Differential Equations Preliminary Examination

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Instructions

This examination consists of two parts, called **Part A** and **Part B**. Part A consists of exercises concerning *Ordinary Differential Equations* and Part B consists of exercises concerning *Partial Differential Equations.*

To obtain full credit you should fully complete **three** exercises from each part. All problems are weighted equally and a passing score will be 70% of the total possible score.

It is important that your arguments and discussion will be sound and detailed, bearing in mind that too many details are time consuming. Thus your judgement of what is essential will be an important factor in determining the final score.

Good Luck!

Part A - ODE
Exercise 1
    Let
$$f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$
be a continuous mapping which satisfies
$$|f(t,x) - f(t,y)| \le L(t)|x - y|, \ \forall t \in \mathbb{R}, \ \forall x, y \in \mathbb{R}^N, \qquad (0.1)$$
where
$$L : \mathbb{R} \to [0, \infty)$$
is a continuous function and where $|\cdot|$ is a norm in $\mathbb{R}^N$. Consider the initial value problem
$$x' = f(t,x)$$
$$x(t_0) = x_0, \ t_0 \in \mathbb{R}, \ x_0 \in \mathbb{R}^N. \qquad (0.2)$$

1. State the fundamental existence uniqueness theorem for solutions of (0.2).

2. State a general principle about the existence intervals for solutions of (0.2) and the behavior of solutions at the endpoints of these intervals.

3. Use the above and (0.1) (note: $|f(t,x)| \le |f(t,x) - f(t,0)| + |f(t,0)|$) to show that solutions of (0.2) are defined on all of $\mathbb{R}$.

Exercise 2
    Let $A : \mathbb{R} \to \mathbb{R}^{N \times N}$ be a continuous $N \times N$ matrix. Consider the systems

$$x' = A(t)x \qquad (0.3)$$

$$x' = -A^T(t)x, \qquad (0.4)$$

where A^T is the transpose of the matrix A.

1. Define what is meant by a fundamental matrix solution $\Phi(t)$ of (0.3).

2. Use general existence uniqueness principles to give the form of all fundamental matrix solutions of (0.3).

3. Let $\Psi(t)$ be a fundamental matrix solution of (0.4). Prove that

$$\Psi^T(t)\Phi(t)$$

is a nonsingular constant matrix.

4. Derive the *variation of constants* formula (using the fundamental solution matrix Φ) for solutions of

$$x' = A(t)x + h(t) \qquad (0.5)$$

5. Let $\Phi(0) = $ identity $ = \Psi(0)$. Give the variation of constants formula derived above in terms of Φ and Ψ. Do you see any computational advantages in this form of the variation of constants formula.

Exercise 3

Use an indirect argument (and basic existence uniqueness principles) to prove the following:

If $u : \mathbb{R} \to \mathbb{R}$ belonging to $C^2(-\infty, \infty)$ is a one-signed solution (i.e. it either is everywhere nonnegative or everywhere nonpositve) of the equation

$$-u'' + V(t)u = u^4,$$

where $V : \mathbb{R} \to \mathbb{R}$ is continuous, then u is either identically zero or never zero.

Exercise 4 Let

$$f : \mathbb{R}^N \to \mathbb{R}^N$$

be a C^1 mapping and consider the system

$$x' = f(x). \qquad (0.6)$$

Assume there exists a C^1 functional

$$\phi : \mathbb{R}^N \to \mathbb{R}$$

such that

$$\phi'(x) := \nabla\phi(x) \cdot f(x) \le 0, \ \ x \in \mathbb{R}^N.$$

Prove that the $\omega-$limit set of any bounded semiorbit of (0.6) is an invariant set for the flow associated with (0.6) and is contained in the set

$$E = \{x : \phi'(x) = 0\}.$$

Apply this result to the system

$$u' = -v$$
$$v' = -v + u^3$$
$$x = \begin{pmatrix} u \\ v \end{pmatrix},$$

using

$$\phi(u, v) = \frac{v^2}{2} + \frac{u^4}{4}.$$

What is your conclusion?

Exercise 5 Let A be a $N \times N$ matrix none of whose eigenvalues have zero real part and consider the system

$$x' = Ax + h(t), \tag{0.7}$$

where $h : \mathbb{R} \to \mathbb{R}^N$ is a continuous function which satisfies

$$h(t + T) = h(t), \ \forall t \in \mathbb{R}.$$

Show that for all such h (0.7) has a unique solution x such that

$$x(t + T) = x(t), \ \forall t \in \mathbb{R}.$$

What stability properties does this periodic orbit have?

Part B - PDE
Exercise 6

1. Let $G \subset \mathbb{R}^N$ be an open set. Define what is meant by a distribution on G. Let $f \in L^2(G)$ be a real or complex valued function. Show that f may be identified with a distribution T_f.

2. Let T be a distribution on G and let $\alpha = (\alpha_1, \cdots, \alpha_N)$ be a multiindex. Define the concept of the $\alpha-$th distributional derivative of T.

3. Let
$$r : \mathbb{R}^N \to \mathbb{R}$$
be defined by
$$r(x) = 1, \quad |x| < 1$$
$$r(x) = 0, \quad |x| \geq 1,$$
where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^N$. Show r may be thought of as a distribution on $\mathbb{R}^N$ and compute its distributional gradient ∇r.

Exercise 7

1. Let G be an open set. Define the Sobolev spaces $H^m(G)$, $m = 1, 2, \cdots$, and $H_0^m(G)$, $m = 1, 2, \cdots$. Indicate why these spaces are Hilbert spaces.

2. Let G be a bounded open set. State and prove Poincaré's inequality for functions $u \in H_0^1(G)$ and thus verify that $H_0^1(G)$ is also a Hilbert space with respect to the inner product
$$(u, v)_{H_0^1(G)} = \int_G \nabla u \cdot \overline{\nabla v} dx.$$

Exercise 8

1. Let G be an open subset of $\mathbb{R}^N$ and let $f \in L^2(G)$. Show that f defines a continuous linear functional on $H^1(G)$.

2. State the Lax-Milgram theorem for subspaces $V \subset H^1(G)$.

3. Show that the Lax-Milgram theorem may be applied to the form
$$a(u, v) = \sum_{i=1}^N \int_G \partial_i u \partial_i \bar{v} + \int_G u \bar{v}$$

and f as above.

Assume that G is a bounded open set with "very smooth" boundary and let u be the Lax-Milgram solution just determined. What partial differential equation (in the sense of distributions) will be solved by u and what boundary conditions does u satisfy (please give an argument to prove the latter)?

Exercise 9

Characterize the orthogonal complement of $H_0^1(a, b)$ in the space $H^1(a, b)$, where (a, b) is a real interval which is either finite or else $b = \infty$ or else $a = -\infty, \ b = \infty$.

Exercise 10

1. Let H be a Hilbert space and let

$$A : D(A) \to H$$

be a linear operator with domain $D(A)$. State the Hille-Yosida-Phillips theorem giving necessary and sufficient conditions for the unique solvability of the problem

$$u' + Au = 0$$
$$u(0) = u_0 \in D(A).$$

Be sure to define pertinent terms.

2. Apply the above result to the initial boundary value problem

$$u_t - u_x = 0, \ t > 0, \ x \in \mathbb{R}$$
$$u(0, x) = g(x), \ x \in \mathbb{R}.$$

Also solve this problem explicitly.