PhD Preliminary Qualifying Examination: Differential equations (6410/20)

August 2008

Instructions: Answer three questions from part A and three questions from part B. Indicate clearly which questions you wish to be graded.

Part A.

1. Consider the linear non–autonomous first order system

$$\dot{x} = Ax + B(t)x, \quad x \in \mathbf{R}^n$$

with A non-singular and $B(t)$ continuous for $t \geq 0$. Further, assume that

- the eigenvalues of A have non–positive real parts and those with zero real part are non-degenerate
- $\int_0^\infty ||B(t)||dt = c_1$ with c_1 a positive constant.

(a) Let $\Phi(t)$ be the fundamental matrix of the equation $\dot{x} = Ax$ with $\Phi(0) = I$. Derive the variation of constants formula

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-s)B(s)x(s)ds$$

(b) Prove that the solution $x(t)$ is bounded for all times $t > 0$. [Hint: Use part (a) and Gronwall's lemma in the following form: $v(t) \leq v_0 + \int_0^t u(s)v(s)ds$ implies that $v(t) \leq v_0 \exp(\int_0^t u(s)ds)$ for $t > 0$]. What does this imply about the stability of the origin?

2. Consider the T–periodic non-autonomous linear differential equation

$$\dot{x} = A(t)x, \quad x \in \mathbf{R}^n, \quad A(t) = A(t+T)$$

Let $\Phi(t)$ be a fundamental matrix with $\Phi(0) = \mathbf{I}$.

(a) Show that there exists at least one nontrivial solution $\chi(t)$ such that

$$\chi(t+T) = \mu\chi(t)$$

where μ is an eigenvalue of $\Phi(T)$.

(b) Suppose that $\Phi(T)$ has n distinct eigenvalues μ_i, $i = 1, \ldots, n$. Show that there are then n linearly independent solutions of the form

$$x_i = p_i(t)\mathrm{e}^{\rho_i t}$$

where the $p_i(t)$ are T–periodic. How is ρ_i related to μ_i?

(c) Consider the equation $\dot{x} = f(t)A_0 x$, $x \in \mathbf{R}^2$, with $f(t)$ a scalar T–periodic function and A_0 a constant matrix with real distinct eigenvalues. Determine the corresponding Floquet multipliers.

3. Consider the scalar equation

$$\ddot{x} + \dot{x} = -\varepsilon(x^2 - x), \quad 0 < \varepsilon \ll 1$$

Using the method of multiple scales show that the $\mathcal{O}(1)$ solution is

$$x_0(t, \tau) = A(\tau) + B(\tau)\mathrm{e}^{-t},$$

where $\tau = \varepsilon t$, and identify any resonant terms at $\mathcal{O}(\varepsilon)$. Show that the non–resonance condition for the amplitude A is

$$A_\tau = A - A^2$$

and hence determine the asymptotic behavior of x_0. Comment on the domain of validity of the asymptotic expansion.

4. Consider the scalar differential equation

$$\ddot{x} + x = -\varepsilon f(x, \dot{x})$$

with $|\varepsilon| \ll 1$. Let $y = \dot{x}$.

(a) Show that if $E(x, y) = (x^2 + y^2)/2$ then

$$\dot{E} = -\varepsilon f(x, y)y.$$

Hence show that an approximate periodic solution of the form $x = A\cos t + \mathcal{O}(\varepsilon)$ exists if

$$\int_0^{2\pi} f(A\cos t, -A\sin t)\sin t dt = 0.$$

(b) Let $E_n = E(x(2\pi n), y(2\pi n))$ with $x(t) = A_n \cos t + \mathcal{O}(\epsilon)$ for $2\pi n \leq t < 2\pi(n+1)$. Show that to lowest order E_n satisfies a difference equation of the form

$$E_{n+1} = E_n + \varepsilon F(E_n)$$

and write down $F(E_n)$ explicitly as an integral. Hence deduce that a periodic orbit with approximate amplitude $A^* = \sqrt{2E^*}$ exists if $F(E^*) = 0$ and this orbit is stable if

$$\varepsilon \frac{dF}{dE}(E^*) < 0.$$

(c) Using the above result, find the approximate amplitude of the periodic orbit of the van der Pol equation

$$\ddot{x} + x + \varepsilon(x^2 - 1)\dot{x} = 0$$

and verify that it is stable.

5. The displacement x of a spring–mounted mass under the action of dry friction is assumed to satisfy

$$\ddot{x} + x = F_0 \mathrm{sgn}(v_0 - \dot{x})$$

(a) Calculate the phase paths in the (x, y) plane and draw the phase diagram. Deduce that the system ultimately converges into a limit cycle oscillation. What happens when $v_0 = 0$?

(b) Suppose $v_0 = 0$ and the initial conditons at $t = 0$ are $x = x_0 > 3F_0$ and $\dot{x} = 0$. Subsequently, whenever $x = -\alpha$, where $2F_0 - x_0 < -\alpha < 0$ and $\dot{x} > 0$, a trigger operates to increase suddenly the forward velocity so that the kinetic energy increases by a constant amount E. Show that if $E > 8F_0^2$ then a periodic motion is approached, and show that the largest value of x in the periodic motion is equal to $F_0 + E/(4F_0)$.

Part B.

1. Consider the scalar linear equation

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = \alpha u$$

with a, b, α differentiable functions of x, y. Suppose that u is prescribed on some arc Γ (Cauchy data).

(a) By introducing a test function ψ that vanishes on an arbitrary curve γ and applying Green's theorem to the domain D bounded by $\partial D = \Gamma \cup \gamma$, derive the integral equation for a weak solution to the Cauchy problem.

(b) Extend the analysis to the case where u is discontinuous across an open curve C_0 within the domain D. Hence show that C_0 is a characteristic projection.

(c) Write down the generalization of part (b) to the case of the quaslinear equation

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = c$$

where P, Q, c are differentiable functions of x, y, u, and hence derive the Rankine–Hugonoit condition.

2. Paint flowing down a wall has thickness $u(x,t)$ satisfying (for $t > 0$)

$$\frac{\partial u}{\partial t} + u^2\frac{\partial u}{\partial x} = 0.$$

(a) Show that the characteristics are straight lines and that the Rankine–Hugoniot condition on a shock $x = S(t)$ is

$$\frac{dS}{dt} = \frac{[u^3/3]_-^+}{[u]_-^+}.$$

(b) A stripe of paint is applied at $t = 0$ so that

$$u(x,0) = \begin{cases} 0, & x < 0 \text{ or } x > 1 \\ 1, & 0 < x < 1 \end{cases}$$

For sufficiently small t, determine u in the domains $x < 0$, $0 < x < t$, $t < x < S(t)$ and $S(t) < x$, where the shock is $x = S(t) = 1 + t/3$

(c) Explain why the solution changes at $t = 3/2$ and show that thereafter $\dot{S} = S/3t$.

3. (a) Consider the three–dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

with Cauchy data

$$u(\mathbf{x}, 0) = 0, \frac{\partial u}{\partial t}(\mathbf{x}, 0) = g(\mathbf{x})$$

(a) Show that radially symmetric solutions are in the form of outgoing and incoming waves

$$u(r,t) = \frac{1}{r} F(r \pm ct).$$

[Hint: Perform the substitution $v = ur$].

(b) Writing the general solution to the Cauchy problem as a superposition of outgoing waves

$$u(\mathbf{x}, t) = \int_{\mathbf{R}^3} \frac{\delta(r - ct)}{r} f(x', y', z') dx' dy' dz',$$

where $r^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$, derive the retarded potential solution

$$u(\mathbf{x}, t) = ct \int_0^{2\pi} \int_0^{\pi} f(x + ct \sin\theta \cos\phi, y + ct \sin\theta \sin\phi, z + ct \cos\theta) \sin\theta d\theta d\phi$$

and show that $f = g/4\pi c$.

(c) Assuming that the initial data g is only nonzero in a bounded domain D, use a graphical construction to derive Huygen's principle.

4. (a) Show that if the real symmetric matrix $\mathbf{A}$ has real eigenvalues λ_i and orthogonal eigenvectors $\mathbf{x}_i$, then for any vector $\mathbf{y} = \sum_i c_i \mathbf{x}_i$, the smallest eigenvalue satisfies

$$\lambda_0 \leq \frac{\mathbf{y}^T \mathbf{A} \mathbf{y}}{\mathbf{y}^T \mathbf{y}}.$$

(b) Show that the eigenfunctions ϕ and eigenvalues $-\lambda$ of the problem

$$\nabla^2 \phi + \lambda \phi = 0 \text{ in a region } D,$$

with

$$\frac{\partial \phi}{\partial n} + \alpha \phi = 0 \text{ on } \partial D,$$

where $\partial / \partial n$ is the outward normal derivative, satisfy

$$\lambda \int_D \phi^2 d\mathbf{x} = \int_D |\nabla \phi|^2 d\mathbf{x} + \alpha \int_{\partial D} \phi^2 ds.$$

Assuming that the eigenfunctions ϕ form a complete orthonormal set, derive an upper bound for the smallest eigenvalue in terms of an appropriate energy integral (Rayleigh quotient).

5. (a) Solve Poisson's equation

$$\nabla^2 u = f \text{ in } D, \;\; u = g \text{ on } \partial D$$

in terms of an appropriately defined Green's function.

(b) Show that the two–dimensional Green's function for the Laplacian in an unbounded domain is

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}'|.$$

(c) Use the method of images to derive the Green's function for the Laplacian in the half–space $x \in \mathbf{R}, y > 0$ with a Dirichlet boundary condition on $y = 0$, and evaluate the corresponding solution of part (a) when $f = 0$.