DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY

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Instructions: Do seven problems with at least three (3) problems from section A and three (3) problems from section B. You need at least two problems completely correct from each section to pass. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first problems answered will be scored.

A. Answer at least three and no more than four of the following questions. Each question is worth ten points. All manifolds are smooth.

1. Let $SL(2, \mathbf{R})$ denote the subset of M^2 (2×2 real matrices), with determinant equal to one. Prove that $SL(2, \mathbf{R})$ is a regular submanifold of $M^2 = \mathbf{R}^4$, and find it's dimension.

2. Let G be a finite group that acts smoothly, and freely (without fixed points) on a manifold M. Prove that the orbit space M/G admits a smooth structure (that is, has the structure of a smooth manifold).

3. Let M and N be compact orientable n-dimensional manifolds and $f : M \to N$ a smooth map. Suppose that
$$\int_M f^*\Omega \neq 0$$
for some n-form Ω. Prove that f is surjective.

4. Let M be a compact manifold. Prove that there does not exist a smooth submersion $f : M \to \mathbf{R}^n$ for any $n > 0$.

5. Let M be a compact oriented Riemannian manifold with boundary. Suppose that u and v are harmonic functions that are equal on ∂M. Show that $u = v$ on M. Recall that u is harmonic on M if $\Delta u = div(\nabla u) = 0$ on M.

6. Show that the group of diffeomorphisms of a compact manifold M acts transitively on M. That is, if $p, q \in M$ there exists a diffeomorphism $\phi : M \to M$ such that $\phi(p) = q$. Hint: First assume p and q are in the same coordinate neighborhood U and construct ϕ as the flow of a vector field supported in U.

B. Answer at least three and no more than four of the following questions so that the total number of questions you have answered is seven. Each question is worth ten points.

7. Let S^2 be the 2-sphere and T the torus.
   (a) Show that every map from S^2 to T is null-homotopic.
   (b) Show that there are infinitely many homotopy classes of maps from T to S^2.

8. Let X be a compact topological space and $\tilde{X}$ its universal cover. Show that $\pi_1(X)$ is finite if and only if $\tilde{X}$ is compact.

9. Let X_n be the space obtained by attaching a disk D to the torus $T = S^1 \times S^1$ where the attaching map is a degree n map from ∂D to the circle $S^1 \times \{p\}$ in T.
   (a) Calculate $\pi_1(X_n)$.
   (b) Calculate the homology and cohomology of X_n with $\mathbb{Z}$-coefficients and with $\mathbb{Z}_m$-coefficients where $\mathbb{Z}_m$ is the cyclic group of order m.

10. Show that $S^2 \vee S^4$ and $\mathbb{C}P^2$ are not homotopy equivalent.

11. Construct a Δ-complex structure for the torus and use it to calculate the homology, cohomology and cup product structure for the torus.

12. Let M be a closed, orientable manifold of dimension $2k$ and assume that $H_{k-1}(M)$ is torsion free. Show that $H_k(M)$ is torsion free.