DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY

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Instructions: Do seven problems with at least three (3) problems from section A and three (3) problems from section B. You need at least two problems completely correct from each section to pass. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first problems answered will be scored.

A. Answer at least three and no more than four of the following questions. Each question is worth ten points. All manifolds are smooth.

1. Let $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $f(x, y) = (x \cdot y, |x|^2 - |y|^2)$, where $x \cdot y$ is the dot product of x and y in $\mathbb{R}^3$ and $|x|$ is the Euclidean norm of x. Suppose that $(a, b) \neq (0, 0)$. Show that
$$\{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : f(x, y) = (a, b)\}$$
is a smooth submanifold, and compute it's dimension.

2. Let U be the open set in $\mathbb{R}^3$, $U = \{(x, y, z) : x > 0,\ y > 0,\ z > 0\}$ and let D be the distribution in U spanned by the vector fields:
$$V_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \quad V_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$
Show that D is involutive. What does the Frobenius Theorem imply about D?

3. Let M be a manifold. Prove that there exists a Riemannian metric on M.

4. Let M and N be manifolds. Show that $M \times N$ is orientable if and only if M and N are orientable.

5. Show that S^3 (the 3-sphere) is the covering space of a manifold M which is not diffeomorphic to $\mathbb{R}P^3$. (Hint: find an appropriate finite group action on S^3.)

6. Let $\Lambda^k(M)$ denote the space of smooth k-forms on a manifold M. Suppose that $\omega \in \Lambda^p(M)$ and $\eta \in \Lambda^q(M)$ are closed forms. Show that the de Rham cohomology class of $\omega \wedge \eta$ depends only on the de Rham cohomology classes of ω and η.

B. Answer at least three and no more than four of the following questions so that the total number of questions you have answered is seven. Each question is worth ten points.

7. Let X be the union of circles in R^2 with centers at $(0, \frac{1}{2n})$ and radius $1/n$. Show X doesn't have a universal cover (show that every connected covering space of X, including X itself, has non-trivial π_1).

8. Let $X \subset \mathbb{R}^3$ be the union of n-lines through the origin. Calculate $\pi_1(\mathbb{R} \backslash X)$.

9. Construct a Δ-complex structure on $\mathbb{R}P^2$ and use it to calculate the homology and cohomology of of $\mathbb{R}P^2$ with $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients.

10. Let X be a contractible CW-complex and attach an n-cell to X for each positive even dimension n to form a new CW-complex X'.

11. Apply the Lefschetz fixed point theorem to show that every map $f : \mathbb{C}P^n \longrightarrow \mathbb{C}P^n$ has a fixed point when n is even (state the Lefschetz fixed point theorem and any fact about the ring structure of $H^*(\mathbb{C}P^n)$ you are using). Construct a fixed point free map $f : \mathbb{C}P^n \longrightarrow \mathbb{C}P^n$ when n is odd.

12. (a) Let M be a closed, orientable 3-manifold such that

$$H_1(M; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2.$$

Calculate the remaining homology and cohomology groups for M with $\mathbb{Z}$ coefficients.

(b) Now assume that M is a closed, non-orientable 3-manifold. Show that $H_1(M; \mathbb{Z})$ is infinite.