Instructions: The examination has two parts consisting of six problems in Part A
and five in Part B. You are to work three problems from part A
and three problems from part B. If you work more than the required
number of problems, then state which problems you wish to be graded,
otherwise the first three will be graded.

In order to receive maximum credit, solutions to problems must
be clearly and carefully presented and should be as detailed as
possible. All problems are worth 20 points. A passing score is 72.

A. Ordinary differential Equations: Do three problems for full credit

A1. Let $f(t, x)$ be continuous function on $\mathbb{R} \times \mathbb{R}^n$ that satisfies

$$\|f(t, x_1) - f(t, x_2)\| \le \|x_2 - x_1\|$$

for all $t \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}^n$. Let $x_0 \in \mathbb{R}^n$ be a point. Show that there is and $\epsilon > 0$ and a
unique function $y \in \mathcal{C}^1([0, \epsilon], \mathbb{R}^n)$ that satisfies the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = f(x, t), \\ x(0) = x_0. \end{cases}$$

A2. Suppose $A(t)$ is a real $n \times n$ matrix function which is smooth in t and periodic of period
$T > 0$. Consider the linear differential equation in $\mathbb{R}^n$

$$\begin{cases} \dfrac{dx}{dt} = A(t)\, x, \\ x(0) = x_0. \end{cases} \tag{1}$$

Let $\Phi(t)$ be the fundamental matrix solution with $\Phi(0) = I$.

(a) Define: *Floquet Matrix, Floquet Multiplier and Floquet Exponent.* How are these re-
lated to $\Phi(t)$? State the necessary and sufficient conditions so that (1) has a nonzero
T-periodic solution.

(b) Prove that the zero solution is unstable for the system $\dot{x} = A(t)x$, where

$$A(t) = \begin{pmatrix} \frac{1}{2} - \cos t & 9.11 \\ \\ 2001 & \frac{3}{2} + \sin t \end{pmatrix}.$$

A3. Let A be an $n \times n$ real matrix whose eigenvalues λ_i satisfy $\Re \lambda_i < 0$ for all $i = 1, \ldots, n$.
Consider the initial value problem for $x_0 \in \mathbb{R}^n$,

$$\begin{cases} \dfrac{dx}{dt} = Ax + \dfrac{t\|x\|^3}{1+t}\, v, \\ x(0) = x_0 \end{cases} \tag{2}$$

where $v \in \mathbb{R}^n$ is a unit vector.

(a) Show that if $|x_0|$ is small enough, there is a bounded solution to (2) that exists for all time.

[Hint: Suppose $g(t)$ and $u(t)$ are nonnegative functions and $c_0 \geq 0$ is a constant that satisfy $u(t) \leq c_0 + \int_0^t g(s)\,u(s)\,ds$ for all $t \geq 0$. Then Gronwall's Inequality implies $u(t) \leq c_0 \exp\left(\int_0^t g(s)\,ds\right)$ for all $t \geq 0$.]

(b) Let $z(t) = 0$ for all t be the zero solution to (2) . Define what it means for z to be *Lyapunov Stable* (some authors call this notion *stable* .) Show that z is Lyapunov stable for this equation.

A4. Consider Griffith's model for a genetic control system, where x and y are proportional to concentration of protein and the messenger RNA from which it is translated, respectively, and $\mu > 0$ is a rate constant

$$\dot{x} = y - \mu x$$
$$\dot{y} = \frac{x^2}{1 + x^2} - y.$$

(a) Show that the system has three fixed points when $\mu < \mu_c$ and one when $\mu > \mu_c$ where μ_c is to be determined.

(b) What is the nature of the bifurcation at $\mu = \mu_c$?

A5. Consider a Brusselator system for chemical reactants x, y given by

$$\dot{x} = 1 - 4x + x^2 y,$$
$$\dot{y} = 3x - x^2 y$$

(a) Show that the quadrilateral bounded by the lines $x = 0$, $y = 0$, $y = 8+x$ and $y = 10-x$ is a forward invariant subset.

(b) Find the fixed points and determine their stability.

(c) Show that the system has at least one nonconstant periodic solution.

A6. (a) State the Center Manifold Theorem for rest points. Briefly explain its importance in bifurcation theory.

(b) Construct an approximation to the center manifold at the origin and use it to determine the local behavior of solutions.

$$\dot{x} = -xy,$$
$$\dot{y} = -y + x^2 - 2y^2.$$

B. Partial Differential Equations. Do three problems to get full credit.

B1. Consider the first-order equation

$$u = x_1 u_{x_1} + x_2 u_{x_2} + \frac{1}{2}(u_{x_1}^2 + u_{x_2}^2),$$

for the function $u(x_1, x_2)$.

(a) Write down the characteristic equations

$$\begin{aligned} \dot{p} &= -D_x F - D_z F p, \\ \dot{z} &= D_p F \cdot p, \\ \dot{x} &= D_p F, \end{aligned}$$

associated with this problem.

(b) Verify that the Cauchy data $u(x_1, 0) = \frac{1}{2}(1 - x_1^2)$, $x_1 \in \mathbb{R}$, is noncharacteristic, and find an *admissible* set of initial conditions.

(c) Find a solution $u(x_1, x_2)$ to the original equation which satisfies the given Cauchy data.

B2. In Kirchhoff-Love Plate Theory, the vertical displacement $w(x)$ of an isotropic static plate under pure bending satisfies

$$Lw := D \left(\frac{\partial^4 w}{\partial x_1^4} + 2 \frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 w}{\partial x_2^4} \right) = -f,$$

where $D > 0$ is a known constant bending stiffness, and $-f(x)$ is a given applied transverse load. Assume that the plate lies within a simply connected bounded region $U \subset \mathbb{R}^2$ with smooth boundary ∂U. Define the bilinear form

$$B(w, v) = \int_U (Lw) v \, dx.$$

(a) Suppose that the boundary of the plate is *clamped*, that is, the displacement w satisfies

$$\begin{aligned} w &= 0, \quad \text{on } \partial U, \\ \frac{\partial w}{\partial \eta} &= 0, \quad \text{on } \partial U, \end{aligned}$$

where η is the unit outward normal. Let X be the space of smooth functions satisfying the above boundary conditions. Show that B is symmetric and positive semidefinite over $X \times X$.

(b) Using an elementary estimate for solutions of Poisson's equation, show that B is bounded and coercive over $X \times X$ in a well-chosen norm.

(c) By extending X appropriately, use the Lax-Milgram Theorem to prove that the boundary value problem above admits a unique weak solution for all sufficiently smooth f.

B3. Let $U \subset \mathbb{R}^n$ be a bounded open set with smooth boundary ∂U, and consider the problem

$$\begin{cases} -\triangle u + c(x)u &= \lambda u + f, \quad \text{in } U, \\ u &= 0, \quad \text{on } \partial U, \end{cases}$$

where $c(x) \in L^\infty(U)$, and $f \in C_0^\infty(\partial U)$.

(a) In the case $f = 0$, by examining the bilinear form over $H_0^1(U)$ associated with the operator $-\triangle + c - \lambda$, determine a general condition on $\lambda \in \mathbb{R}$ which guarantees that $u = 0$ is the only weak solution of the problem.

(b) Still assuming $f = 0$, suppose that $\lambda = \lambda_1, \lambda_2$ have been chosen such that there exist nonzero weak solutions $u_1, u_2 \in H_0^1(U)$ respectively (that is, (λ_1, u_1) and (λ_2, u_2) are eigenpairs). Prove that λ_j must be real, and if $\lambda_1 \neq \lambda_2$, then u_1 and u_2 are orthogonal.

(c) Suppose λ_1 is an eigenvalue as in part (b), and let $E_1 \subset H_0^1(U)$ be the set of all associated eigenfunctions. Setting $\lambda = \lambda_1$, for general $f \in C_0^\infty(U)$, find the least restrictive constraint on f which guarantees a weak solution to the problem exists. Is the solution unique?

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B4. Let $U \subset \mathbb{R}^n$ be an open bounded connected set with smooth boundary ∂U. Consider the initial boundary-value problem for the heat equation:

$$\begin{cases} u_t - \triangle u & = & 0, \quad \text{in } U \times (0, T] \\ u & = & 0, \quad \text{on } \partial U \times [0, T] \\ u & = & g, \quad \text{on } U \times \{t = 0\}. \end{cases}$$

(a) Use the mean value property for the heat equation

$$u(x, t) = \frac{1}{4r^n} \iint_{E(x,t;r)} u(y, s) \frac{|x - y|^2}{(t - s)^2} \, dy ds,$$

where $E(x, t; r) = \{(y, s) \in \mathbb{R}^{n+1} : s \leq t, \ \phi(x - y, t - s) \geq 1/r^n\}$ and $\phi(x, t)$ is the fundamental solution of the heat equation, to prove the *strong maximum principle* for the heat equation.

(b) Show that the strong maximum principle implies the principle of *infinite propagation speed* for the problem above.

B5. Assume that $u \in C^2(\mathbb{R}^n \times [0, \infty])$ satisfies the initial value problem for the wave equation

$$\begin{cases} u_{tt} - \triangle u = 0, \quad \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g, \ u_t = h, \quad \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Recall that when $n = 3$, using spherical means and the Euler-Poisson-Darboux equation, one can derive Kirchhoff's formula

$$\begin{aligned} u(x, t) & = & \frac{\partial}{\partial t} \left(t \fint_{\partial B(x,t)} g \, dS \right) + t \fint_{\partial B(x,t)} h \, dS \\ & = & \fint_{\partial B(x,t)} th(y) + g(y) + Dg(y) \cdot (y - x) \, dS(y), \quad t > 0. \end{aligned}$$

(a) Outline the procedure (the "method of descent") for deriving a similar formula (Poisson's formula) for the initial value problem in the case $n = 2$.

(b) Explain how a disturbance originating at a point $x \in \mathbb{R}^2$ at time $t = 0$ propagates in relation to the cone $C = \{(y, t) : t > 0, \ |x - y| < t\}$. In particular, using the Kirchhoff and Poisson formulas, show that there is a qualitative difference in propagation behavior between the cases $n = 2$ and $n = 3$.