UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Differential Equations

January 4, 2013.

Instructions: The examination has two parts consisting of six problems each. You are to work three problems from part A and three problems from part B. If you work more than the required umber of problems, then state which problems you wish to be graded, otherwise the first three will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be as detailed as possible. All problems are worth 20 points. A passing score is 72.

A. Ordinary differential Equations: Do three problems for full credit

A1. Let $f(x)$ be continuously differentiable for all $\mathbf{x} \in \mathbb{R}^n$ and let $x_0 \in \mathbb{R}^n$ be a point. Show that there is and $\epsilon > 0$ and a function $y \in \mathcal{C}^1([0,\epsilon],\mathbb{R}^n)$ that satisfies the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = f(x), \\ x(0) = x_0. \end{cases}$$

A2. Let A be an $n \times n$ real matrix whose eigenvalues λ_i satisfy $\Re\mathrm{e}\,\lambda_i < 0$ for all $i = 1, \ldots, n$. Answer the questions about the linear system

$$\begin{cases} \dfrac{dx}{dt} = A\,x, \\ x(0) = x_0. \end{cases} \tag{1}$$

(a) Define: $V(x)$ is a *Liapunov Function.*

(b) Find a Liapunov Function.

(c) Define: $z(t) = 0$ is an *Asymptotically Stable* solution for (1).

(d) Using your Liapunov Function, show that $z(t) = 0$ is an asymptotically stable solution.

A3. Show that there is a solution that goes to ∞ as $t \to \infty$.

$$y'' - \cos^2(t)\,y' + \sin^2(t)\,y = 0.$$

A4. Let A be an $n \times n$ real matrix whose eigenvalues λ_i satisfy $\Re\mathrm{e}\,\lambda_i < 0$ for all $i = 1, \ldots, n$. Let $g(x) \in \mathcal{C}^1(\mathbb{R}^n,\mathbb{R}^n)$ such that $g(x) = \mathbf{O}(|x|^\beta)$ for some $\beta > 1$ as $x \to 0$. Consider the initial value problem for $x_0 \in \mathbb{R}^n$,

$$\begin{cases} \dfrac{dx}{dt} = Ax + g(x), \\ x(0) = x_0. \end{cases} \tag{2}$$

Show that if $|x_0|$ is small enough, there is a solution to (2) which exists for all time.

[Hint: Suppose $g(t)$ and $u(t)$ are nonnegative functions and $c_0 \ge 0$ is a constant that satisfy $u(t) \le c_0 + \int_0^t g(s)\,u(s)\,ds$ for all $t \ge 0$. Then Gronwall's Inequality implies $u(t) \le c_0 \exp\left(\int_0^t g(s)\,ds\right)$ for all $t \ge 0$.]

A5. Show that the planar system

$$\dot{x} = x(x^2 + y^2 - 3x - 1) - y,$$
$$\dot{y} = y(x^2 + y^2 - 3x - 1) + x$$

has at least one nonconstant periodic solution.

A6. (a) State the Center Manifold Theorem for rest points. Briefly explain its importance in bifurcation theory.

(b) Using an approximate center manifold, determine the nature of the bifurcation at the origin when $\varepsilon = 0$.

$$\dot{x} = \varepsilon x - x^3 + y^2,$$
$$\dot{y} = -y + x^2.$$

B. Partial Differential Equations. Do three problems to get full credit

B1. Suppose that $\rho(x,t)$ is the number density of cars per unit length along a road, x being the distance along the road such that

$$\frac{\partial \rho}{\partial t} + \frac{\partial[\rho(1-\rho)]}{\partial x} = 0.$$

(a) Show that ρ is constant along characteristics

$$\frac{dx}{dt} = 1 - 2\rho,$$

and derive the following Rankine-Hugoniot conditions for the speed of a shock $x = S(t)$:

$$\frac{dS}{dt} = \frac{[\rho(1-\rho)]_-^+}{[\rho]_-^+}.$$

(b) A queue is building up at traffic light $x = 1$ so that, when the light turns to green at $t = 0$,

$$\rho(x,0) = \begin{cases} 0, & \text{if } x < 0 \text{ and } x > 1; \\ x, & \text{if } 0 < x < 1. \end{cases}$$

Solve the characteristic equations, and sketch the resulting characteristic curves. Deduce that a collision first occurs at $x = 1/2$ when $t = 1/2$, and that thereafter there is a shock such that

$$\frac{dS}{dt} = \frac{S + t + 1}{2t}.$$

B2. Let $D \in \mathbb{R}^n$ be a bounded, connected domain with smooth boundary. Let ϕ be an eigenfuncion of the Laplacian on D with Dirichlet boundary conditions ($\phi = 0$ on ∂D) and with eigenvalue λ.

$$\Delta\phi + \lambda\phi = 0. \tag{3}$$

(a) Show that there is a finite constant C so that for every nonzero continuously differentiable $u \in C^1(D)$ such that $u = 0$ on ∂D satisfies

$$\int_D u^2 \leq C \int_D |Du|^2.$$

(b) By integrating eigenfunctions of (3), conclude that eigenvalues are positive $\lambda > 0$.

(c) Denote by

$$\lambda_1(D) = \inf_{u \in C_0^1(D),\ u \not\equiv 0} \frac{\int_D |Du|^2}{\int_D u^2}$$

the least value of the Rayleigh quotient (which is at least $1/C$). Assuming that $\lambda_1(D)$ is an eigenvalue of D with Dirichlet boundary conditions, show that if two such domains satisfy $D_1 \subset D_2$, then $\lambda_1(D_1) \geq \lambda_1(D_2)$.

B3. Consider the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \qquad t > 0, x \in \mathbb{R}.$$

along with initial conditions

$$u(x,0) = g(x), \qquad u_t(x,0) = h(x).$$

(a) Assuming c is constant, derive d'Alembert's Formula

$$u(x,t) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)\,d\xi.$$

(b) Briefly explain the concepts of *domain of dependence*, and *finite propogation speed*, and prove that these properties follow d'Alembert's Formula.

(c) Suppose that we restrict the spatial domain to $x > 0$ and impose the boundary condition $u(0,t) = 0$ for $t \geq 0$. If the initial conditions satisfy $g(0) = h(0) = 0$, using d'Alembert's Formula as a starting point, find a closed form solution for $u(x,t)$ with $x, t > 0$.

B4. Let $D \in \mathbb{R}^d$ be a bounded domain with smooth boundary ∂D. Assume that that a complete orthonormnal set of eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ for the Laplacian on D with Dirichlet boundary conditions are known, in other words

$$-\Delta \phi_n = \lambda_n\, \phi_n, \qquad \text{in } D,\ n = 1, 2, 3 \ldots, \tag{4}$$

with $\phi_n(x) = 0$ for $x \in \partial D$, and $\int_D \phi_n^2\, dx = 1$.

(a) Given $f \in L^2(D)$, find a closed form series solution $u(x,t)$ for the initial, boundary value problem for the heat equation

$$\begin{aligned} -u_t + \Delta u &= 0, \qquad && t > 0,\ x \in D \\ u(x,t) &= 0, && x \in \partial D, \\ u(x,0) &= 0, && x \in D. \end{aligned}$$

(b) Show that there is a constant c such that

$$\left| \int_D u(x,t)\, \phi_n(x)\, dx \right| \leq c\, e^{-\lambda_n t}$$

for all n. Find $\lim_{t \to \infty} u(x,t)$.

(c) Give an example (the case $d = 1$ will suffice) which shows that solutions of the heat equation do not exhibit finite propagation speed.

B5. Suppose that $u(x_1, x_2)$ is a $\mathcal{C}^2$ harmonic function in the domain $\Omega \subset \mathbb{R}^2$, so $\Delta u = 0$ in Ω.

(a) Prove the *mean value property:* if $y = (y_1, y_2) \in \Omega$ and $r > 0$ is chosen such that $B_r(y) \subset \Omega$ (ball of radius r centered at y) then

$$u(y_1, y_2) = \frac{1}{2\pi} \int_0^{2\pi} u(y_1 + r\cos\theta, y_2 + r\sin\theta)\, d\theta.$$

(b) Assuming Ω is connected, prove that u can attain its maximum value at an interior point $x \in \Omega$, only if u is constant.