UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph. D. Preliminary Examination in Differential Equations

January 7, 2009

```
Instructions:  The examination has two parts consisting of six problems each.
               You are to work three problems from part A and three problems
               from part B. If you work more than the required umber of problems,
               then state which problems you wish to be graded, otherwise
               the first three will be graded.

               In order to receive maximum credit, solutions to problems must
               be clearly and carefully presented and should be as detailed as
               possible.  All problems are worth 20 points.  A passing score is 72.
```

A. Ordinary differential equations: Do three problems for full credit

A1. Suppose $f(x,t) \in C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ and $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$. Show that for the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = f(x,t), \\ x(t_0) = x_0. \end{cases} \tag{1}$$

there exists a unique local solution, *i.e.*, for some $\epsilon > 0$ there is a unique function $y \in C^1((t_0 - \epsilon, t_0 + \epsilon), \mathbb{R}^n)$ which satisfies (1) for $t \in (x_0 - \epsilon, x_0 + \epsilon)$.

A2. Consider the T-periodic non-autonomous linear differential equation in $\mathbb{R}^n$

$$\frac{dx}{dt} = A(t)\, x \tag{2}$$

where $A(t)$ is a real, smooth $n \times n$ matrix function such that $A(t+T) = A(T)$ for all t. Let $\Phi(t)$ be the fundamental matrix with $\Phi(0) = I$.

(a) Show that there is a T-periodic function $P(t)$ and a matrix B so that

$$\Phi(t) = P(t)\, e^{tB}.$$

(b) Suppose the eigenvalues ω of $\Phi(T)$ satisfy $|\omega| \le 1$ and that all ω with $|\omega| = 1$ possess a full set of eigenvactors. Show that then all solutions of (2) are bounded.

A3. Let A be an $n \times n$ real matrix whose eigenvalues λ_i satisfy $\Re\mathrm{e}\, \lambda_i < 0$ for all $i = 1, \ldots, n$. Let $g(x,t) \in C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ such that $|g(x,t)| \le |x|^2$ for all $x \in \mathbb{R}^n$. Consider the ODE

$$y' = Ay + g(y,t).$$

(a) Show that if the initial value $\|x(0)\|$ is small enough, then the solution exists for all time and $\lim\limits_{t\to\infty} x(t) = 0$. [Hint: Gronwall's Inequality for nonnegative functions $u(t)$ and $v(t)$ that satisfy $v(t) \le c_0 + \int_0^t u(s)\, v(s)\, ds$ is $v(t) \le c_0 \exp\left(\int_0^t u(s)\, ds\right)$.]

(b) Let $z(t) = 0$ for all t be the zero solution. Define what it means for z to be *Stable* (often called *Liapunov Stable.*) Discuss the stability of z for this equation.

A5. Consider the scalar equation

$$\ddot{u} + \epsilon(u^2 - 1)\dot{u} + u = 0, \qquad 0 < \epsilon \ll 1.$$

Apply the method of multiple scales, with variables t and $\eta = \epsilon t$ to show that if $u(0) = a > 0$ and $\dot{u}(0) = 0$ then for $t = O(\epsilon^{-1})$,

$$u = \frac{2\cos t}{\sqrt{1 + \left(\frac{4}{a^2} - 1\right) e^{-\epsilon t}}} + O(\epsilon).$$

A6. Consider the scalar differential equation

$$\ddot{u} - \epsilon u^3 + u = 0, \qquad 0 < \epsilon \ll 1. \tag{3}$$

(a) Suppose that (3) has a T-periodic solution $u(t;\epsilon)$. Let $\mathcal{E} = \frac{1}{2}(u^2 + \dot{u}^2)$. Using the fact that $\mathcal{E}(T) = \mathcal{E}(0)$, find the amplitude $u(0;\epsilon) = a(\epsilon)$ with $\dot{u}(0;\epsilon) = 0$ and period $T(\epsilon)$ for an approximate periodic solution.

(b) Determine the stability of the periodic solution.

A7. Show that the system

$$\begin{aligned} \dot{x} &= x + y - x^3 \\ \dot{y} &= -x + y - y^3 \end{aligned}$$

has at least one nonzero periodic solution.

B. Partial Differential Equations. Do three problems to get full credit

B1. Consider the scalar first order PDE

$$\begin{aligned} u\, u_x + u_y &= 0, \\ u(x, 0) &= h(x). \end{aligned} \tag{4}$$

(a) Define what it means to be a weak solution of (4).

(b) Suppose that $u_0 > 0$ and

$$h(x) = \begin{cases} u_0, & \text{if } x \le 0; \\ u_0(1 - x), & \text{if } 0 < x < 1; \\ 0, & \text{if } x \ge 1. \end{cases}$$

Show that a shock develops at a finite time and describe the global weak solution.

B2. Traffic flow may be modeled by the equation

$$\frac{\partial}{\partial x}\Big(G(\rho)\Big) + \frac{\partial}{\partial t}\rho = 0$$

where $\rho(x,t)$ is the density of traffic flow along a highway. If $\rho_{\max}$ is the maximum density of cars in a bumper to bumper situation, then

$$G(\rho) = c\rho\left(1 - \frac{\rho}{\rho_{\max}}\right)$$

where $c > 0$ is the free speed constant. Suppose that the initial concentration is given by

$$\rho(x, o) = \begin{cases} \frac{1}{2}\rho_{\max}, & \text{if } x < 0; \\ 0, & \text{if } x \ge 0. \end{cases}$$

Find the shock curve and describe the weak solution. Interpret your result in terms of traffic flow.

B3. Let $D \in \mathbb{R}^n$ be a bounded, connected domain with smooth boundary. Let ϕ be an eigenfuncion of the Laplacian on D with with Dirichlet boundary conditions ($\phi = 0$ on ∂D) and with eigenvalue λ,

$$\Delta\phi + \lambda\phi = 0. \tag{5}$$

(a) Show that there is a finite constant C so that for every nonzero continuously differentiable $u \in C^1(D)$ such that $u = 0$ on ∂D satisfies

$$\int_D u^2 \le C \int_D |Du|^2.$$

(b) By integrating eigenfunctions of (5), conclude that eigenvalues are positive $\lambda > 0$.

(c) Denote by

$$\lambda_1(D) = \inf_{u \in C_0^1(D),\ u \not\equiv 0} \frac{\int_D |Du|^2}{\int_D u^2}$$

the least value of the Rayleigh quotient (which is at least $1/C$). Assuming that $\lambda_1(D)$ is an eigenvalue of D with Dirichlet boundary conditions, show that if two such domains satisfy $D_1 \subset D_2$, then $\lambda_1(D_1) \ge \lambda_1(D_2)$.

B4. Consider the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad t > 0,\ x \in \mathbb{R}.$$

along with the intial conditions

$$u(x,0) = g(x), \quad u_t(x,0) = h(x).$$

(a) Assuming c is constant, derive d'Alembert's formula

$$u(x,t) = \frac{1}{2}(g(x+ct) + g(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)\, d\xi.$$

(b) Briefly explain the concepts of *domain of dependence*, and *finite propagation speed*, and prove that these properties follow from d'Alembert's formula.

(c) Suppose that we restrict the spatial domain to $x > 0$ and impose the boundary condition $u(0,t)$ for $t \ge 0$. If the initial conditions satisfy $g(0) = h(0) = 0$, using d'Alembert's formula as a starting point, find a closed-form solution for $u(x,t)$, with $x, t > 0$.

B5. Let $D \subset \mathbb{R}^d$ be a bounded domain with smooth boundary ∂D. Assume that the complete, orthonormal set of eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ for the negative Dirichlet Laplacian on D is known, in other words,

$$-\triangle \phi_n = \lambda_n \phi_n, \quad \text{in } D, n = 1, 2, 3, \ldots,$$

with $\phi_n(x) = 0$ for $x \in \partial D$, and $\int_D \phi_n^2 \, dx = 1$.

(a) Given $f \in L^2(D)$, find a closed-form series solution $u(x,t)$ for the initial boundary value problem for the heat equation

$$\begin{aligned} u_t - \triangle u &= 0, \quad t > 0, \ x \in D \\ u(x,t) &= 0, \quad x \in \partial D, \\ u(x,0) &= f, \quad x \in D. \end{aligned}$$

(b) Show that there exists a constant C such that $|\int_D u(x,t)\phi_n(x)\,dx| \le Ce^{-\lambda_n t}$, for all n. Find $\lim_{t\to 0} u(x,t)$.

(c) Give an example (the case $d = 1$ will suffice) which shows that the heat equation *does not* exhibit finite propagation speed.

B6. Suppose that $u(x_1, x_2)$ is a C^2 harmonic function in the domain $\Omega \subset \mathbb{R}^2$, so $\triangle u = 0$ in Ω.

(a) Prove the *mean value property*: if $y = (y_1, y_2) \in \Omega$ and $r > 0$ is chosen such that $B_r(y)$ (the ball of radius r, centered at y) satisfies $B_r(y) \subset \Omega$, then

$$u(y_1, y_2) = \frac{1}{2\pi} \int_0^{2\pi} u(y_1 + r\cos\theta, y_2 + r\sin\theta)\, d\theta.$$

(b) Assuming Ω is connected, prove that u can attain a maximum value at an interior point $x \in \Omega$, only if u is constant.