DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY
August 2012

Instructions: Do all problems from section A and all problems from section B. Be sure to provide all relevant definitions and statements of theorems cited.

A. Answer all of the following questions.

1. $13xdx + y^2dy + xyzdz$ is a form on $\mathbb{R}^3$. What's its exterior derivative?

2. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be the function $f(x, y, z) = (xy, z)$. Find the pullback form $f^*(dy \wedge dz + x^2dx \wedge dy)$.

3. Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be the function $g(x, y, z) = (x^2y, 3xz, y + z)$. Find $D_{(2,3,5)}g : \mathbb{R}^3 \to \mathbb{R}^3$, the derivative of g at the point $(2, 3, 5)$.

4. Let $h : \mathbb{R}^3 \to \mathbb{R}$ be the function $h(x, y, z) = xyz$. Let s be the vector field on $\mathbb{R}^3$ given by $s(x, y, z) = xy\frac{\partial}{\partial x} + (y - z^3)\frac{\partial}{\partial y} + 3\frac{\partial}{\partial z}$. Find $L_s(f)(1, 1, 2)$, the Lie derivative of f in the direction of s at the point $(1, 1, 2)$.

5. Find the following bracket of two vector fields in $\mathbb{R}^3$
$$\left[(x + y)\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}\ ,\ yz\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial z}\right]$$

6. Let S^2 be the vectors of length 1 in $\mathbb{R}^3$. Find $\int_{S^2} dx \wedge dy$.

7. Let Γ be a group of diffeomorphisms acting on a smooth manifold M. When is $\Gamma\backslash M$ a manifold?

8. Let $f : M \to N$ be a smooth map of smooth manifolds. If $Q \subseteq N$ is an embedded submanifold, and f is transverse to Q, prove that $f^{-1}(Q)$ is a manifold.

9. Let M be a compact, connected smooth manifold, and suppose that N is a connected smooth manifold. If $F : M \times [0, 1] \to N$ is smooth and $m \mapsto F(m, 0)$ is an immersion of M into N, then show there is some $\varepsilon > 0$ such that $m \mapsto F(m, \delta)$ is an immersion of M into N for any fixed $\delta < \varepsilon$.

10. Let M be a smooth compact manifold and let X be a smooth vector field with a corresponding 1-parameter flow group $\{\theta_t^X : M \to M\}_{t\in\mathbb{R}}$. Let f be a diffeomorphism of M and let f_*X be its pushforward. Show that the flow group for f_*X is $f \circ \theta_t^X \circ f^{-1}$.

B. Answer all of the following questions.

11. Let f be the homeomorphism of the annulus $S^1 \times [0,1]$ given by

$$f(z,s) = (ze^{2\pi i s}, s)$$

where we view S^1 as the set of unit norm complex numbers.

(a) Construct an explicit isotopy (i.e. homotopy through homeomorphisms) between f and the identity that does not move the points of $S^1 \times \{0\}$.

(b) Prove that there is no homotopy between f and the identity that does not move points on both $S^1 \times \{0\}$ and $S^1 \times \{1\}$.

12. Consider the disk with 2 holes

$$P = \{(x,y) \in \mathbb{R}^2 \mid ||(x,y)|| \leq 4, ||(x,y) - (2,0)|| \geq 1, ||(x,y) + (2,0)|| \geq 1\}$$

Orient each boundary component counterclockwise. Let X be the space obtained from P by identifying all boundary components via orientation preserving homeomorphisms. Find a presentation of $\pi_1(X)$.

13. Give a definition of the chain homotopy between chain maps and prove that chain homotopic chain maps induce the same homomorphism in homology.

14. A map $f : S^n \to S^n$ is said to be *even* if $f(-x) = f(x)$ for every $x \in S^n$. Show that even maps have even degree, and in fact that the degree must be 0 when n is even.

15. (a) Describe the simplest cell structures on $\mathbb{C}P^2$ (including attaching maps).

(b) Prove that $H_i(\mathbb{C}P^2; G)$ is isomorphic to $H_i(S^2 \vee S^4; G)$ for every abelian group G and all i, and the same is true for cohomology.

(c) Prove that $\mathbb{C}P^2$ is not homotopy equivalent to $S^2 \vee S^4$ by considering ring structures in cohomology.

16. Prove that when $g < h$ every map $S_g \to S_h$ between orientable closed surfaces of genus g and h respectively is equal to 0.