.

Instructions: This examination has two parts consisting of five problems in part A and five in part B. You are to work three problems from part A and three problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be detailed as possible. All problems are worth 20 points.

A. Ordinary Differential Equations: Do three problems for full credit

A1. Let $f \in C^1(U, \mathbb{R}^n)$ for $U \subset \mathbb{R}^n$ and $x_0 \in U$. Given the Banach space $X = C([0,T], \mathbb{R}^n)$ with norm $\|x\| = \max_{0 \leq t \leq T} |x(t)|$, let

$$K(x)(t) = x_0 + \int_0^t f(x(s))ds$$

for $x \in X$. Define $V = \{x \in X \mid \|x - x_0\| \leq \epsilon\}$ for fixed $\epsilon > 0$ and suppose $K(x) \in V$ (which holds for sufficiently small T), so that $K : V \to V$ with V a closed subset of X.

(a) Give the definition of a locally Lipschitz function in an open set U of a normed vector space.

(b) Using the fact that f is locally Lipschitz in U with Lipschitz constant L_0, and taking $x, y \in V$ show that

$$|K(x(t)) - K(y(t))| \leq L_0 t \|x - y\|.$$

Hence, show that

$$\|K(x) - K(y)\| \leq L_0 T \| |x - y\| \quad x, y \in V.$$

(c) State the contraction mapping principle on a Banach space.

(d) Choosing $T < 1/L_0$, apply the contraction mapping principle to show that the integral equation has a unique continuous solution $x(t)$ for all $t \in [0, T]$ and sufficiently small T. Hence establish existence and uniqueness of the initial value problem

$$\frac{dx}{dt} = f(x), \quad x(0) = x_0.$$

A2. Consider the T–periodic non-autonomous linear differential equation

$$\dot{x} = A(t)x, \quad x \in \mathbf{R}^n, \quad A(t) = A(t + T)$$

Let $\Phi(t)$ be a fundamental matrix with $\Phi(0) = \mathbf{I}$.

(a) Show that there exists at least one nontrivial solution $\chi(t)$ such that

$$\chi(t + T) = \mu \chi(t)$$

where μ is an eigenvalue of $\Phi(T)$.

(b) Suppose that $\Phi(T)$ has n distinct eigenvalues μ_i, $i = 1, \ldots, n$. Show that there are then n linearly independent solutions of the form

$$x_i = p_i(t)\mathrm{e}^{\rho_i t}$$

where the $p_i(t)$ are T–periodic. How is ρ_i related to μ_i?

(c) By determining the time derivative of the Wronskian $W(t) = \det[\Phi(t)]$, prove that the product of the Floquet multipliers μ_i satisfies

$$\prod_{i=1}^{n} \mu_i = \exp\left(\int_0^T \mathrm{tr}[A(s)]ds\right)$$

(d) Using part (c) show that $\mu_1\mu_2 = 1$ in the case of Mathieu's equation for a parametric oscillator:

$$\ddot{x} + (\alpha + \beta\cos t)x = 0.$$

A3. Consider the van der Pol equation

$$\ddot{x} + x + \varepsilon(x^2 - 1)\dot{x} = \Gamma\cos(\omega t), \quad 0 < \varepsilon \ll 1$$

with $\Gamma = \mathcal{O}(1)$ and $\omega \neq 1/3, 1, 3$. Using the method of multiple scales show that the solution is attracted to

$$x(t) = \frac{\Gamma}{1-\omega^2}\cos\omega t + \mathcal{O}(\varepsilon)$$

when $\Gamma^2 \geq 2(1-\omega^2)^2$ and

$$x(t) = 2\left[1 - \frac{\Gamma^2}{2(1-\omega^2)^2}\right]^{1/2}\cos t + \frac{\Gamma}{1-\omega^2}\cos\omega t + \mathcal{O}(\varepsilon)$$

when $\Gamma^2 < 2(1-\omega^2)^2$. Explain why this result breaks down when $\omega = 1/3, 1, 3$.

A.4. (a) Give definitions for the following: *invariant set, attracting set, ω-limit set.*

(b) Use the Poincare'-Bendixson (PB) Theorem and the fact that the planar system

$$\dot{x} = x - y - x^3, \quad \dot{y} = x + y - y^3$$

has only the one critical point at the origin to show that this system has a periodic orbit in the annular region $A = \{x \in R^2 \,|\, 1 < |x| < \sqrt{2}\}$.

(c) Show that the system

$$\dot{x} = x - rx - ry + xy, \quad \dot{y} = y - ry + rx - x^2$$

can be written in polar coordinates as $\dot{r} = r(1-r), \dot{\theta} = r(1-\cos\theta)$. Show that it has an unstable node at the origin and a saddle node at $(1, 0)$. Use this information and the PB Theorem to sketch the phase portrait and then deduce that for all $(x, y) \neq (0, 0)$, the flow $\phi_t(x, y) \to (1, 0)$ as $t \to \infty$ but that $(1, 0)$ is not linearly stable.

A.5. The displacement x of a spring–mounted mass under the action of dry friction is assumed to satisfy

$$m\ddot{x} + kx = -F_0\mathrm{sgn}(\dot{x})$$

An example would be a mass m connected to a fixed support by a spring of stiffness k and resting on a surface with frictional force F_0, $F_0 > 0$. Set $m = k = 1$ for convenience and let $y = \dot{x}$.

(a) Consider the initial conditions $x = x_0 > 0, \dot{x} = 0$ at $t = 0$. Show that the phase path will spiral exactly n times before reaching equilibrium if

$$(4n-1)F_0 < x_0 < (4n+1)F_0.$$

(b) Suppose the initial conditions at $t = 0$ are $x = x_0 > 3F_0$ and $\dot{x} = 0$. Subsequently, whenever $x = -\alpha$, where $2F_0 - x_0 < -\alpha < 0$ and $\dot{x} > 0$, a trigger operates to increase suddenly the forward velocity so that the kinetic energy increases by a constant amount E. Show that if $E > 8F_0^2$ then a periodic motion is approached, and show that the largest value of x in the periodic motion is equal to $F_0 + E/(4F_0)$.

B. Partial Differential Equations. Do three problems to get full credit

Useful facts:

1. Green's integral identity is $\int_\Omega (v\nabla^2 u - u\nabla^2 v)dx = \int_{\partial\Omega}(vu_\nu - uv_\nu)d\sigma$ where u_ν refers to the outward normal derivative.

2. The laplacian in polar coordinates is

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}. \tag{1}$$

B.1. Solve (and sketch the solution of) the differential equation $u_t + uu_x = 0$ on the domain $-\infty < x < \infty$, $t > 0$ subject to initial data $u(x,0) = u_0(x)$ where

(a)

$$u_0(x) = \begin{cases} 0 & x < 0 \\ x & 0 < x < 1 \\ 1 & x > 1 \end{cases}. \tag{2}$$

(b)

$$u_0(x) = \begin{cases} 1 & x < 0 \\ 1 - x & 0 < x < 1 \\ 0 & x > 1 \end{cases}. \tag{3}$$

For which (if any) of these initial profiles does the solution of the problem $u_t + uu_x = \epsilon u_{xx}$, $\epsilon > 0$, converge to a traveling wave profile? What is the speed of the traveling wave solution?

B.2. (a) Does a solution of the equation $u_x + xu_y = u$ with $u(x, x^2) = x$ exist for $-\infty < x < \infty$, $y > 0$? Explain why or why not.

(b) Does a solution of the equation $xu_x - yu_y = u$ with $u(x, x^2) = x$ exist for $-\infty < x < \infty$, $y > 0$? Explain why or why not.

B.3. Find the canonical form for the partial differential equation

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = xy^2 u_x. \tag{4}$$

What type of partial differential equation is this?

B.4. Under what conditions on λ, if any, does a solution $u(r, \theta)$ of

$$\nabla^2 u = 1, \tag{5}$$

on the annulus $1 < r < 2$, with boundary conditions $u_r = \cos\theta$ at $r = 1$, and $u_r = \lambda \cos^2\theta$ at $r = 2$? Find a solution when it exists. Is it unique?

B.5. The fundamental solution for the Laplacian in two dimensional space is $G(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \ln(|\mathbf{x} - \mathbf{y}|)$, where $\mathbf{x} = (x_1, x_2)$. What is the Green's function for the Laplacian operator on the first quadrant $0 < x_1, x_2 < \infty$, subject to boundary conditions $G(x_1, 0) = 0$, $G_{x_1}(0, x_2) = 0$? Be sure to verify the validity of your solution.