UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Differential Equations

August 15, 2011.

---

Instructions: The examination has two parts consisting of six problems in Part A and five in Part B. You are to work three problems from part A and three problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be as detailed as possible. All problems are worth 20 points. A passing score is 72.

---

**A. Ordinary differential Equations: Do three problems for full credit**

A1. Let $f(t,x)$ be continuously differentiable for $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ and let $x_0 \in \mathbb{R}^n$ be a point. Show that there is and $\epsilon > 0$ and a unique function $y \in \mathcal{C}^1([-\epsilon,\epsilon],\mathbb{R}^n)$ that satisfies the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = f(x,t), \\ x(0) = x_0. \end{cases}$$

A2. Suppose $A(t)$ is a real $n \times n$ matrix function which is smooth in $t$ and periodic of period $T > 0$. Consider the linear differential equation in $\mathbb{R}^n$

$$\begin{cases} \dfrac{dx}{dt} = A(t)\,x, \\ x(0) = x_0. \end{cases} \tag{1}$$

Let $\Phi(t)$ be the fundamental matrix solution with $\Phi(0) = I$.

(a) Define: *Floquet Matrix, Floquet Multiplier and Floquet Exponent.* How are these related to $\Phi(t)$? State the necessary and sufficient conditions on (1) so that all solutions are bounded.

(b) Prove that all solutions of the Mathieu Equation (2) are bounded

$$\ddot{y} + \left(1 + \frac{1}{20}\sin 4t\right) y = 0. \tag{2}$$

A3. Let $A$ be an $n \times n$ real matrix whose eigenvalues $\lambda_i$ satisfy $\Re\mathrm{e}\,\lambda_i < 0$ for all $i = 1, \ldots, n$. Let $f(t,x)$ be a $\mathcal{C}^2$ function such that $|f(t,x)| = \mathbf{o}(|x|)$ as $x \to 0$, uniformly in $t$. Consider the initial value problem for $x_0 \in \mathbb{R}^n$,

$$\begin{cases} \dfrac{dx}{dt} = Ax + f(t,x), \\ x(0) = x_0 \end{cases} \tag{3}$$

(a) Show that if $|x_0|$ is small enough, there is a bounded solution to (3) that exists for all time.
[Hint: Suppose $g(t)$ and $u(t)$ are nonnegative functions and $c_0 \geq 0$ is a constant that satisfy $u(t) \leq c_0 + \int_0^t g(s)\,u(s)\,ds$ for all $t \geq 0$. Then Gronwall's Inequality implies $u(t) \leq c_0 \exp\left(\int_0^t g(s)\,ds\right)$ for all $t \geq 0$. ]

(b) Let $z(t) = 0$ for all $t$ be the zero solution. Define what it means for $z$ to be *Lyapunov Stable* (some authors call this notion *stable* .) Show that $z$ is Lyapunov stable for this equation.

A4. A nonconstant periodic orbit near the origin is created or destroyed as the parameter $\mu$ passes through zero in the one parameter family of equations

$$\begin{aligned}\dot{x} &= y + \mu x\\ \dot{y} &= -x + \mu y - x^2 y.\end{aligned}$$

(a) Determine whether the origin is stable or unstable for the system when $\mu = 0$.

(b) For which values of $\mu$ near zero does the system have a nonconstant periodic orbit near the origin? Is this orbit attracting or repelling? Why?

A5. Consider a modified predator-prey system for populations $x$, $y$ given by

$$\begin{aligned}\dot{x} &= x \cdot \left(1 - \frac{x}{4} - \frac{y}{1+x}\right),\\ \dot{y} &= y \cdot \left(\frac{x}{1+x} - \frac{1}{2}\right)\end{aligned}$$

(a) Show that the triangle with vertices $(0,0)$, $(5,0)$ and $(0,5)$ is a forward invariant subset.

(b) Find the fixed points and determine their stability.

(c) Show that the system has at least one nonconstant periodic solution.

A6. Suppose that $f, g : \mathbb{R}^3 \to \mathbb{R}$ are smooth functions. Consider the parameter dependent system

$$\begin{aligned}\dot{x} &= f(x, y; \varepsilon)\\ \dot{y} &= g(x, y; \varepsilon).\end{aligned} \tag{4}$$

(a) Suppose $\Gamma_0$ is a nonconstant $T$-periodic solution of (4) when $\varepsilon = 0$. Define *Poincarè Section, Poincaré Map* for $\Gamma_0$.

(b) Find a nontrivial periodic solution $\Gamma_0$ for the system (5) when $\varepsilon = 0$. Show that a nontrivial periodic solution $\Gamma_\varepsilon$ for the $\varepsilon$-system can be continued from $\Gamma_0$ as the parameter varies through small $|\varepsilon|$.

$$\begin{aligned}\dot{x} &= y + x(1 - x^2 - y^2)\\ \dot{y} &= -x + y(1 - x^2 - y^2) + \varepsilon y.\end{aligned} \tag{5}$$

## B. Partial Differential Equations. Do three problems to get full credit.

B1. Assume that $u \in C^2(\mathbb{R})$, and $f \in C_0^2(\mathbb{R})$.

(a) Assume $u$ satisfies $u'' = 0$. Without using an explicit representation for the solution, show that the mean value property

$$u(x) = \frac{1}{2}(u(x+r) + u(x-r)) = \frac{1}{2r}\int_{x-r}^{x+r} u(t)\,dt, \quad \text{for each interval } (x-r, x+r),$$

is satisfied, and conversely show that the mean value property implies $u'' = 0$.

(b) Assuming $u'' = 0$ in the bounded interval $(a, b)$, use part (a) to prove the *strong maximum principle* for $u$.

(c) Find a radially symmetric *fundamental solution* $\Phi(x)$ for the problem $u'' = 0$, and use it to prove that

$$u(x) = \int_{-\infty}^{\infty} \Phi(x-y) f(y)\, dy$$

satisfies $-u'' = f$ in $\mathbb{R}$.

B2. Given $f \in C^\infty(\mathbb{R}^n)$, and $g \in C^\infty(\mathbb{R}^{n-1})$, consider the eikonal equation

$$|Du| = f(x), \quad \text{in } \mathbb{R}^n,$$

with the boundary condition $u = g$ on the plane $\Gamma = \{x_n = 0\}$.

(a) Write down the characteristic equations

$$\begin{aligned} \dot{p} &= -D_x F - D_z F p, \\ \dot{z} &= D_p F \cdot p, \\ \dot{x} &= D_p F, \end{aligned}$$

associated with this problem.

(b) Given an initial point $x^0 \in \Gamma$, if possible, determine a set of initial conditions for the characteristic equations which satisfy the *compatibility conditions* (in other words, the initial conditions are *admissible*).

(c) Determine conditions on $g$ and $f$ which guarantee a unique local solution to the boundary value problem near $x^0$.

B3. Compute the unique entropy solution of

$$\begin{aligned} u_t + \left(\frac{u^2}{2}\right)_x &= 0 \quad \text{in } \mathbb{R} \times (0, \infty), \\ u &= g \quad \text{on } \mathbb{R} \times \{0\}, \end{aligned}$$

where

$$g(x) = \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } 0 \le x \le 1, \\ 0, & \text{if } x > 1. \end{cases}$$

Draw a sketch documenting your answer, illustrating what happens at all times $t > 0$.

B4. Let $u$ solve the initial value problem

$$\begin{aligned} u_{tt} - u_{xx} = 0, &\quad \text{in } \mathbb{R} \times (0, \infty), \\ u = g, \quad u_t = h, &\quad \text{on } \mathbb{R} \times \{0\}. \end{aligned}$$

Suppose $g$, $h$ have compact support. Define the kinetic energy $k(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\, dx$ and the potential energy $p(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\, dx$. Prove

(a) $k(t) + p(t)$ is constant in $t$.

(b) $k(t) = p(t)$ for all $t$ sufficiently large.

B5. Let $a(x) \in C^\infty(\bar{U})$, where $U$ is a smooth bounded connected domain, $U \subset \mathbb{R}^n$. Assume $a(x)$ is uniformly bounded away from zero on $U$. Consider the initial-boundary value problem

$$(*) \begin{cases} u_t &= \operatorname{div}(aDu), \quad \text{in } U \times (0, \infty), \\ u &= 0, \quad \text{on } \partial U \times (0, \infty), \\ u &= g, \quad \text{on } U \times \{0\}. \end{cases}$$

(a) If $g$ minimizes $\lambda = \int_U a|Dg|^2 \, dx$ over all $g \in H_0^1(U)$ satisfying $\int_U g^2 \, dx = 1$, characterize $\int_U u(x,1)^2 \, dx$ in terms of $\lambda$.

(b) Let $J(g) = \int_U u(x,1)^2 \, dx$, where $u$ solves $(*)$ with initial condition $g$, and denote $B = \{g \in L^2(U) : \|g\|_2 = 1\}$. Denote by $g_1$ the minimizer of $\lambda$ from part (a). Prove

$$\sup_{g \in B} J(g)$$

is attained by $g_1$. Does $J(g)$ have a minimizer over $B$? What is $\inf_{g \in B} J(g)$?