DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY
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Instructions: Do all problems from section A and all problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. To pass the exam you need to have at least 3 completely correct solutions in **both** parts.

A. Answer all of the following questions.

1. Identify $\mathcal{M}(2)$, the set of two-by-two matrices, with $\mathbb{R}^4$. Let $SL(2,\mathbb{R}) \subset \mathcal{M}(2)$ be the set of matrices with determinant one. Show that $SL(2,\mathbb{R})$ is a smooth submanifold and calculate the tangent space $T_{id}SL(2,\mathbb{R})$ as a subspace of $T_{id}\mathcal{M}(2) = \mathbb{R}^4$. Is the line $\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$, $t \in \mathbb{R}$ transverse to $SL(2,\mathbb{R})$? Justify your answer.

2. Define $\phi : \mathbb{R}^2 \to \mathbb{R}^2\backslash\{0\}$ by $\phi(x,y) = (e^x \cos y, e^x \sin y)$. Let ω be a 1-form on $\mathbb{R}^2\backslash\{0\}$ such that $dy = \phi^*\omega$. Show that ω is closed but not exact on $\mathbb{R}^2\backslash\{0\}$.

3. Let X and Y be closed submanifolds of $\mathbb{R}^n$. Show that for almost all $a \in \mathbb{R}^n$ the translate $X + a = \{X + a | x \in X\}$ intersects Y transversally.

4. Let $U \subset \mathbb{R}^n$ be a connected open set with $p, q \in U$. Show that there exists a diffeomorphism ϕ of $\mathbb{R}^n$ such that $\phi(p) = q$ and ϕ is the identity outside of U. (Hint: First assume that U is convex with compact closure and find a vector field with support in U whose flow takes p to q.)

5. Let M be a differentiable manifold. Prove that its tangent bundle TM and and its cotangent bundle are isomorphic as smooth vector bundles.

6. Give two definitions of a differentiable manifold being orientable; one via charts and the other via forms. Show that the two definitions are equivalent.

B. Answer all of the following questions.

7. Let X be a path connected, locally path connected, and semi-locally simply connected space. (Recall that X is semi-locally simply connected if every point in X has a neighborhood U such that every closed loop in U can be contracted to a point in X.) Outline the construction of the universal cover $p : \tilde{X} \to X$. Specifically,

   (a) Define $\tilde{X}$ as a set and the projection $p : \tilde{X} \to X$,

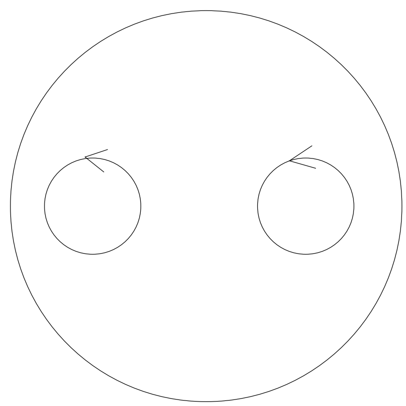
   (b) Define the topology on $\tilde{X}$ (you don't have to check that this is a topology),

   (c) Give an argument that $\tilde{X}$ is simply connected.

   Indicate where each of the three assumptions on X is used.

8. Describe a cell structure on the real projective space $\mathbb{R}P^n$. Explicitly describe the attaching maps of all cells. Use this cell structure to compute $H_i(\mathbb{R}P^n; \mathbb{Z}/2)$ for $i \geq 0$.

9. Let X be the compact surface
$$X = \left\{ z \in \mathbb{C} \mid |z| \leq 1, |z - \frac{1}{2}| \geq \frac{1}{4}, |z + \frac{1}{2}| \geq \frac{1}{4}| \right\}$$
obtained from the disk by deleting two disjoint open disks. There are two essentially different ways to identify the three boundary components by homeomorphisms: fix orientations on two boundary components as in the Figure below and use two possible orientations on the third. Show that the two quotient spaces have non-isomorphic fundamental groups. Hint: Abelianize.



10. Let X be the space obtained from the circle S^1 by attaching two 2-cells, one with degree 3 attaching map and the other with degree 5 attaching map.

    (a) Using van Kampen's theorem show that X is simply connected.

    (b) Show that X is homotopy equivalent to the 2-sphere. Hint: Use repeateadly the fact that the spaces $Y \cup_f e^2$ and $Y \cup_g e^2$ obtained from Y by attaching a single 2-cell with homotopic attaching maps f, g are homotopy equivalent.

11. Suppose a space X can be written as the union $\cup_{i=1}^n U_i$ of open subsets U_i. Also assume that every U_i, as well as every nonempty intersection $U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_k}$, is contractible. Prove that $\tilde{H}_i(X; \mathbb{Z}) = 0$ for $i \geq n - 1$. (Hint: For $i = 1, \ldots, n-1$ let $V_i = U_n \cap U_i$ and apply induction.)

12. Show that the degree of every map $\mathbb{C}P^2 \to \mathbb{C}P^2$ of the complex projective plane to itself is nonnegative.