UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph. D. Preliminary Examination in Geometry / Topology
January 4, 2020.

**Instructions** This exam has two parts, A and B, covering material from Math 6510 and 6520, respectively. To pass the exam, you will have to pass each part. To pass each part you will have to demonstrate mastery of the material. Not all questions are equally difficult.

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**Part A.** Answer five of the following questions. Indicate which of the questions are to be graded. If you do more than five, only the first five will be graded. Each question is worth 20 points. A passing score is 60 on this part.

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1. Show that $\mathcal{C} \cap \mathcal{S}$ is an embedded submanifold in $\mathbb{R}^m \times \mathbb{R}^n$ and find its dimension, where
$$\mathcal{C} = \left\{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n : |x|^2 = |y|^2\right\}, \qquad \mathcal{S} = \left\{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n : |x|^2 + |y|^2 = 1\right\}.$$

2. Let $X, Y \subset \mathbb{R}^3$ be smooth 1-dimensional submanifolds. Show that there is a $v \in \mathbb{R}^3$ such that $X$ is disjoint from $Y + v = \{y + v : y \in Y\}$.

3. Let $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere. Let
$$\omega = \begin{cases} \dfrac{dy \wedge dz}{x}, & \text{if } x \neq 0; \\[2ex] \dfrac{dz \wedge dx}{y}, & \text{if } y \neq 0; \\[2ex] \dfrac{dx \wedge dy}{z}, & \text{if } z \neq 0. \end{cases}$$
   (a) Show that $\omega$ is a well defined two form on $\mathbb{S}^2$
   (b) Find $\displaystyle\int_{\mathbb{S}^2} \omega$.
   (c) Is $\omega$ exact? Why?

4. Consider the Lie subgroup $Z$ of the Heisenberg group $H$. Find all connected two dimensional Lie subgroups of $H$ that contain $Z$. Describe them as both submanifolds of $H$ and as groups.
$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}, \qquad Z = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{R} \right\}$$

5. Prove that $\mathbb{RP}^2$ is not orientable.

6. Let $M$ be a smooth compact manifold without boundary. Prove that if $M$ has a smooth nowhere vanishing closed one form, then $H^1_{\mathrm{dR}}(M) \neq \{0\}$. Deduce that if there is a smooth submersion of $M$ onto $\mathbb{S}^1$, then $H^1_{\mathrm{dR}}(M) \neq \{0\}$.

**Part B.** Answer as many questions as you can. You need to get three questions completely correct to pass this part.

1. Construct a $CW$-complex whose fundamental group is the free product $\mathbb{Z} * \mathbb{Z}_2$.
2. Let $X$ be the wedge sum of two copies of $\mathbb{R}\mathrm{P}^2$. Show that there is only one connected degree three cover of $X$ and determine if it is regular or irregular.
3. Recall that if $X$ is a topological space its *suspension* $SX$ is the product $X \times [0,1]$ with both $X \times \{0\}$ and $X \times \{1\}$ collapsed to points. Show that $\tilde{H}_{n+1}(SX) = \tilde{H}_n(X)$ for $n \geq 0$.
4. Let $X_n$ be the topological space obtained by attaching a disk $D$ to the torus $T = S^1 \times S^1$ where the attaching map is a degree $n$ map from $\partial D$ to $S^1 \times \{p\}$ in $T$.
   (a) Calculate $\pi_1(X_n)$.
   (b) Calculate the homology and cohomology of $X_n$ with $\mathbb{Z}$ and $\mathbb{Z}_m$-coefficients.
5. Construct a $\Delta$-complex structure for $\mathbb{R}\mathrm{P}^2$ and use it to calculate the homology, cohomology and cup product structure for $\mathbb{R}\mathrm{P}^2$ with $\mathbb{Z}_2$-coefficients.
6. Prove that $S^1 \vee S^3$ is not homotopy equivalent to a compact, orientable 3-manifold without boundary.