DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY

August 2009

Instructions: Do all problems from section A and four (4) problems from section B. Be sure to provide all relevant definitions and statements of theorems cited.

A. Answer all of the following questions.

1. Very briefly for each, explain why the following spaces are smooth manifolds: S^n, $\mathbb{P}^n(\mathbb{R})$, T^n, $\mathrm{SL_n}(\mathbb{R})$, and $\mathrm{GL_n}(\mathbb{R})$.

2. (a) Let X be a vector field on a smooth manifold M and let $f : M \to N$ be a diffeomorphism. Define $f_*(X)$ – the pushforward of X on N.
   (b) Let $f : \mathbb{R}^3 \to \mathbb{R}\times(-\frac{\pi}{2}, \frac{\pi}{2})\times\mathbb{R}$ be defined by $f(x, y, z) = (e^x, \tan^{-1}(y), -2z)$. Calculate the bracket $[f_*(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}), f_*(\frac{\partial}{\partial y} - \frac{\partial}{\partial z})]$.

3. Let V and W be vector fields on $\mathbb{R}^3$ given by
$$V(x, y, z) = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z} \qquad \text{and} \qquad W(x, y, z) = \frac{\partial}{\partial y} + x^2\frac{\partial}{\partial z}$$
.
   (a) Find $[V, W](x, y, z)$.
   (b) What is the definition of an integrable, k-dimensional plane field on a manifold M?
   (c) Prove or disprove: $\Delta(x, y, z) = \mathrm{span}\{\, V(x, y, z)\, ,\, W(x, y, z)\, \}$ is an integrable plane field.
   (d) Give an example of three foliations on a two dimensional torus: one with every leaf compact; one with every leaf noncompact; and one with some compact and some noncompact leaves.

4. (a) Let G be the Lie subgroup of $\mathrm{GL}_3(\mathbb{R})$ that consists of the following matrices:
$$\begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix}$$
where $ad-bc = 1$. Find a basis for the Lie algebra of G, considered as a Lie subalgebra of $\mathrm{M}_{3\times3}(\mathbb{R})$.
   (b) Let $\mathfrak{gl}_2(\mathbb{R})$ be the Lie algebra of $\mathrm{GL}_2(\mathbb{R})$. For
$$\begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \in \mathfrak{gl}_2(\mathbb{R})$$
find
$$\left[\begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\right]$$

5. (a) State Stokes' Theorem
   (b) Given two 3-forms on S^7, denoted θ and ω, prove that $\int_{S^7} d\theta \wedge \omega = \int_{S^7} \theta \wedge d\omega$.

6. Let $S^n = \{v \in \mathbb{R}^{n+1} \mid ||v|| = 1\}$ and let $f : S^n \to S^n$ be the restriction of the map $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by $v \mapsto -v$. Suppose ω is an n-form on S^n with $\int_{S^n} \omega = \frac{\pi^2}{6}$. What is $\int_{S^n} f^*\omega$. Why?

7. (a) State Sard's Theorem.
   (b) Show that any smooth map $f : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{S}^4$ is homotopic to a constant map.

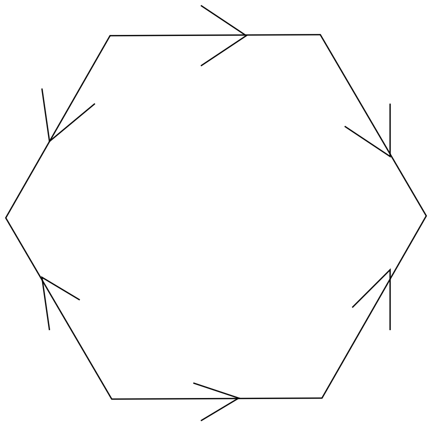
8. Let $M = T^2 \times S^2$, and suppose ω is a 4-form on M such that $\int_M \omega = 7$.
   (a) Define a smooth map $f : M \to M$ that has degree 2.
   (b) Define a 4-form θ on M such that $\int_M \theta = 56$, using both f and ω nontrivially.

9. (a) What theorem from multivariable calculus is the proof of the regular value theorem dependent on?
   (b) What theorem from single variable calculus is the proof of Stokes' theorem dependent on?
   (c) What theorem from differential topology is used to prove that transverse intersection of manifolds is a manifold?

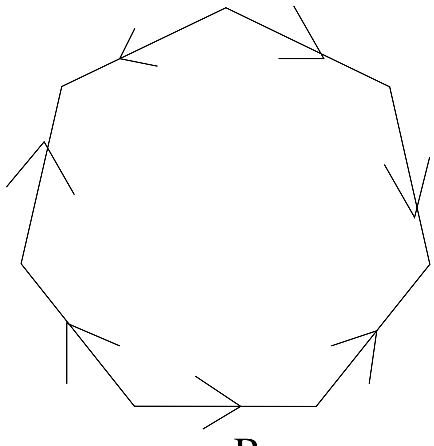
10. Let M and N be smooth, closed, connected n-manifolds, and let $f : M \to N$ be a smooth map. If every point of N is a regular value, show that the cardinality of the pre-image of a point in N defines a constant function on N.

B. Answer four of the following questions.

11. Consider the following two 2-complexes, A and B, defined as identification spaces of polygons, with side identifications as indicated (so all edges are identified with each other).



A

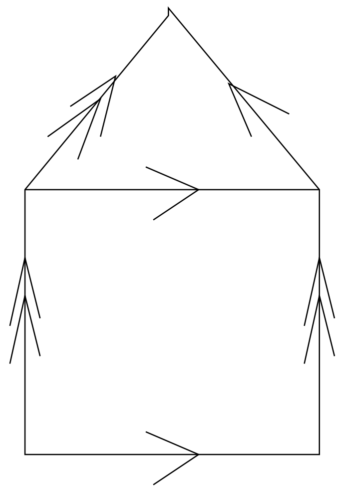


B

Is either space homotopy equivalent to a 1-complex?

12. Is there a covering map $p : X \to Y$ where
   (a) $X = S_3$, $Y = S_2$ where S_g is a closed orientable surface of genus g?
   (b) X is S_3 with a small open disk removed and Y is S_2 with a small open disk removed.
   When p exists, what is the number of sheets?

13. Using Δ-homology, compute homology and cohomology (with $\mathbb{Z}$ coefficients) of the following space (solid square and triangle with identifications as indicated).



14. Let M be a closed connected 5-manifold such that $\pi_1(M) = \mathbb{Z}/7\mathbb{Z}$ and $H_2(M;\mathbb{Z}) = \mathbb{Z}^2$. Compute $H_k(M;\mathbb{Z})$, $H^k(M;\mathbb{Z})$ and $H_k(M;\mathbb{Z}/7\mathbb{Z})$ for all k.

15. Describe an explicit cell decomposition of $\mathbb{C}P^n$ (including the precise subsets corresponding to cells; you don't have to check all properties of cell structures). From the cell decomposition read off all homology and cohomology groups with $\mathbb{Z}$ coefficients.

16. Describe the procedure for computing (i.e. writing down a presentation of) the fundamental group of a cell complex with one 0-cell. Using this, show that $\mathbb{C}P^n$ is simply connected and $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$.

17. Compute the fundamental group of the space $X_{2,3}$ obtained from the cylinder $S^1 \times [-1, 1]$ via identifications: $(z, -1) \sim (-z, -1)$ and $(z, 1) \sim (z\ e^{\frac{2\pi i}{3}}, 1) \sim (z\ e^{\frac{4\pi i}{3}}, 1)$ for all $z \in S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. (That is, in the circle $S^1 \times \{-1\}$ antipodal points are identified, and in $S^1 \times \{1\}$ two points are identified if they differ by an order 3 rotation.) Show that this group is infinite.

18. Use the Mayer-Vietoris sequence to compute homology groups (with $\mathbb{Z}$-coefficients) of the space $X_{2,3}$ from problem #17. Do the same for the space $X_{2,2}$ defined similarly except that on $S^1 \times \{1\}$ antipodal points are identified.