UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph. D. Preliminary Examination in Geometry / Topology

August 13, 2019.

Instructions This exam has two parts, A and B, covering material from Math 6510 and 6520, respectively. To pass the exam, you will have to pass each part. To pass each part you will have to demonstrate mastery of the material. Please answer as many questions as you can. Not all questions are equally difficult.

Part A. For Part A, each question is worth 20 points. A passing score is 60 on this part.

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1. Let $M^m \subset \mathbb{R}^n$ be a smooth embedded submanifold. Show that the unit tangent bundle

$$UM = \{(p, V) : p \in M \text{ and } V \in T_pM \text{ such that } |V| = 1\}$$

is a smooth embedded submanifold of $T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$. What is its dimension?

2. Consider the Lie Group $G = \mathbf{O}(n) = \{A \in \mathcal{M}_n : A^T A = I\}$ where $\mathcal{M}_n \cong \mathbb{R}^{n^2}$ is the collection of real $n \times n$ matrices and A^T is the transpose of A.

   (a) Describe the set of matrices which form the tangent space $T_P G$ at the point $P \in G$.

   (b) Given $V \in T_I G$ and $P \in G$, find $X_V(P) \in T_P G$ where X_V is the left invariant vector field on G such that $X_V(I) = V$. Show that your X_V is left invariant.

   (c) Let θ_t^X be the flow for the left invariant vector field $X = X_V$ for some $V \in T_I G$. Find and verify a formula for $\theta_t^X(P)$ in terms of P, t and V. Compute $\theta_t^X(I)$ in case $G = \mathbf{O}(2)$ and $V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

3. Let M and N be smooth manifolds. Without using the de Rham Theorem, show that if the de Rham cohomology of M doesn't vanish $H^k_{\mathrm{dR}}(M) \neq \{0\}$ then $H^k_{\mathrm{dR}}(M \times N) \neq \{0\}$.

4. Let M^m be a smooth compact orientable manifold and $\omega \in \Omega^m(M)$ a smooth, nonvanishing m-form that determines the orientation of M. Show that $\int_M \omega > 0$.

5. Suppose the three sphere $\mathbb{S}^3 = \{(x, y, z, w) \in \mathbf{R}^4 : x^2 + y^2 + z^2 + w^2 = 1\}$ is embedded into four space the standard way. Two independent vector fields are given by $U = (y, -x, w, -z)$ and $V = (w, z, -y, -x)$. Determine whether it is possible to find a smooth embedded two dimensional submanifold $N \subset \mathbb{S}^3$ such that the vector fields U and V are tangent to N at every point of N, *i.e.*, $U(p), V(p) \in T_pN$ for every $p \in N$?

Part B. For Part B answer as many questions as you can. You need to get three questions completely correct to pass this part.

1. Let X be the wedge product of a circle and a projective plane. Find all connected three sheeted covers of X.

2. Let n_1, n_2 and n_3 be integers. Given an example of a topological space X with
$$\pi_1(X, x_0) = \langle a, b | a^{n_1} b^{n_2} a^{n_3} b \rangle.$$
What is $H_1(X)$? Under what conditions is $H_1(X)$ torsion free?

3. Give an example of a chain complex whose homology and cohomology are different.

4. For $n \geq 3$ let $X = \mathbb{S}^{n-1} \times \mathbb{S}^1$ be the product of an $(n-1)$-sphere and a circle and let Y be obtained by attaching an n-cell to X where the attaching map is a degree k from the boundary of the cell to the $(n-1)$-sphere $\mathbb{S}^{n-1} \times \{p\} \subset X$ via a degree k map. Calculate the homology of Y.

5. Let X be obtained from attaching two tori along a circle in their product structure. In particular each individual torus is $\mathbb{S}^1 \times \mathbb{S}^1$ and we attach the two tori along circles $\mathbb{S}^1 \times \{p\}$ in the first torus and $\mathbb{S}^1 \times \{q\}$ in the second torus. Calculate the cohomology and cup product structure on X with $\mathbb{Z}$ coefficients. You can use the cup product structure of a torus as a given. (Hint: X retracts onto each of the individual tori.)

6. (a) Let M be a closed, connected, orientable 3-manifold such that
$$H_1(M; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2.$$
Calculate the remaining homology and cohomology groups with $\mathbb{Z}$ coefficients.

   (b) Show that the Euler characteristic of any closed 3-manifold is zero.

   (c) Now assume that M is a closed, non-orientable 3-manifold. Show that $H_1(M; \mathbb{Z})$ is infinite.