UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Differential Equations

January 5th, 2017.

Instructions: This examination has two parts consisting of five problems in part A and five in part B. You are to work three problems from part A and three problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be as detailed as possible. All problems are worth 20 points.

A. Ordinary Differential Equations: Do three problems for full credit

A1. (a) Find the first three successive approximations $u_1(t), u_2(t), u_3(t)$ for the initial value problem (IVP)

$$\dot{x} = x^2, \quad x(0) = 1.$$

Use mathematical induction to show that for all $n \geq 1$, $u_n(t) = 1 + t + \cdots + t^n + \mathcal{O}(t^{n+1})$ as $t \to 0$.

(b) Solve the IVP in part (a) and show that the function $x(t) = 1/(1-t)$ is a solution to the IVP on the interval $(-\infty, 1)$. Compare with the approximation of part (a)

(c) Let $f \in C^1(U, \mathbb{R}^n)$ for $U \subset \mathbb{R}^n$ and $x_0 \in U$. Given the Banach space $X = C([0,T], \mathbb{R}^n)$ with norm $\|x\| = \max_{0 \leq t \leq T} |x(t)|$, let

$$K(x)(t) = x_0 + \int_0^t f(x(s))ds$$

for $x \in X$. Define $V = \{x \in X \mid \|x - x_0\| \leq \epsilon\}$ for fixed $\epsilon > 0$ and suppose $K(x) \in V$ (which holds for sufficiently small T), so that $K : V \to V$ with V a closed subset of X. Using the fact that f is locally Lipschitz in U with Lipschitz constant L_0, and taking $x, y \in V$ show that

$$|K(x(t)) - K(y(t))| \leq L_0 t \|x - y\|.$$

Hence, show that

$$\|K(x) - K(y)\| \leq L_0 T \| |x - y\| \quad x, y \in V.$$

A2. Suppose $A(t)$ is a real $n \times n$ matrix function which is smooth in t and periodic of period $T > 0$. Consider the linear differential equation in $\mathbf{R}^n$

$$\begin{cases} \dfrac{dx}{dt} = A(t)\, x, \\ x(0) = x_0. \end{cases} \tag{1}$$

Let $\Phi(t)$ be the fundamental matrix solution with $\Phi(0) = I$.

(a) Suppose that $\Phi(T)$ has n distinct eigenvalues μ_i, $i = 1, \ldots, n$. Show that there are then n linearly independent solutions of the form

$$\mathbf{x}_i = \mathbf{p}_i(t) e^{\rho_i t}$$

where the $\mathbf{p}_i(t)$ are T–periodic. How is ρ_i related to μ_i?

(b) Prove that the zero solution is unstable for the system $\dot{x} = A(t)x$, where

$$A(t) = \begin{pmatrix} 1 & 1 \\ 0 & \dot{h}(t)/h(t) \end{pmatrix},$$

and $h(t) = 2 + \sin t - \cos t$.

(c) Suppose that the autonomous nonlinear equation $\dot{\mathbf{x}} = f(\mathbf{x})$ exhibits a limit cycle. Explain how Floquet theory can be used to determine the linear stability of the limit cycle

A3. Consider the following linear equation for $\mathbf{x} \in \mathbb{R}^n$:

$$\dot{\mathbf{x}} = \mathbf{A} + \mathbf{B}(\mathbf{t})\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

where $\mathbf{A}, \mathbf{B}(t)$ are $n \times n$ matrices. Suppose that all eigenvalues λ_j, $j = 1, \ldots, n$, of the matrix $\mathbf{A}$ satisfy $\mathrm{Re}(\lambda_j) < 0$, and let $\mathbf{B}(t)$ be continuous for $0 \leq t < \infty$ with $\int_0^\infty \|\mathbf{B}(t)\| dt < \infty$.

(a) Using the variation of constants formula show that there exist constants $K, \sigma > 0$ such that

$$|\mathbf{x}(t)| \leq K\mathrm{e}^{-\sigma(t-t_0)}|\mathbf{x}_0| + K\int_{t_0}^t \mathrm{e}^{-\sigma(t-s)}|\mathbf{x}(s)|\|\mathbf{B}(s)\|ds.$$

(b) Let $u(t) = \mathrm{e}^{\sigma t}|\mathbf{x}(t)|$, $v(t) = \|\mathbf{B}(t)\|$ and $c = K\mathrm{e}^{\sigma t_0}|\mathbf{x}_0|$. Show that the inequality of part (a) can be rewritten as

$$u(t) \leq c + \int_{t_0}^t v(s)u(s)ds.$$

(c) From Gronwall's inequality we have

$$u(t) \leq c\exp\left(\int_{t_0}^t v(s)ds\right).$$

Use this to show that the zero solution of the IVP is asymptotically stable.

A4. Consider the differential operator acting on $L^2(\mathbb{R})$,

$$L = -\frac{d^2}{dx^2}, \quad 0 \leq x < \infty$$

with self-adjoint boundary conditions $\psi(0)/\psi'(0) = \tan\theta$ for some fixed angle θ.

(a) Show that when $\tan\theta < 0$ there is a single negative eigenvalue with a normalizable eigenfunction $\psi_0(x)$ localized near the origin, but none when $\tan\theta > 0$.

(b) Show that there is a continuum of eigenvalues $\lambda = k^2$ with eigenfunctions $\psi_k(x) = \sin(kx + \eta(k))$, where the phase shift η is found from

$$\mathrm{e}^{i\eta(k)} = \frac{1 + ik\tan\theta}{\sqrt{1 + k^2\tan^2\theta}}.$$

(c) Evaluate the integral

$$I(x, x') = \frac{2}{\pi}\int_0^\infty \sin(kx + \eta(k))\sin(kx' + \eta(k))dk,$$

and interpret the result with regards the relationship to the Dirac Delta function and completeness, that is, $\delta(x-x') - I(x,x') = \psi_0(x)\psi_0(x')$. You will need the following standard integral

$$\int_{-\infty}^{\infty} \mathrm{e}^{ikx} \frac{1}{1+k^2t^2} \frac{dk}{2\pi} = \frac{1}{2|t|} \mathrm{e}^{-|x/t|}.$$

HINT: you should monitor how the bound state contribution (for $\tan\theta < 0$) switches on and off as θ is varied. Keeping track of the modulus signs $|\ldots|$ in the standard integral is crucial for this.

A5. (a) Consider the dynamical system

$$\begin{aligned} \dot{x} &= -y + x(1 - z^2 - x^2 - y^2) \\ \dot{y} &= x + y(1 - z^2 - x^2 - y^2) \\ \dot{z} &= 0. \end{aligned}$$

Determine the invariant sets and attracting set of the system. Give a general definition of the ω–limit set, and determine it in the case of a trajectory for which $|z(0)| < 1$. Sketch the flow.

(b) Use the Poincare-Bendixson (PB) Theorem and the fact that the planar system

$$\dot{x} = x - y - x^3, \quad \dot{y} = x + y - y^3$$

has only the one critical point at the origin to show that this system has a periodic orbit in the annular region $A = \{x \in R^2 \,|\, 1 < |x| < \sqrt{2}\}$.

B. Partial Differential Equations. Do three problems to get full credit

B1. Does the partial differential equation

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = -u$$

subject to the restriction that $u = \exp(-x^2)$ on the curve $t = x^2$ have a solution? If it does have a solution, indicate for what ranges of x and t this solution is uniquely defined and if it does not have a solution, explain why.

B2. The Ornstein-Uhlenbeck (OU) differential equation is

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}(kxp) + D\frac{\partial^2 p}{\partial x^2}.$$

(a) What type of partial differential equation is this? What calculation is needed to justify this statement?

(b) If t has units of time and x has units of length, what are the units of the parameters k and D?

(c) Express this equation in nondimensional variables? How many nondimensional parameters remain after nondimensionalization?

(d) Find the solution of the OU equation on the domain $-\infty < x < \infty$, $t \geq 0$ with initial data $p(x, 0) = \delta(x - x_0)$ for arbitrary x_0. Give an interpretation of this solution in terms of a probability.

B3. Suppose the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0, \qquad x \in \mathbb{R}^n, \qquad t > 0,$$

subject to initial conditions $u(x, 0) = 0$, $\frac{\partial u}{\partial t}(x, 0) = h(x)$, for $x \in \mathbb{R}^n$ has a solution denoted by u_h. Find (and verify) the solution of the same equation subject to the initial conditions $u(x, 0) = g(x)$, $\frac{\partial u(x,0)}{\partial t} = h(x)$, for $x \in \mathbb{R}^n$ in terms of u_h and u_g.

B4. Consider the partial differential equation defined for $-\infty < x < \infty$

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \chi(x)f(u), \qquad \chi(x) = \begin{cases} 0 & 0 < x < L \\ 1 & \text{elsewhere} \end{cases}$$

with $f(u)$ a bistable function, $f(0) = f(a) = f(1) = 0$, $0 < a < 1$, $f'(0) < 0$, $f'(1) < 0$ and $\int_0^1 f(u)du > 0$.

(a) Prove that for all $L > 0$ sufficiently large, there is a standing, monotone increasing C^1 profile with limiting behavior $u(-\infty) = 0$, $u(\infty) = 1$.

(b) State a comparison theorem that can be used to show that if L is sufficiently large, the region $0 < x < L$ acts as a "blocking region" to propagation for this equation.

B5. (a) Suppose an initial profile $u_0(x)$ is monotone with $\lim_{x\to-\infty} u_0(x,) = u_L$, $\lim_{x\to\infty} u_0(x) = u_R$. Under what conditions on u_L and u_R does the solution of Burger's equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0,$$

evolve into a shock, and what is its asymptotic speed?

(b) Consider the viscous Burger's equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \epsilon\frac{\partial^2 u}{\partial x^2}.$$

with $\epsilon > 0$. Under what conditions on u_L and u_R, where $\lim_{x\to-\infty} u(x,t) = u_L$, $\lim_{x\to\infty} u(x,t) = u_R$ does this equation have a travelling wave solution, and what is its speed? Verify your result.