Preliminary Examination, Numerical Analysis, Spring 2006

Instructions: This exam is closed books and notes. The time allowed is three hours and you need to work on any two out of questions 1-3 and any three out of questions 4-7. All questions have equal weights and the passing score will be determined after all the exams are graded. Indicate clearly the work that you wish to be graded.

1. **(Sensitivity Analysis)**

   (a) Let $X \in R^{n\times n}$ such that $||X|| < 1$ for some induced norm. Show that $I - X$ is nonsingular and

   $$||(I-X)^{-1}|| \leq \frac{1}{1-||X||}.$$

   (b) Assume that $A \in R^{n\times n}$ is nonsingular, and the linear system

   $$A\mathbf{x} = \mathbf{b}$$

   is perturbed to

   $$(A+\delta A)(\mathbf{x}+\delta\mathbf{x}) = \mathbf{b}+\delta\mathbf{b},$$

   and the perturbations are small enough so that

   $$||\delta A|| \leq \epsilon||A||, \quad ||\delta\mathbf{b}|| \leq \epsilon||\mathbf{b}||,$$

   for some small $\epsilon > 0$, and $\epsilon\kappa(A) = r < 1$, where $\kappa(A)$ is the condition number of A. Show that $A + \delta A$ is nonsingular.

   (c) Let $\tilde{\mathbf{x}} = \mathbf{x} + \delta\mathbf{x}$, show that

   $$||\tilde{\mathbf{x}}|| \leq \frac{1+r}{1-r}||\mathbf{x}||;$$

   and

   $$\frac{||\delta\mathbf{x}||}{||\mathbf{x}||} \leq \frac{2\epsilon}{1-r}\kappa(A).$$

2. **(SVD)** Let A be a real $m \times n$ matrix ($m \geq n$) of full rank. We are interested in the singular value decomposition of $A = U\Sigma V^T$. It is known that algorithms for SVD can be derived by turning the SVD problem into an eigenvalue problem. One such approach is to find the eigenvalue decomposition of the $(m+n)\times(m+n)$ symmetric matrix

   $$\begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}.$$

   Establish the connections between the singular values, left and right singular vectors of A and the eigenvalues, eigenvectors of the above matrix. You should pay particular attention to the case $m > n$. Comment on the advantages and disadvantages of this approach compared with an approach based on the eigendecomposition of A^TA.

3. **(Numerical Quadrature)** Consider the quadrature formula

$$\int_a^b w(x)f(x)dx = \sum_{i=1}^{n} \alpha_i f(\xi_i) + E,$$

for the integral of f with a given positive weight function $w(x) > 0$. We would like to use n knot values $(f(\xi_i), i = 1, \ldots, n)$ to achieve the exact integral if f is a polynomial of degree up to $2n-1$.

   (a) Start from the special case $a = -1, b = 1$ and $w(x) = 1$, define the orthogonal polynomials over $[-1, 1]$.

   (b) Use those orthogonal polynomials and polynomial division to simplify the integral. How would you pick the knots ξ_i and the weights α_i such that the formula is exact (i.e., $E = 0$) for all polynomials of degree up to $2n-1$? Verify that your choice works.

   (c) Show that you can extend the approach to more general integrals, with arbitrary $a < b$ and positive weight function w.

4. **(ODE)** Consider the initial value problem for a system of ODEs

$$y' = Ay$$

where A has eigenvalues ranging from -10^4 to -1. Which of the following schemes would be the best choice for solving this problem? Justify your answer in terms of stability, accuracy, and efficiency.

$$y^{n+1} = y^n + hAy^n, \tag{1}$$

$$y^{n+1} = y^n + hAy^{n+1}, \tag{2}$$

$$y^{n+1} = y^n + \frac{h}{2}(Ay^n + Ay^{n+1}). \tag{3}$$

5. **(Elliptic Equations)** Consider the standard five-point difference approximation (centered difference for both the gradient and divergence operators) for the variable coefficient Poisson's equation

$$-\nabla \cdot (a\nabla u) = f$$

with Dirichlet boundary conditions, in a two-dimensional rectangular region. We assume that $a(\mathbf{x}) \geq \delta > \mathbf{0}$. The approximate solution $\{u_{i,j}\}$ satisfies a linear system $Au = b$.

(a) State and prove the maximum principle for the numerical solution $u_{i,j}$.

(b) Use the maximum principle to show that the coefficient matrix A is nonsingular.

(c) Derive the matrix A in the one-dimensional case and show that it is symmetric and positive definite.

(d) With this linear system, do we get convergence with Jacobi iterations? What about Gauss-Seidel iterations? Give reasons for your answers.

6. **(Wave Equation)** For the PDE $v_t + cv_x = 0$, derive the Lax-Wendroff scheme

$$u_j^{n+1} = u_j^n - \frac{1}{2}\left(\frac{c\Delta t}{\Delta x}\right)\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{1}{2}\left(\frac{c\Delta t}{\Delta x}\right)^2\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right).$$

Analyze this scheme in terms of accuracy and stability. Does this scheme satisfy a maximum principle? If it does, explain why. If it does not, give an example to show this.

7. **(Convection-Diffusion Equation)** For the convection-diffusion equation

$$v_t + av_x = bv_{xx},$$

where $a > 0$ and $b > 0$, a natural Crank-Nicolson scheme is

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a}{2}\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x}\right)$$
$$= \frac{b}{2}\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} + \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\Delta x)^2}\right).$$

Let $\nu = b\Delta t/(\Delta x)^2$ and $\mu = a\Delta t/(2\Delta x)$. Show that the scheme is stable in the max norm if $\mu \le \nu \le 1$. Also use Fourier analysis to derive the stability condition in the l_2 norm and show that the scheme is always stable in the l_2 norm. Suppose u stands for some volume fraction of a certain species, which norm is more appropriate? Explain why this scheme is so restrictive that it is not a good choice for many applications.

Preliminary Examination, Numerical Analysis, August 2005

Instructions: This exam is closed books and notes. The time allowed is three hours and you need to work on any two out of questions 1-3 and any three out of questions 4-8. All questions have equal weights and a score of 75% is considered a pass. Indicate clearly the work that you wish to be graded.

1) Ordinary Differential Equations:

Consider the initial value problem for a system of two ODEs

$$y' = Ay$$

where A has eigenvalues -10^4 and 1. Which of the following schemes would be the best choice for solving this problem? Justify your answer in terms of stability, accuracy, and efficiency.

$$y^{n+1} = y^n + hAy^n, \tag{1}$$

$$y^{n+1} = y^n + hAy^{n+1}, \tag{2}$$

$$y^{n+1} = y^n + \frac{h}{2}(Ay^n + Ay^{n+1}). \tag{3}$$

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2) Wave Equation:
Consider the Lax-Friedrichs scheme below for the PDE $v_t + cv_x = 0$, where you may assume that $c > 0$:

$$u_j^{n+1} = \frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right) - \frac{ck}{2h}\left(u_{j+1}^n - u_{j-1}^n\right),$$

where h is the spacestep and k is the timestep. Note that the scheme can also be written

$$u_j^{n+1} = u_j^n - \frac{ck}{2h}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{1}{2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right).$$

a) Analyze the stability of this scheme in terms of $\alpha = \frac{ck}{h}$.

b) Recall that a modified equation for a scheme is the PDE satisfied, to a certain order, by a smooth solution $u(x,t)$ of the *difference scheme*. To zeroth order in k and h, the modified equation for the Lax-Friedrichs scheme is $u_t + cu_x = 0$. Derive the modified equation through terms of first order in k and h. You may assume that α is constant. Does this modified equation predict the same stability condition as your analysis in part (a)? Explain your answer. What does the modified equation suggest about the numerical solution that will be obtained for the step-function initial data $u(x,0) = 1$ for $x \leq 0$ and $u(x,0) = 0$ for $x > 0$?

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3) Heat Equation:
Consider the variable coefficient diffusion equation

$$v_t = (\beta v_x)_x, \qquad 0 < x < 1,\ t > 0$$

with Dirichlet boundary conditions

$$v(0,t) = 0.0, \qquad v(1,t) = 0.0$$

and initial data $v(x,0) = f(x)$. Assume that $\beta(x) \geq \beta_0 > 0$, and that $\beta(x)$ is smooth. For any fixed value of $\theta \in [0,1]$, consider the θ-scheme for this problem:

$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{1}{h^2} \left\{ (1-\theta)\left(\beta_{j-1/2} u_{j-1}^{n+1} - (\beta_{j-1/2} + \beta_{j+1/2}) u_j^{n+1} + \beta_{j+1/2} u_{j+1}^{n+1}\right) \right.$$
$$\left. + \theta \left(\beta_{j-1/2} u_{j-1}^{n} - (\beta_{j-1/2} + \beta_{j+1/2}) u_j^{n} + \beta_{j+1/2} u_{j+1}^{n}\right) \right\},$$

with appropriate treatment of the boundary conditions in the relevant equations. Here, h is the spacestep, $(J+1)h = 1$, k is the timestep, and $\beta_{j+1/2} = \beta(x_{j+1/2})$, where $x_{j+1/2} = (j+1/2)h$. Let M be the symmetric $J \times J$ matrix defined by

$$M = \begin{pmatrix} -(\beta_{1/2}+\beta_{3/2}) & \beta_{3/2} & \cdots & \cdots & \cdots & \cdots \\ \beta_{3/2} & -(\beta_{3/2}+\beta_{5/2}) & \beta_{5/2} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \beta_{J-3/2} & -(\beta_{J-3/2}+\beta_{J-1/2}) & \beta_{J-1/2} \\ \cdots & \cdots & \cdots & \cdots & \beta_{J-1/2} & -(\beta_{J-1/2}+\beta_{J+1/2}) \end{pmatrix}$$

Analyze the stability of this scheme in terms of θ, $\frac{k}{h^2}$, and the eigenvalues of M. Are there values of θ for which the scheme is stable for all choices of $\frac{k}{h^2} > 0$? (Note that answering this question should not require lots of writing.)

4) Sensitivity:
Consider a 6×6 symmetric positive definite matrix A with singular values $\sigma_1 = 1000$, $\sigma_2 = 500$, $\sigma_3 = 300$, $\sigma_4 = 20$, $\sigma_5 = 1$, $\sigma_6 = 0.01$.
a) Suppose you use a Cholesky factorization package on a computer with a machine epsilon 10^{-14} to solve the system $Ax = b$ for some nonzero vector b. How many digits of accuracy do you expect in the computed solution? Justify your answer in terms of condition and stability. You may assume that the entries of A and b are exactly represented in the computer's floating-point system.

b) Suppose that instead you use an iterative method to find an approximate solution to $Ax = b$ and you stop iterating and accept iterate $x^{(k)}$ when the residual $r^{(k)} = Ax^{(k)} - b$ has 2-norm less than 10^{-9}. Give an estimate of the maximum size of the relative *error* in the final iterate? Justify your answer.

5) Iterative Methods:
Consider the fixed-point problem $x = Tx + c$, where T is a real $N \times N$ matrix and x and c are real N-vectors.
a) Show that the fixed point iteration

$$x^{(k+1)} = Tx^{(k)} + c$$

converges for any choice of initial vector $x^{(0)}$ if and only if the spectral radius of T, $\rho(T)$, satisfies $\rho(T) < 1$.

b) Consider the linear system $Ax = b$ where A is a strictly diagonally dominant matrix, that is, $a_{ii} \geq \sum_{j \neq i} |a_{ij}|$ for all i, and strict inequality holds for at least one value of i. Show that the Jacobi iterative scheme converges for this problem.

6) Interpolation and Integration:

a) Consider equally spaced points $x_j = a + jh$, $j = 0, \ldots, J+1$ on the interval $[a, b]$, where $(J+1)h = b - a$. Let $f(x)$ be a function defined on $[a, b]$. Consider the problem of finding a cubic spline approximation $s(x)$ to $f(x)$ that interpolates f at the points x_j, is twice continuously differentiable, and satisfies $s''(a) = s''(b) = 0$. Does this problem always have a solution? If your answer is yes, derive formulas by which to determine the spline. If your answer is no, explain your reasoning.

b) Let $I_n(f)$ denote the result of using the composite Trapezoidal rule to approximate $I(f) \equiv \int_a^b f(x)dx$ using n equally sized subintervals of length $h = (b-a)/n$. It can be shown that the integration error $E_n(f) \equiv I(f) - I_n(f)$ satisfies

$$E_n(f) = d_2 h^2 + d_4 h^4 + d_6 h^6 + \ldots$$

where $d_2, d_4, d_6, \ldots$ are numbers that depend only on the values of f and its derivatives at a and b. Suppose you have a black-box program that, given f, a, b, and n calculates $I_n(f)$. Show how to use this program to obtain an $O(h^4)$ approximation and an $O(h^6)$ approximation to $I(f)$.

7) Projectors:
Recall that a square matrix P that satisfies $P^2 = P$ is called a projector and that the subspaces range(P) and null(P) are complementary subspaces. Recall also that a projector P is an orthogonal projector if these two subspaces are perpendicular to one another, or equivalently, if P is Hermitian.
a) Let P be a nonzero projector. Show that $||P||_2 \geq 1$, with equality if and only if P is an orthogonal projector.

b) Given an $m \times n$ matrix A $(m \geq n)$ with rank r. Describe a stable algorithm for constructing an orthogonal projector onto range(A). Why do you think your algorithm is stable?

8) Eigenvalue Problems:

a) Let A be a real $n \times n$ matrix with simple eigenvalue λ, right eigenvector x (*i.e.*, $Ax = \lambda x$), and left eigenvector y (*i.e.*, $y^* A = \lambda y^*$). Assume that $\|x\|_2 = \|y\|_2 = 1$. Show that the condition number for λ to changes in A is given by $s(\lambda) = 1/|y^* x|$. Give an example of a matrix A which has a badly conditioned eigenvalue, and another matrix A which has only well-conditioned eigenvalues.

b) Consider the problem of computing an eigenvalue/eigenvector pair of a real symmetric $m \times m$ matrix A with eigenvalues of distinct magnitudes. One algorithm for trying to do this is Rayleigh Quotient Iteration:

Guess $v^{(0)}$ with $\|v^{(0)}\|_2 = 1$.
Set $\lambda^{(0)} = (v^{(0)})^T A v^{(0)}$.
For $k = 1, 2, 3, \ldots$
  Solve $(A - \lambda^{(k-1)} I)w = v^{(k-1)}$ for w
  $v^{(k)} = w/\|w\|_2$
  $\lambda^{(k)} = (v^{(k)})^T A v^{(k)}$
Until convergence.

Suppose that the Rayleigh Quotient Iteration iterates $\lambda^{(k)}$ converge to a simple eigenvalue λ of A. Analyze the speed of convergence of $\lambda^{(k)}$ to λ.

Preliminary Examination, Numerical Analysis, January 2005

Instructions: This exam is closed books and notes. Time: 3 hours. Answer 3 of questions 1-5 and answer both of questions 6 and 7. Indicate clearly which of your answers you wish to have graded. A score of 75% constitutes a pass. All questions are valued equally. On the last page of the exam are some facts that you may find useful.

1) Sensitivity Analysis:
In this problem, we consider a real, non-singular $n \times n$ matrix A and vectors $b,\ x \in \mathbf{R}^n$. We are concerned with the error in solving the linear system $Ax = b$. It turns out that the numerical solution $\hat{x}$ is only an approximate solution to this equation. However, it satisfies a perturbed equation $\hat{A}\hat{x} = b$ exactly, where $\hat{A} = A + \delta A$. Let the error be defined as $\delta x = \hat{x} - x$, we are interested in bounding $||\delta x||$ in terms of $||\delta A||||A||$, or $||\delta A||||\hat{A}||$, and $||x||$, or $||\hat{x}||$, where $||\cdot||$ is some suitable norm, and we proceed as follows:

a) First prove the identity

$$A^{-1} - \hat{A}^{-1} = A^{-1} \cdot \delta A \cdot \hat{A}^{-1},$$

and hence deduce that

$$||A^{-1} - \hat{A}^{-1}|| \leq ||A^{-1}|| \cdot ||\delta A|| \cdot ||\hat{A}^{-1}||.$$

b) Assuming that $\hat{\delta} = ||\delta A||||\hat{A}^{-1}|| < 1$, show that

$$||A^{-1}|| \leq \frac{1}{1-\hat{\delta}}||\hat{A}^{-1}||, \qquad ||A^{-1} - \hat{A}^{-1}|| \leq \frac{\hat{\delta}}{1-\hat{\delta}}||\hat{A}^{-1}||.$$

c) By comparing the equations $Ax = b$ and $(A + \delta A)(x + \delta x) = b$, show that

$$||\delta x|| \leq \frac{\hat{\delta}}{1-\hat{\delta}}||\hat{x}||.$$

Similarly, show that if $\delta = ||\delta A||||A^{-1}|| < 1$,

$$||\delta x|| \leq \frac{\delta}{1-\delta}||x||.$$

2) Linear Least Squares:
The Linear Least Squares problem for an $m \times n$ real matrix A and $b \in \mathbf{R}^m$ is the problem:

$$\text{Find} \quad x \in \mathbf{R}^n \text{ such that } \|Ax - b\|_2 \text{ is minimized.}$$

a) Suppose that you have data $\{(t_j, y_j)\}$, $j = 1, 2, \ldots, m$ that you wish to approximate by an expansion

$$p(t) = \sum_{k=1}^{n} x_k \phi_k(t).$$

Here, the functions $\phi_k(t)$ are given functions. Which norm on the difference between the approximation function p and the data gives rise to a linear least squares problem for the unknown expansion coefficients x_k? What is the matrix A in this case, and what is the vector b?

b) Suppose that A is a real $m \times n$ matrix of full rank and let $b \in \mathbf{R}^m$. Show that the Least Squares problem has a unique solution and describe the *geometry* of the Least Squares problem (in terms of vectors and/or subspaces of $\mathbf{R}^m$ and/or $\mathbf{R}^n$).

c) For the matrix A and vector b of part (b), answer the following questions: What are the 'normal equations' for the Least Squares problem? How can they be used to solve the Least Squares problem? What is the QR factorization of A and how can it be used to solve the Least Squares problem? Compare and contrast these approaches for numerically solving the Least Square problem.

3) Singular Value Decomposition:
The *singular value decomposition* (SVD) of a real $m \times n$ matrix A (you may assume $m \geq n$) is defined by

$$A = U \Sigma V^T$$

where U is $m \times m$ and orthogonal, Σ is $m \times n$ and diagonal, V is $n \times n$ and orthogonal.

a) Prove that every matrix A has an SVD.

b) Use the SVD of A to prove the famous result from linear algebra that the null space of A is the orthogonal complement of the range of A^T (the adjoint of A).

c) For a nonsingular $m \times m$ matrix A, the solution to the system $Ax = b$ can be expressed in terms of A's SVD. Write $b = \sum_{j=1}^{m} c_j u_j$ where u_j is the j^{th} column of U. Then the solution x is

$$x = \sum_{j=1}^{m} \frac{c_j}{\sigma_j} v_j$$

where v_j is the j^{th} column of V and σ_j is the j^{th} singular value. Verify this formula. Consider the special case $b = u_1$. What perturbation δb in b, with $\|\delta b\|_2 = 1$, causes the largest change δx in the solution x?

4) Eigenvalue Problems:

a) Let A be a real $n \times n$ matrix with simple eigenvalue λ, right eigenvector x (*i.e.*, $Ax = \lambda x$), and left eigenvector y (*i.e.*, $y^*A = \lambda y^*$). Assume that $\|x\|_2 = \|y\|_2 = 1$. Show that the condition number for λ to changes in A is given by $s(\lambda) = 1/|y^*x|$. Give an example of a matrix A which has a badly conditioned eigenvalue, and another matrix A which has only well-conditioned eigenvalues.

b) Consider the problem of computing an eigenvalue/eigenvector pair of a real symmetric $m \times m$ matrix A with eigenvalues of distinct magnitudes. One algorithm for trying to do this is Rayleigh Quotient Iteration:

Guess $v^{(0)}$ with $\|v^{(0)}\|_2 = 1$.
Set $\lambda^{(0)} = (v^{(0)})^T A v^{(0)}$.
For $k = 1, 2, 3, \ldots$
  Solve $(A - \lambda^{(k-1)} I)w = v^{(k-1)}$ for w
  $v^{(k)} = w/\|w\|_2$
  $\lambda^{(k)} = (v^{(k)})^T A v^{(k)}$
End

Suppose that the Rayleigh Quotient Iteration iterates $\lambda^{(k)}$ converge to a simple eigenvalue λ of A. Analyze the speed of convergence of $\lambda^{(k)}$ to λ.

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5) Iterative Methods for Linear Systems:
Consider the system $Ax = b$, in which A is a symmetric positive definite real $m \times m$ matrix, and denote by x_* its true solution. The conjugate gradient (CG) algorithm for finding the solution to the system is:

Set $x_0 = 0$, $r_0 = b$, and $p_0 = r_0$.
For $n = 1, 2, 3, \dots$
$$\begin{aligned} \alpha_n &= r_{n-1}^T r_{n-1} / p_{n-1}^T A p_{n-1} \\ x_n &= x_{n-1} + \alpha_n p_{n-1} \\ r_n &= r_{n-1} - \alpha_n A p_{n-1} \\ \beta_n &= r_n^T r_n / r_{n-1}^T r_{n-1} \\ p_n &= r_{n-1} + \beta_n p_{n-1} \end{aligned}$$
End

The vectors generated by this algorithm have the property that, provided $r_{n-1} \neq 0$.
$$\begin{aligned} \mathcal{K}_n \equiv \; & \text{span}\{b, Ab, A^2b, \dots, A^{n-1}b\} & = \text{span}\{x_1, x_2, \dots, x_n\} \\ = \; & \text{span}\{r_0, r_1, \dots, r_{n-1}\} & = \text{span}\{p_0, p_1, \dots, p_{n-1}\} \end{aligned}$$
and $r_n^T r_j = 0$ and $p_n^T A p_j = 0$ for all $j < n$.

Let $e_n = \|x_n - x_*\|_A$, where the A-norm of a vector x is defined by $\|x\|_A = (x^T A x)^{1/2}$.

a) Prove the following:

Suppose the CG algorithm is applied to the symmetric positive definite system $Ax = b$ and that it has not yet converged ($r_{n-1} \neq 0$). Then $e_n < \|y - x_*\|_A$ for all $y \in \mathcal{K}_n$, $y \neq x_n$; $\|e_n\|_A \leq \|e_{n-1}\|_A$; and $e_N = 0$ for some $N \leq m$.

b) In practice this result is not very useful. Why is this, and what determines the speed of convergence of x_n to x_*? What is 'preconditioning' and how is it related to the speed of convergence of the CG algorithm?

6) Numerical ODEs:

a) The difference equation

$$y_{n+k} + a_1 y_{n+k-1} + \ldots + a_k y_n = 0$$

can, after the substitution $\mathbf{y}_j = \left(y_j, y_{j+1}, \ldots, y_{j+k-1}\right)^\mathsf{T}$, be written in the form $\mathbf{y}_{n+1} = A\mathbf{y}_n$. What does the matrix A look like? Describe the condition for $\mathbf{y}$ to remain bounded in terms of A.

b) The differential equation

$$y^{(k)} + a_1 y^{(k-1)} + \ldots + a_k y = 0$$

can, after the substitution $\mathbf{y} = \left(y, y', \ldots, y^{(k-1)}\right)^\mathsf{T}$, be written in the form $\mathbf{y}' = A\mathbf{y}$. What does the matrix A look like? Describe the condition for $\mathbf{y}$ to remain bounded in terms of A.

c) Show that all solutions of the difference equation

$$y_{n+1} - 2\lambda y_n + y_{n-1} = 0$$

are bounded, as $n \to \infty$, if λ is real and $|\lambda| < 1$.

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7) Finite-Difference Methods: Consider the initial boundary value problem

$$v_t = \beta v_{xx}, \quad 0 < x < 1, \quad 0 < t, \quad v(x,0) = f(x)$$

with Dirichlet boundary conditions

$$v(0,t) = 0, \quad v(1,t) = 0.$$

Suppose that the interval $[0,1]$ is partitioned by meshpoints $x_j = jh$ for $j = 0, 1, \ldots, N$ and $h = 1/N$. A possible discretization of the above problem uses the Crank-Nicolson scheme for the differential equation:

$$(I - \frac{\beta k}{2} D_+ D_-) u_j^{n+1} = (I + \frac{\beta k}{2} D_+ D_-) u_j^n$$

for $j = 1, \ldots, N-1$, combined with the boundary conditions:

$$u_0^n = 0 \quad \text{and} \quad u_N^n = 0$$

for each time level n. Initial data is obtained from by setting $u_j^0 = f_j$ for $j = 0, 1, \ldots, N$. (Note that D_+ and D_- are the usual forward and backward difference operators.)
Set up the matrix system that must be solved in order to advance the numerical solution from time level n to time level $n+1$. Show that the suggested discretization is stable in the l_2 norm

$$\|u\|_2 = \left(\sum_{j=0}^{N} h |u_j|^2 \right)^{1/2}$$

for *all* choices of timestep k and spacestep h. (Note that von Neumann analysis is not enough to prove stability for the initial **boundary** value problem.) What other issues are important to consider in deciding whether this scheme is a good choice for the problem?

Fact 1: A real symmetric $n \times n$ matrix A can be diagonalized by an orthogonal similarity transformation, and A's eigenvalues are real.

Fact 2: The $(N-1) \times (N-1)$ matrix M defined by

```
[ -2  1  0  0  0  .  .  .  0  0  0  0 ]
[  1 -2  1  0  0  .  .  .  0  0  0  0 ]
[  0  1 -2  1  0  .  .  .  0  0  0  0 ]
[  0  0  1 -2  1  .  .  .  0  0  0  0 ]
[  .  .  .  .  .  .  .  .  .  .  .  . ]
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[  .  .  .  .  .  .  .  .  .  .  .  . ]
[  0  0  0  0  0  .  .  .  1 -2  1  0 ]
[  0  0  0  0  0  .  .  .  0  1 -2  1 ]
[  0  0  0  0  0  .  .  .  0  0  1 -2 ]
```

has eigenvalues $\mu_l \; = \; -4\sin^2(\frac{\pi l}{2N})$, $l = 1, 2, \ldots, N-1$.

Fact 3: The $(N+1) \times (N+1)$ matrix:

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[ -1  1  0  0  0  .  .  .  0  0  0  0 ]
[  1 -2  1  0  0  .  .  .  0  0  0  0 ]
[  0  1 -2  1  0  .  .  .  0  0  0  0 ]
[  0  0  1 -2  1  .  .  .  0  0  0  0 ]
[  .  .  .  .  .  .  .  .  .  .  .  . ]
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has eigenvalues $\mu_l = -4\sin^2\left(\frac{\pi l}{2(N+1)}\right), \quad l = 0, 1, \ldots, N.$

Fact 4: For a real $n \times n$ matrix A, the Rayleigh quotient of a vector $x \in R^n$ is the scalar

$$r(x) = \frac{x^T A x}{x^T x}.$$

The gradient of $r(x)$ is

$$\nabla r(x) = \frac{2}{x^T x}(Ax - r(x)x).$$

If x is an eigenvector of A then $r(x)$ is the corresponding eigenvalue and $\nabla r(x) = 0$.

Preliminary Examination, Numerical Analysis, Fall 2004

Instructions: This exam is closed books and notes. The time allowed is three hours and you need to work on any two out of questions 1-3 and any other three out of questions 4-7. All questions have equal weights and a score of 75% is considered a pass. Indicate clearly the work that you wish to be graded.

1. **(Sensitivity Analysis, 20%)** In this problem, we consider a real, non-singular $n \times n$ matrix A and vectors $f,\ b,\ x \in \mathbb{R}^n$.

   (a) (6%) First we consider the conditioning of matrix-vector multiplication $f = Ax$, that is, find the smallest factor $\kappa > 0$ such that when x is perturbed by any small δx, the perturbation for f satisfies

   $$\frac{||\delta f||}{||f||} \leq \kappa \frac{||\delta x||}{||x||}.$$

   Here we assume no perturbation in matrix A.

   (b) (8%) For the linear system $Ax = b$, a numerical solution $\hat{x}$ actually solves a slightly different problem due to roundoff errors:

   $$(A + \delta A)\hat{x} = b + \delta b.$$

   If we denote $\delta x = \hat{x} - x$, derive an upper bound for the relative error

   $$E = \frac{||\delta x||}{||x||}$$

   in terms of the condition number of A, $\kappa(A)$, and the norms of $A, \delta A, b$ and δb, where $||\cdot||$ is a suitable vector norm.

   (c) (6%) In practice, x is not available so $\hat{x}$ is often used in the above bound to generate an approximate bound for δx. Comment on this practice as what the advantages and disadvantages are, point out cases where this approximation may be inappropriate.

2. **(LS Problems and SVD, 20%)** Let A be a real $m \times n$ matrix ($m > n$) of full rank and let $b \in \mathbb{R}^m$, the least square problem is to find $x \in \mathbb{R}^n$ such that

   $$||Ax - b||_2$$

   is minimized.

   (a) (13%) Describe the following three approaches to solve the problem: 1) normal equations, 2) QR decomposition, and 3) singular value decomposition (SVD). Discuss briefly the advantages and disadvantages of each approach. In particular, give a geometric interpretation of the normal equation approach, and explain why it is often to be avoided.

(b) (7%) In addition to minimizing the above norm, we impose the constraint: $||x||_2 \leq \beta$, for some constant $\beta > 0$. This is a constrained LS problem. Use SVD to formulate and solve this problem, notice that there are different cases where a Lagrange multiplier may be needed.

3. **(Eigenvalue Problems, 20%)**

(a) (8%) QR iteration can be viewed as an extension of the power method. We assume $n \times n$ real symmetric, nonsingular matrix A, with distinct eigenvalues. Here is the algorithm of orthogonal iteration for two linearly independent vectors z_1 and z_2, listed in an $n \times 2$ matrix $Z_0 = [z_1, z_2]$:

$i = 0$

repeat

$Y_{i+1} = AZ_i$

QR *Factor* $Y_{i+1} = Z_{i+1}R_{i+1}$

$i = i + 1$

until convergence

Notice that Y_i, Z_i are $n \times 2$ matrices, and R_i is a 2×2 upper triangular matrix. Our assumptions allow us to write $A = S\Lambda S^{-1}$, where Λ is a diagonal matrix containing all eigenvalues $|\lambda_1| > |\lambda_2| > ... > |\lambda_n|$, S=$[s_1, s_2, ..., s_n]$ contains all the eigenvectors.

Show that span$\{Z_{i+1}\}$=span$\{Y_{i+1}\}$=span$\{AZ_i\}$, so span$\{Z_i\}$=span$\{A^i Z_0\}$= span$\{S\Lambda^i S^{-1} Z_0\}$. Analyze it by writing

$$S\Lambda^i S^{-1} Z_0 = \lambda_2^i S D_i (S^{-1} Z_0).$$

Derive the diagonal D_i and the $n \times 2$ matrix $Z_0' = S^{-1}Z_0$, find the limiting span of Z_i as $i \to \infty$. How do we obtain the approximating eigenvalues λ_1, λ_2 and approximating eigenvectors s_1 and s_2, once iterations are terminated?

(b) (6%) Assume that we can continue this extension to n linearly independent vectors, contained in an $n \times n$ nonsingular Z_0, in particular when $Z_0 = I$, we will have the method of QR iteration. Describe the QR algorithm (without shift) and argue for the rate of convergence based on the analysis in part (a).

(c) (6%) To speed up convergence, we should use shifting. Describe the QR iteration with a shift, and make a suggestion as what shift should be used. How fast is the convergence?

4. **(Numerical ODEs, 20%)**

(a) (8%) Consider the general system of ODEs for $y(x) \in \mathbb{R}^n$, $x \in \mathbb{R}$, and $f = [f_1(x, y), ..., f_n(x, y)]^T$:

$$y' = f(x, y),$$

where we assume f is continuously differentiable in both x and y. To study local stability, we compare two solutions $y_1(x)$ and $y_2(x)$ where $y_1(x_0) \approx y_2(x_0)$. Derive the linearized system for $e(x) = y_1(x) - y_2(x)$ near $x = x_0$ to show that

$$e' \approx Ae$$

and find this square matrix A. What is the most important property of A we need to find that will affect the choice of numerical schemes and step sizes?

(b) (12%) Now consider the initial value problem

$$y'' + 2\mu y' + (1 + \mu^2)y = 0,$$

$$y(0) = 0, \ y'(0) = 1,$$

where $\mu > 0$ is a given constant. If we use a simple explicit Euler's method, determine the maximum step size h that is allowed for the solution to be stable.

5. **(Iterative Methods for Linear Systems, 20%)**

(a) (6%) Consider the linear system $Ax = b$ and a splitting of $A : A = M - K$, with M being nonsingular, an iterative method is to iterate

$$x_{m+1} = Rx_m + c,$$

where $R = M^{-1}K$ and $c = M^{-1}b$. Show that the method converges for any x_0 if $||R|| < 1$ where $|| \cdot ||$ is any operator norm.

(b) (14%) To solve the boundary value problem

$$-u''(x) = f(x), \quad \text{for } 0 < x < 1,$$

with $u'(0) = 0$, and $u(1) = 0$, we use second-order central differences for the discretization of the equation at mesh points $x_j = j/N$:

$$-u_{j-1} + 2u_j - u_{j+1} = f_j = \frac{f(x_j)}{N^2}, \quad j = 0, 1, ..., N-1$$

and the central difference approximation for u' at $x_0 = 0 : \ u_1 - u_{-1} = 0$, to eliminate the ghost point.

Show that the finite difference equations and the boundary conditions lead to the following linear system $Au = b$, where

$$A = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \cdot & \cdot & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad u = \begin{pmatrix} u_0 \\ u_1 \\ \cdot \\ u_{N-2} \\ u_{N-1} \end{pmatrix}, \quad b = \frac{1}{N^2}\begin{pmatrix} \frac{1}{2}f_0 \\ f_1 \\ \cdot \\ f_{N-2} \\ f_{N-1} \end{pmatrix}.$$

Notice that equations are scaled so that A is symmetric. Determine if A has eigenvectors

$$v^{(k)} = \begin{bmatrix} 1 \\ \cos \frac{1}{2}k\pi x_1 \\ \cdot \\ \cdot \\ \cos \frac{1}{2}k\pi x_j \\ \cdot \\ \cdot \\ \cos \frac{1}{2}k\pi x_{N-2} \\ \cos \frac{1}{2}k\pi x_{N-1} \end{bmatrix}, \qquad k = 1, 3, 5, ..., 2N-3, 2N-1$$

and use them to find the eigenvalues of the iteration matrix R for the Jacobi method. Analyze the convergence properties of the Jacobi method applied to this problem, and express the convergence speed as a function of N. How does the number of iterations required to reduce the initial error by a factor of δ depend on N? In reality, should you use an iterative method to solve this matrix problem? If not, what other method is preferred?

6. **(Wave Equations, 20%)** Consider the standard wave equation in one space dimension:

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < 1, \quad t > 0,$$

with initial conditions

$$u(x,0) = f(x), \ u_t(x,0) = g(x), \quad 0 < x < 1$$

for given smooth functions f and g, and boundary condition

$$u(0,t) = u(1,t) = 0, \quad t > 0.$$

The leap-frog scheme takes advantage of the fact that the equation can be reformulated as a system of first order equations

$$u_t + cv_x = 0,$$
$$v_t + cu_x = 0.$$

Denote $x_j = j\Delta x$, $U_j^n \approx u(x_j, n\Delta t)$, and $V_{j+1/2}^{n+1/2} \approx v(x_{j+1/2}, (n+\frac{1}{2})\Delta t)$, use central difference approximations for both derivatives to derive the leap-frog scheme for the system of equations. How do you implement the initial and boundary conditions for both u and v? Derive the truncation error for this approximation and determine the order of truncation error. For stability, consider solutions in the form

$$\begin{bmatrix} U^n \\ V^{n+1/2} \end{bmatrix} = \lambda^n e^{ikx} \begin{bmatrix} \tilde{U} \\ \tilde{V} \end{bmatrix},$$

for some constants $\tilde{U}$ and $\tilde{V}$. Substitute this solution into the finite difference equations, find out what condition λ has to satisfy in order for $\tilde{U}$ and $\tilde{V}$ to have nontrivial solutions. For those λ values, Compute $|\lambda|$ to see if the stability condition $|\lambda| \leq 1$ is satisfied.

7. **(Convection-Diffusion Equation, 20%)** For the convection-diffusion equation

$$u_t + au_x = bu_{xx},$$

where $a > 0$ and $b > 0$, a natural Crank-Nicolson scheme is

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{a}{2}\left(\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} + \frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x}\right)$$
$$= \frac{b}{2}\left(\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2} + \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{(\Delta x)^2}\right).$$

Denote $\nu = b\Delta t/(\Delta x)^2$ and $\mu = a\Delta t/(2\Delta x)$, show that the scheme is stable in the max norm if $\mu \leq \nu \leq 1$. Also use Fourier analysis to derive the stability condition in the l_2 norm and show that the scheme is always stable in the l_2 norm. Suppose u stands for some chemical concentration, which norm is more appropriate? Explain why this scheme is so restrictive that it is not often used.

Numerical Analysis Prelim, January 2004

Answer any five of the seven questions below. Clearly indicate which questions you want to be counted towards your score. 80% constitute a passing score. Time for this exam is three hours. Do not use books, notes, or a calculator.

-1- (Householder Reflections.) A Householder Reflection H is an $n \times n$ orthogonal matrix of the form

$$H = I - 2uu^T \tag{1}$$

where

$$||u||_2 = 1 \tag{2}$$

and I is the $n \times n$ identity matrix. Let $a \in \mathbb{R}^n$ be a given (non-zero) vector. Show how to pick u in (1) such that

$$Ha = ||a||_2 e_1 \tag{3}$$

where e_1 is the first unit vector. Describe how to use this technique to compute the QR factorization of an $n \times n$ matrix A.

-2- (Frobenius Norm.) The Frobenius Norm $||A||_F$ of an $n \times n$ matrix A is defined by

$$||A||_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2}. \tag{4}$$

Let $H = I - 2uu^T$ be a Householder reflection as in the preceding problem and compute the Frobenius Norm of H.

-3- (Gaussian Quadrature.) Consider the quadrature formula

$$\int_a^b w(x) f(x) \mathrm{d}x = \sum_{i=1}^{n} \alpha_i f(\xi_i) + E. \tag{5}$$

w is a given positive weight function. How do you pick the knots ξ_i and the weights α_i such that the formula is exact (i.e., $E = 0$) for all polynomials of degree up to $2n - 1$? Show that your choice works and that you can find suitable weights and knots for all intervals $[a, b]$ and positive weight functions w.

-4- (Bernstein Bézier Form.) Let T be a non-degenerate triangle with vertices V_1, V_2, and V_3. The barycentric coordinates b_1, b_2, and b_3 of a point $x \in \mathbb{R}^2$ with respect to T are defined by

$$x = \sum_{i=1}^{3} b_i V_i \quad \text{where} \quad \sum_{i=1}^{3} b_i = 1. \tag{6}$$

A polynomial p of degree d can be written uniquely in its Bernstein-Bézier form as

$$p(x) = \sum_{i+j+k=d} \frac{d!}{i!j!k!} c_{ijk} b_1^i b_2^j b_3^k \tag{7}$$

Assuming you know the coefficients c_{ijk} of p as a polynomial of degree d, show how to write p as a polynomial of degree $d+1$ in Bernstein-Bézier form. What are the coefficents of p if $p(x) = 1$?

-5- (Sensitivity Analysis.) Consider the (square, non-singular) linear system

$$Ax = b. \tag{8}$$

Suppose you compute (by any method) a numerical solution $\hat{x}$ and you are concerned about the relative error

$$E = \frac{||x - \hat{x}||}{||x||} \tag{9}$$

where $||.||$ denotes a suitable vector norm. Derive upper and lower bounds on E in terms of the residual $r = b - A\hat{x}$ and the condition number of A with respect to $||\cdot||$.

-6- (ODEs.) Consider the initial value problem of ODEs:

$$y' = f(x, y), \quad y(a) = y_0. \tag{10}$$

Describe the Trapezoidal Rule applied to this problem and analyze its local accuracy and its region of absolute stability.

-7- (PDEs.) Consider the one-dimensional Heat Equation

$$u_t = u_{xx}, \quad t \geq 0, \quad x \in [0, 1], \quad u(x, 0) = f(x), \quad u(0, t) = u(1, t) = 0. \tag{11}$$

Describe how to discretize this problem by applying the Method of Lines and the Backward Euler Method. Analyze the stability and local accuracy of your method.

Prelims Numerical Analysis, August 2003

Answer any 9 of the 10 questions below. Clearly indicate which question you do not want to be counted. 80% constitute a passing score. Time for this exam is three hours. It is closed books and notes.

-1- (Newton's Method.) Describe how (the two-variable version of) Newton's Method can be used to solve the nonlinear system

$$\begin{aligned} x^2 + y^2 &= 1 \\ e^x - y &= 0 \end{aligned} \tag{1}$$

Discuss the selection of a starting point and the advantages and disadvantages of Newton's method.

-2- (The QR factorization.) Describe the QR factorization of an $m \times n$ matrix A, how to compute it, and what it can be used for.

-3- (The QR Algorithm.) Outline the basic ingredients of the QR algorithm for solving eigenvalue problems.

-4- (The Singular Value Decomposition.) Define the term "singular value decomposition" and show that every $m \times n$ matrix A has a singular value decomposition.

-5- (The Jacobi Method.) Let A be a strictly diagonally dominant $n \times n$ matrix, i.e.,

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, \quad i = 1, \ldots, n. \tag{2}$$

Describe the Jacobi method applied to the linear system

$$Ax = b \tag{3}$$

and prove that it converges for all initial approximations of the solution.

-6- (Spline Spaces.) Let T be a triangulation of a domain Ω, and S the linear space of functions on T that are continuous on Ω and that on each triangle in T can be written as a (bivariate) polynomial of degree d. Derive the dimension of S.

-7- (Numerical ODEs.) Define the terms linear multistep method (LMM), order, consistency, zero-stability, local truncation error, and region of absolute stability of an LMM. What is the highest order implicit 1 step method and discuss its region of absolute stability?

-8- (Numerical PDEs.) Consider the Initial-Boundary Value Problem

$$u_t = u_{xx}, \quad x \in [0,1], \quad t \geq 0, \quad \text{u given for } x = 0,\ x = 1, \text{ and } t = 0 \tag{4}$$

Describe the Crank-Nicolson Method, and analyze its local truncation error and stability.

-9- (Ill Conditioning.) Define the condition number of a square matrix A and its significance for the solution of the linear system

$$Ax = b. \tag{5}$$

Can you define a similar number that describes the conditioning of the eigenvalue problem

$$Ax = \lambda x? \tag{6}$$

-10- (Multigrid.) Describe the basic ideas of the multigrid technique for the solution of elliptic PDEs.

Preliminary Examination, Numerical Analysis, Fall 2002

Instructions: This exam is closed books and notes. The time allowed is 3 hours and you need to work on any 2 out of questions 1-3 and any other 3 out of questions 4-7. All questions have equal weights and a score of 75 % is considered a pass. Indicate clearly the work that you wish to be graded.

1. **(Sensitivity Analysis.)** For the linear system $Ax = b$ where A is a square, real non-singular matrix, a numerical solution $\hat{x}$ actually solves a slightly different problem due to roundoff errors:

$$(A + \delta A)\hat{x} = b + \delta b.$$

   (a) If we denote $\delta x = \hat{x} - x$, derive an upper bound for the relative error

$$E_1 = \frac{||\delta x||}{||\hat{x}||}$$

   in terms of the condition number of A, $\kappa(A)$, and the norms of $A, \hat{x}, \delta A, \delta b$, where $||\cdot||$ is a suitable vector norm.

   (b) We can define another relative error

$$E_2 = \frac{||\delta x||}{||x||},$$

   based on the exact solution. Derive an upper bound for E_2 in terms of $\kappa(A)$ and the norms of $\delta A, A, \delta b$ and b.

   (c) Strictly speaking, the relative error is given by E_2, even though in many applications we simply use E_1 since it is more readily available. Point out cases where this approximation may not be appropriate. Explain the significance of the condition number $\kappa(A)$, and express the relative distance of A to its nearest singular matrix in terms of $\kappa(A)$.

2. **(Linear Least Squares and Singular Value Decomposition.)** Let A be a real $m \times n$ matrix ($m > n$) of full rank and let $b \in \mathbf{R}^m$, the least square problem is to find $x \in \mathbf{R}^n$ such that

$$||Ax - b||_2$$

   is minimized.

   Describe the following three approaches to solve the problem: 1) normal equations, 2) QR decomposition, and 3) singular value decomposition (SVD). Discuss briefly the advantages and disadvantages of each approach. In particular, give a geometric interpretation of the normal equation approach, and explain why it is often to be avoided. Characterize the difficulties in a rank-deficient least square problem and make a suggestion about which method you should use.

3. **(Eigenvalue Problems.)**

(a) For the general eigenvalue problem of a nonsingular matrix A, the simplest method is the power method, which converges to the largest eigenvalue in magnitude, assuming it is simple. The algorithm is the following:

$i = 0$

repeat

$$y_{i+1} = Ax_i$$
$$x_{i+1} = y_{i+1}/||y_{i+1}||_2$$
$$\tilde{\lambda}_{i+1} = x_{i+1}^T A x_{i+1}$$
$$i = i + 1$$

until convergence

The inverse iteration algorithm takes advantage of the fact the if σ (the shift) is closest to one of the simple eigenvalues λ, then $1/(\lambda - \sigma)$ will be the largest eigenvalue in magnitude of the matrix $(A - \sigma I)^{-1}$. Use this fact to construct the inverse iteration algorithm. Note that we will not need to construct the explicit inverse of $A - \sigma I$. How fast does it converge to the desired eigenvalue? When do you expect the convergence to be slow? How do we find the corresponding eigenvector?

(b) Now consider the symmetric eigenvalue problem where A is symmetric. What is the Rayleigh quotient? The Rayleigh quotient method is the inverse iteration algorithm described above applied to a symmetric matrix, when the shift is chosen to be the Rayleigh quotient ρ_i of A and the updated vector x_i. Explain the rationale of choosing ρ_i to be the shift and determine the speed of convergence of ρ_i to λ.

4. **(Numerical ODEs.)**

(a) Consider the general system of ODEs for $y(x) \in R^n$, $x \in R$ and $f = [f_1, ..., f_n]^T$:

$$y' = f(x, y),$$

where we assume f to be Lipschitz continuous. To study the numerical stability, we linearize f near $x = x_0, y = y_0$ to obtain:

$$y' \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + \nabla_y f(x_0, y_0)(y - y_0)$$

If we denote $A = \nabla_y f(x_0, y_0)$, then we claim that the local stability of the above system can be analyzed by studying the following constant coefficient system

$$y' = Ay.$$

Justify this claim by analyzing the simple scalar equation

$$y' = a + bx + cy,$$

where a, b, c are constants, in the explicit Euler's method, to show that the stability is determined by how we solve $y' = cy$. What is the most important property of A we need to find that will affect the choice of numerical schemes and step sizes?

(b) Now consider the initial value problem

$$y'' + 2\mu y' + (1+\mu^2)y = 0,$$

$$y(0) = 0, \; y'(0) = 1,$$

where $\mu > 0$ is a given constant. If we use a simple explicit Euler's method, determine the maximum step size h that is allowed for the solution to be stable.

5. **(Iterative Methods for Linear Systems.)**

(a) Consider the linear system $Ax = b$ and a splitting of $A : A = M - K$, with M being nonsingular, an iterative method is to iterate

$$x_{m+1} = Rx_m + c,$$

where $R = M^{-1}K$ and $c = M^{-1}b$. Show that the method converges for any x_0 if $||R|| < 1$ where $||\cdot||$ is any operator norm.

(b) To solve the boundary value problem

$$-u''(x) = f(x), \quad \text{for } 0 < x < 1,$$

with $u(0) = 0$, and $u'(1) = 0$, we use second-order central differences for the discretization at the mesh points $x_j = j/N$:

$$-u_{j-1} + 2u_j - u_{j+1} = f_j = \frac{f(x_j)}{N^2}, \quad j = 1, 2, ..., N$$

and the central difference for u' at x_N to eliminate the ghost point x_{N+1} : $u_{N+1} = u_{N-1}$.

Use all the finite difference equations and the boundary conditions to obtain the linear system $Au = b$, where

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \cdot & \cdot & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{bmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \cdot \\ u_{N-1} \\ u_N \end{pmatrix}, \quad b = \frac{1}{N^2}\begin{pmatrix} f_1 \\ f_2 \\ \cdot \\ f_{N-1} \\ f_N \end{pmatrix}.$$

Verify that A has eigenvectors

$$v^{(k)} = \begin{bmatrix} \sin \frac{k\pi}{2} x_1 \\ \sin \frac{k\pi}{2} x_2 \\ \cdot \\ \cdot \\ \sin \frac{k\pi}{2} x_j \\ \cdot \\ \cdot \\ \sin \frac{k\pi}{2} x_{N-1} \\ \sin \frac{k\pi}{2} x_N \end{bmatrix}, \qquad k = 1, 3, 5, ..., 2N-3, 2N-1,$$

and use them to find the eigenvalues of the iteration matrix R for the Jacobi method. Analyze the convergence properties of the Jacobi method applied to this problem, and express the convergence speed as a function of N. How does the number of iterations required to reduce the initial error by a factor of δ depend on N?

6. **(Wave Equations.)** Consider the first-order wave equation in one space dimension:

$$u_t + au_x = 0, \quad -\infty < x < \infty, \quad t > 0,$$

where a is a constant, with the initial condition

$$u(x, 0) = u_0(x), \quad -\infty < x < \infty$$

for a given function u_0. We use the usual discretization

$$U_j^n \approx u(x_j, t^n),$$

where $x_j = j\Delta x$ and $t^n = n\Delta t$. For a finite difference scheme to solve this problem, what is the CFL condition? Give an intuitive explanation for the condition. Derive the first-order upwind scheme for this equation and its truncation error. For numerical stability, derive the conditions for stability in the maximum norm (maximum principle) and the l_2 norm (Fourier analysis).

7. **(Heat Equation.)** Derive the Crank-Nicolson scheme for the following problem

$$u_t = u_{xx}, \quad t > 0, \quad 0 < x < 1,$$

with the boundary conditions

$$u(0, t) = u(1, t) = 0, \quad t \geq 0,$$

and the initial data

$$u(x, 0) = u_0(x), \quad 0 \leq x \leq 1,$$

for some given function u_0. Use the usual discretization with uniform mesh points $x_j = jh$ where $h = 1/N$ and $j = 0, 1, 2, ..., N$. Show that the method is second order by deriving the truncation error. What other conditions do we need to guarantee the convergence of the numerical solution? Derive the stability condition in l_2 norm by using Fourier analysis. For any value of $\nu = \Delta t/(\Delta x)^2$, do we always have the maximum principle satisfied? If we have some initial data such that $0 \leq u_0(x) \leq 1$, derive the condition that should be imposed on ν so that $0 \leq u_j^n \leq 1$ also holds for all j and $n > 0$.

Preliminary Examination, Numerical Analysis, Fall 2001

Instructions: This exam is closed books and notes. Time: 3 hours. Answer 5 of the following 7 questions; at least 1 of the questions you choose must be from questions 6 and 7. Indicate clearly which of your answers you wish to have graded. A score of 70% constitutes a pass. All questions have equal weight. On the last page of the exam are some facts that you may find useful.

1) Linear Least Squares: The Linear Least Squares problem for an $m \times n$ real matrix A and $b \in \mathbf{R}^m$ is the problem:

$$\text{Find} \quad x \in \mathbf{R}^n \text{ such that } \|Ax - b\|_2 \text{ is minimized.}$$

a) Suppose that you have data $\{(t_j, y_j)\}$, $j = 1, 2, \ldots, m$ that you wish to approximate by an expansion

$$p(t) = \sum_{k=1}^{n} x_k \phi_k(t).$$

Here, the functions $\phi_k(t)$ are given functions. Which norm on the difference between the approximation function p and the data gives rise to a linear least squares problem for the unknown expansion coefficients x_k? What is the matrix A in this case, and what is the vector b?
b) Suppose that A is a real $m \times n$ matrix of full rank and let $b \in \mathbf{R}^m$. Show that the Least Squares problem has a unique solution and describe the *geometry* of the Least Squares problem (in terms of vectors and/or subspaces of $\mathbf{R}^m$ and/or $\mathbf{R}^n$).
c) For the matrix A and vector b of part (b), answer the following questions: What are the 'normal equations' for the Least Squares problem? How can they be used to solve the Least Squares problem? What is the QR factorization of A and how can it be used to solve the Least Squares problem? Compare and contrast these approaches for numerically solving the Least Square problem.

2) Singular Value Decomposition: The *singular value decomposition* (SVD) of a real $m \times n$ matrix A (you may assume $m \geq n$) is defined by

$$A = U \Sigma V^T$$

where U is $m \times m$ and orthogonal, Σ is $m \times n$ and diagonal, V is $n \times n$ and orthogonal.

a) Prove that every matrix A has an SVD.
b) Use the SVD of A to prove the famous result from linear algebra that the null space of A is the orthogonal complement of the range of A^T (the adjoint of A).
c) For a nonsingular $m \times m$ matrix A, the solution to the system $Ax = b$ can be expressed in terms of A's SVD. Write $b = \sum_{j=1}^{m} c_j u_j$ where u_j is the j^{th} column of U. Then the solution x is

$$x = \sum_{j=1}^{m} \frac{c_j}{\sigma_j} v_j$$

where v_j is the j^{th} column of V and σ_j is the j^{th} singular value. Verify this formula. Consider the special case $b = u_1$. What perturbation δb in b, with $\|\delta b\|_2 = 1$, causes the largest change δx in the solution x?

3) Iterative Methods for Linear Systems: Consider the fixed-point iteration

$$u^{(k+1)} = Tu^{(k)} + c$$

for finding a solution of the problem

$$u = Tu + c$$

where T is an $m \times m$ real matrix and c is a real m-vector.
a) Show that the fixed point iteration will converge for an arbitrary initial guess $u^{(0)}$ if and only if the spectral radius of T, $\rho(T)$, is less than 1.
b) Consider the boundary value problem

$$-u''(x) = f(x), \quad \text{for } 0 \leq x \leq 1$$

with $u(0) = u(1) = 0$, and the following discretization of it:

$$-U_{j-1} + 2\, U_j - U_{j+1} = F_j$$

for $j = 1, 2, \ldots, N-1$ where $Nh = 1$ and $F_j \equiv h^2\, f(jh)$.
Analyze the convergence properties of the Jacobi iterative method for this problem. In particular, express the speed of convergence as a function of the discretization stepsize h. How does the number of iterations required to reduce the initial error by a factor δ depend on h? In practice, would you use this method to solve the given problem? If so, explain why this is a good idea? If not, how would you solve it in practice?

4) Eigenvalue Problems:
a) Let A be a real $n \times n$ matrix with simple eigenvalue λ, right eigenvector x (*i.e.*, $Ax = \lambda x$), and left eigenvector y (*i.e.*, $y^* A = \lambda y^*$). Assume that $\|x\|_2 = \|y\|_2 = 1$. Show that the condition number for λ to changes in A is given by $s(\lambda) = 1/|y^* x|$. Give an example of a matrix A which has a badly conditioned eigenvalue, and another matrix A which has only well-conditioned eigenvalues.
b) Consider the problem of computing an eigenvalue/eigenvector pair of a real symmetric $m \times m$ matrix A with eigenvalues of distinct magnitudes. One algorithm for trying to do this is Rayleigh Quotient Iteration:

Guess $v^{(0)}$ with $\|v^{(0)}\|_2 = 1$.
Set $\lambda^{(0)} = (v^{(0)})^T A v^{(0)}$.
For $k = 1, 2, 3, \ldots$
  Solve $(A - \lambda^{(k-1)} I) w = v^{(k-1)}$ for w
  $v^{(k)} = w / \|w\|_2$
  $\lambda^{(k)} = (v^{(k)})^T A v^{(k)}$
End

Suppose that the Rayleigh Quotient Iteration iterates $\lambda^{(k)}$ converge to a simple eigenvalue λ of A. Analyze the speed of convergence of $\lambda^{(k)}$ to λ.

5) Krylov Subspace Methods: Consider the system $Ax = b$, in which A is a symmetric positive definite real $m \times m$ matrix, and denote by x_* its true solution. The conjugate gradient (CG) algorithm for finding the solution to the system is:

Set $x_0 = 0$, $r_0 = b$, and $p_0 = r_0$.
For $n = 1, 2, 3, \ldots$
$\alpha_n = r_{n-1}^T r_{n-1} / p_{n-1}^T A p_{n-1}$
$x_n = x_{n-1} + \alpha_n p_{n-1}$
$r_n = r_{n-1} - \alpha_n A p_{n-1}$
$\beta_n = r_n^T r_n / r_{n-1}^T r_{n-1}$
$p_n = r_{n-1} + \beta_n p_{n-1}$
End

The vectors generated by this algorithm have the property that, provided $r_{n-1} \neq 0$,

$$\begin{aligned} \mathcal{K}_n \equiv \; & \operatorname{span}\{b, Ab, A^2b, \ldots, A^{n-1}b\} = \operatorname{span}\{x_1, x_2, \ldots, x_n\} \\ = \; & \operatorname{span}\{r_0, r_1, \ldots, r_{n-1}\} = \operatorname{span}\{p_0, p_1, \ldots, p_{n-1}\} \end{aligned}$$

and $r_n^T r_j = 0$ and $p_n^T A p_j = 0$ for all $j < n$.

Let $e_n = \|x_n - x_*\|_A$, where the A-norm of a vector x is defined by $\|x\|_A = (x^T A x)^{1/2}$.

a) Prove the following:

Suppose the CG algorithm is applied to the symmetric positive definite system $Ax = b$ and that it has not yet converged ($r_{n-1} \neq 0$). Then $e_n < \|y - x_*\|_A$ for all $y \in \mathcal{K}_n$, $y \neq x_n$; $\|e_n\|_A \leq \|e_{n-1}\|_A$; and $e_N = 0$ for some $N \leq m$.

b) In practice this result is not very useful. Why is this, and what determines the speed of convergence of x_n to x_*? What is 'preconditioning' and how is it related to the speed of convergence of the CG algorithm?

6) Numerical Methods for ODEs: Consider the Linear Multistep Method

$$y_{n+2} - \frac{4}{3} y_{n+1} + \frac{1}{3} y_n = \frac{2}{3} h f_{n+2}$$

for solving an initial value problem $y' = f(y, x)$, $y(0) = \eta$. You may assume that f is Lipschitz continuous with respect to y for all x.

a) Analyze the consistency, stability, accuracy, and convergence properties of this method.

b) Would it be more reasonable to use this method or Euler's method for the initial value problem:

$$y' = -10^8 [y - \cos(x)] - \sin(x), \quad y(0) = 1?$$

Justify your answer.

7) Finite-Difference Methods: Consider the initial boundary value problem

$$v_t = \beta v_{xx}, \quad 0 < x < 1, \quad 0 < t, \quad v(x, 0) = f(x)$$

with Dirichlet boundary conditions

$$v(0, t) = 0, \quad v(1, t) = 0.$$

Suppose that the interval $[0, 1]$ is partitioned by meshpoints $x_j = jh$ for $j = 0, 1, \ldots, N$ and $h = 1/N$. A possible discretization of the above problem uses the Crank-Nicolson scheme for the differential equation:

$$(I - \frac{\beta k}{2} D_+ D_-) u_j^{n+1} = (I + \frac{\beta k}{2} D_+ D_-) u_j^n$$

for $j = 1, \ldots, N-1$, combined with the boundary conditions:

$$u_0^n = 0 \text{ and } \mathrm{u}_\mathrm{N}^\mathrm{n} = 0$$

for each time level n. Initial data is obtained from by setting $u_j^0 = f_j$ for $j = 0, 1, \ldots, N$. (Note that D_+ and D_- are the usual forward and backward difference operators.)
Set up the matrix system that must be solved in order to advance the numerical solution from time level n to time level $n+1$. Show that the suggested discretization is stable in the l_2 norm

$$\|u\|_2 = \left(\sum_{j=0}^{N} h |u_j|^2 \right)^{1/2}$$

for *all* choices of timestep k and spacestep h. What other issues should be considered in deciding whether the suggested scheme is a good choice for the problem?

Fact 1: A real symmetric $n \times n$ matrix A can be diagonalized by an orthogonal similarity transformation, and A's eigenvalues are real.

Fact 2: The $(N-1) \times (N-1)$ matrix M defined by

```
[ -2  1  0  0  0  .  .  .  0  0  0  0 ]
[  1 -2  1  0  0  .  .  .  0  0  0  0 ]
[  0  1 -2  1  0  .  .  .  0  0  0  0 ]
[  0  0  1 -2  1  .  .  .  0  0  0  0 ]
[  .  .  .  .  .  .  .  .  .  .  .  . ]
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[  .  .  .  .  .  .  .  .  .  .  .  . ]
[  0  0  0  0  0  .  .  .  1 -2  1  0 ]
[  0  0  0  0  0  .  .  .  0  1 -2  1 ]
[  0  0  0  0  0  .  .  .  0  0  1 -2 ]
```

has eigenvalues $\mu_l = -4\sin^2(\frac{\pi l}{2N})$, $l = 1, 2, \ldots, N-1$.

Fact 3: The $(N+1) \times (N+1)$ matrix:

```
[ -1  1  0  0  0  .  .  .  0  0  0  0 ]
[  1 -2  1  0  0  .  .  .  0  0  0  0 ]
[  0  1 -2  1  0  .  .  .  0  0  0  0 ]
[  0  0  1 -2  1  .  .  .  0  0  0  0 ]
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[  0  0  0  0  0  .  .  .  1 -2  1  0 ]
[  0  0  0  0  0  .  .  .  0  1 -2  1 ]
[  0  0  0  0  0  .  .  .  0  0  1 -1 ]
```

has eigenvalues $\mu_l = -4\sin^2\left(\frac{\pi l}{2(N+1)}\right)$, $l = 0, 1, \ldots, N$.

Fact 4: For a real $n \times n$ matrix A, the Rayleigh quotient of a vector $x \in R^n$ is the scalar

$$r(x) = \frac{x^T A x}{x^T x}.$$

The gradient of $r(x)$ is

$$\nabla r(x) = \frac{2}{x^T x}(Ax - r(x)x).$$

If x is an eigenvector of A then $r(x)$ is the corresponding eigenvalue and $\nabla r(x) = 0$.

Preliminary Exam, Numerical Analysis, August 2000

Instructions: This exam is closed books and notes. The time allowed is 3 hours and you need to work on any 3 out of questions 1-4 and any other 2 out of questions 5-7. All questions have equal weight and a score of 75 % is considered a pass. Indicate clearly the work that you wish to be graded.

1. **(Sensitivity Analysis.)** Consider the linear system $Ax = b$ where A is a square, non-singular matrix. A numerical solution $\hat{x}$ actually satisfies

$$(A + \delta A)\hat{x} = b + \delta b$$

If we denote $\delta x = \hat{x} - x$, derive an upper bound for the relative error

$$E_1 = \frac{||\delta x||}{||\hat{x}||}$$

in terms of the condition number of A and the norms of $A, \hat{x}, \delta A, \delta b$, where $||\cdot||$ is a suitable vector norm. Also, we can define another relative error

$$E_2 = \frac{||\delta x||}{||x||}.$$

Derive an upper bound for E_2 in terms of the condition number of A and the norms of $\delta A, A, \delta b$ and b. When are these two errors close to each bound? What's the significance of the condition number?

2. **(Linear Least Squares.)** Let A be a real $m \times n$ matrix of full rank $(m > n)$ and let $b \in \mathbf{R}^m$. The Least Square problem is to find $x \in \mathbf{R}^n$ such that

$$||Ax - b||_2$$

is minimized.

Describe the following three approaches to solve the problem: 1) normal equations, 2) QR decomposition, and 3) Singular Value Decomposition (SVD). Discuss briefly the advantages and disadvantages of these approaches. In particular, give a geometric interpretation of the normal equation approach and explain why it is not often used?

3. **(Eigenvalue Problems.)** Let λ be a simple eigenvalue of A (not necessarily symmetric), with right eigenvector x (*i.e.*, $Ax = \lambda x$), and left eigenvector y (*i.e.*, $y^*A = \lambda y^*$). Assume that $||x||_2 = ||y||_2 = 1$. Let $\lambda + \delta\lambda$ be the corresponding eigenvalue of $A + \delta A$. a) Show that

$$\delta\lambda = \frac{y^*\delta A x}{y^* x} + O(||\delta A||^2),$$

and derive an upper bound for $|\delta\lambda|$. What is the condition number for λ with respect to changes in A?

b) Consider the matrix

$$\begin{bmatrix} 0 & 1 & & & \\ & \cdot & 1 & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ \epsilon & & & & 0 \end{bmatrix},$$

Compute $d\lambda/d\epsilon$ and show that the condition number becomes infinite as ϵ approaches zero. c) Describe the power method for finding the eigenvalue of the largest absolute magnitude, and inverse iteration for the eigenvalue closest to a given value. How fast do you expect these algorithms to converge?

4. **(Singular Value Decomposition.)** Define the singular value decomposition of a real $m \times n$ matrix with $m \geq n$. Is the SVD unique for a given matrix A? How is an SVD problem connected with an eigendecomposition of a symmetric matrix? Sketch the general structure of algorithms for those imported from eigendecomposition, namely, how do you reduce A to some form more manageable? How do you obtain the information about the rank of A through the SVD of A? If A is square and nonsingular, how does the SVD of A help you determine the sensitivity of the solution x for $Ax = b$?

5. **(Iterative Methods for Linear Systems.)** Consider the linear system $Ax = b$ and a splitting of $A : A = M - K$, with M nonsingular, and the iteration

$$x_{m+1} = Rx_m + c,$$

where $R = M^{-1}K$ and $c = M^{-1}b$. Show that the method converges for any x_0 if $||R|| < 1$ where $||\cdot||$ is any operator norm.

Given an $(N-1) \times (N-1)$

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad b = \frac{1}{N^2}\begin{pmatrix} f_1 \\ f_2 \\ \cdot \\ f_{N-2} \\ f_{N-1} \end{pmatrix}$$

Analyze the convergence properties of the Jacobi and Gauss-Seidel methods for this problem. For each method, express the convergence speed as a function of N. How does the number of iterations required to reduce the initial error by a factor of δ depend on N for each method?

6. **(Wave Equations.)** Consider the wave equation in one space dimension:

$$u_t + au_x = 0, \quad -\infty < x < \infty, \quad t > 0,$$

with initial condition

$$u(x, 0) = u_0(x)$$

Consider the usual discretization

$$U_m^k \approx u(x_m, t_k)$$

where $x_m = m\Delta x$ and $t_k = k\Delta t$. For an explicit scheme, what is the CFL condition? Give an intuitive explanation of the condition. Give the first-order upwind scheme for this equation and derive the truncation error.

7. **(Heat Equation.)** Consider the following heat equation problem

$$u_t = u_{xx}, \quad t > 0, \quad 0 < x < 1,$$

$$u(0, t) = u(1, t) = 0, \quad t > 0,$$

$$u(x, 0) = u_0(x), \quad 0 \leq x \leq 1.$$

Using the forward or backward difference for the time derivative and central difference for the space derivative, derive the explicit and implicit schemes. What's the stability condition for each case? Perform a Fourier analysis on both the explicit and implicit methods and use the von Neumann condition to verify these stability conditions.

Prelims Numerical Analysis, August 1999

Answer any five of the six questions below. Clearly indicate which questions you want to be counted. 80% constitute a passing score. Time for this exam is three hours. It is closed books and notes.

-1- (Quadrature.) Consider the quadrature formula

$$\int_a^b w(x)f(x)\mathrm{d}x = \sum_{i=0}^{n} \alpha_i f(\xi_i) + E. \tag{1}$$

w is a given positive weight function. The knots ξ_i and the weights α_i are at your disposal. How do you choose them so that the error E is 0 for all polynomials f of degree as high as possible? How high is possible? Give reasons for your answers.

-2- (Spline Spaces.) Let T be a triangulation of a domain Ω, and S the linear space of functions on T that are continuous on Ω and that on each triangle in T can be written as a (bivariate) polynomial of degree d. Derive the dimension of S?

-3- (A Matrix.) Compute the eigenvalues and eigenvectors of the tridiagonal $N \times N$ matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & \dots & 0 & 0 \\ 1 & 2 & 1 & \dots & 0 & 0 \\ 0 & 1 & 2 & \ddots & 0 & 0 \\ & & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & \ddots & 2 & 1 \\ 0 & 0 & 0 & \dots & 1 & 2 \end{bmatrix} \tag{2}$$

Note that the diagonal entries are indeed meant to be positive 2, not -2.

-4- (Sensitivity Analysis.) Consider the (square, non-singular) linear system

$$Ax = b. \tag{3}$$

Suppose you compute (by any method) a numerical solution $\hat{x}$ and you are concerned about the relative error

$$E = \frac{||x - \hat{x}||}{||x||} \tag{4}$$

where $||.||$ denotes a suitable vector norm. Derive upper and lower bounds on E in terms of the residual $r = b - A\hat{x}$ and the condition number of A with respect to $||\cdot||$.

-5- (Numerical ODEs.) Define the terms linear multistep method (LMM), order, consistency, zero-stability, local truncation error, and region of absolute stability of an LMM. Derive the highest order implicit 2 step method and discuss its region of absolute stability.

-6- (Numerical PDEs.) Consider the PDE

$$u_t + u_x = 0 \tag{5}$$

and the usual discretization

$$U_m^n \approx u(x_m, t_n) \quad \text{where} \quad x_m = mh \quad \text{and} \quad t_n = nk. \tag{6}$$

The *Leap Frog Method* is given by

$$U_m^{n+1} = U_m^{n-1} - \frac{k}{h}\left(U_{m+1}^n - U_{m-1}^n\right). \tag{7}$$

Compute the local truncation error and discuss the stability of this method.

Preliminary Examination, Numerical Analysis, Fall 1998

Instructions: This exam is closed books and notes. Time: 3 hours. Answer 5 of the following 8 questions; at least 2 of the questions you choose must be from questions 6,7, and 8. Indicate clearly which of your answers you wish to have graded. A score of 70% constitutes a pass. All questions have equal weight. On the last page of the exam, are some facts that you may find useful.

1) Linear Least Squares: The Linear Least Squares problem for an $m \times n$ real matrix A and $b \in \mathbf{R}^m$ is the problem:

$$\text{Find} \quad x \in \mathbf{R}^n \text{ such that } \|Ax - b\|_2 \text{ is minimized.}$$

a) Suppose that you have data $\{(t_j, y_j)\}$, $j = 1, 2, \ldots, m$ that you wish to approximate by an expansion

$$p(t) = \sum_{k=1}^{n} x_k \phi_k(t).$$

Here, the functions $\phi_k(t)$ are given functions. Which norm on the difference between the approximation function p and the data gives rise to a linear least squares problem for the unknown expansion coefficients x_k? What is the matrix A in this case, and what is the vector b?
b) Suppose that A is a real $m \times n$ matrix of full rank and let $b \in \mathbf{R}^m$. Show that the Least Squares problem has a unique solution and describe the *geometry* of the Least Squares problem (in terms of vectors and/or subspaces of $\mathbf{R}^m$ and/or $\mathbf{R}^n$).
c) For the matrix A and vector b of part (b), answer the following questions: What are the 'normal equations' for the Least Squares problem? How can they be used to solve the Least Squares problem? What is the QR factorization of A and how can it be used to solve the Least Squares problem? How can the Singular Value Decomposition (SVD) of A be used to solve the Least Squares problem? Compare and contrast these approaches for numerically solving the Least Square problem.

2) Singular Value Decomposition: Suppose that A is a real $m \times n$ matrix with $m \geq n$.
a) What is the *singular value decomposition* (SVD) of A? Does every matrix A of this type have an SVD? If your answer is yes, prove the assertion. If your answer is no, give an example of a matrix A which has no SVD.
b) For a matrix A which does have an SVD, how would having the SVD give easy access to information about the rank of A? If A were $m \times m$ and nonsingular, how would knowing A's SVD help you determine the sensitivity of the solution of a linear system of the following form

$$Ax = b\ ?$$

c) For a nonsingular matrix A which does have an SVD, express the relative distance from A to the nearest rank deficient matrix in terms of A's singular values.

3) Iterative Methods for Linear Systems: a) Consider the fixed-point iteration

$$u^{(k+1)} = Tu^{(k)} + c$$

for finding a solution of the problem

$$u = Tu + c$$

where T is an $m \times m$ real matrix and c is a real m-vector. Show that the fixed point iteration will converge for an arbitrary initial guess $u^{(0)}$ if and only if the spectral radius of T, $\rho(T)$, is less than 1.

b) Consider the boundary value problem

$$-u''(x) = f(x), \quad \text{for } 0 \le x \le 1$$

with $u(0) = u(1) = 0$, and the following discretization of it:

$$-U_{j-1} + 2\,U_j - U_{j+1} = F_j$$

for $j = 1, 2, \ldots, N-1$ where $Nh = 1$ and $F_j \equiv h^2\, f(jh)$.

Analyze the convergence properties of the Jacobi iterative method for this problem. In particular, express the speed of convergence as a function of the discretization stepsize h. How does the number of iterations required to reduce the initial error by a factor δ depend on h? In practice, would you use this method to solve the given problem? If so, explain why this is a good idea? If not, how would you solve it in practice?

4) Eigenvalue Problems:

a) Consider the problem of computing an eigenvalue/eigenvector pair of a real symmetric $m \times m$ matrix A with eigenvalues of distinct magnitudes. Two algorithms for trying to do this are:

Power Iteration	**Rayleigh Quotient Iteration**
Guess $v^{(0)}$ with $\|v^{(0)}\|_2 = 1$	Guess $v^{(0)}$ with $\|v^{(0)}\|_2 = 1$. Set $\lambda^{(0)} = (v^{(0)})^T A v^{(0)}$.
For $k = 1, 2, 3, \ldots$ $w = Av^{(k-1)}$ $v^{(k)} = w/\|w\|_2$ $\lambda^{(k)} = (v^{(k)})^T A v^{(k)}$ End	For $k = 1, 2, 3, \ldots$ Solve $(A - \lambda^{(k-1)} I)w = v^{(k-1)}$ for w $v^{(k)} = w/\|w\|_2$ $\lambda^{(k)} = (v^{(k)})^T A v^{(k)}$ End

i) Analyze the convergence properties of the Power Iteration algorithm for a generic guess v^0.

ii) Suppose that the Rayleigh Quotient Iteration iterates $\lambda^{(k)}$ converge to a simple eigenvalue λ of A. Analyze the speed of convergence of $\lambda^{(k)}$ to λ.

b) Many algorithms for finding eigenvalues of a complex $m \times m$ matrix A are based on the existence of a Schur decomposition of A, namely, $A = QTQ^*$ where Q is unitary and T is upper triangular. Prove that every matrix A has a Schur decomposition and explain how knowing the Schur decomposition of A would help one determine the eigenvalues of A.

5) Krylov Subspace Methods: Consider the system $Ax = b$, in which A is a symmetric positive definite real $m \times m$ matrix, and denote by x_* its true solution. The conjugate gradient (CG) algorithm for finding the solution to the system is:

Set $x_0 = 0$, $r_0 = b$, and $p_0 = r_0$.
For $n = 1, 2, 3, \ldots$
  $\alpha_n = r_{n-1}^T r_{n-1} / p_{n-1}^T A p_{n-1}$
  $x_n = x_{n-1} + \alpha_n p_{n-1}$
  $r_n = r_{n-1} - \alpha_n A p_{n-1}$
  $\beta_n = r_n^T r_n / r_{n-1}^T r_{n-1}$
  $p_n = r_{n-1} + \beta_n p_{n-1}$
End

The vectors generated by this algorithm have the property that, provided $r_{n-1} \neq 0$,

$$\begin{aligned}\mathcal{K}_n \equiv \; & \text{span}\{b, Ab, A^2b, \ldots, A^{n-1}b\} && = \text{span}\{x_1, x_2, \ldots, x_n\} \\ = \; & \text{span}\{r_0, r_1, \ldots, r_{n-1}\} && = \text{span}\{p_0, p_1, \ldots, p_{n-1}\}\end{aligned}$$

and $r_n^T r_j = 0$ and $p_n^T A p_j = 0$ for all $j < n$.

Let $e_n = \|x_n - x_*\|_A$, where the A-norm of a vector x is defined by $\|x\|_A = (x^T A x)^{1/2}$.

a) Prove the following:
Theorem: Suppose the CG algorithm is applied to the symmetric positive definite system $Ax = b$ and that it has not yet converged ($r_{n-1} \neq 0$). Then $e_n < \|y - x_*\|_A$ for all $y \in \mathcal{K}_n$, $y \neq x_n$; $\|e_n\|_A \leq \|e_{n-1}\|_A$; and $e_N = 0$ for some $N \leq m$.

b) In practice this result is not very useful. Why is this, and what determines the speed of convergence of x_n to x_*? What is 'preconditioning' and how is it related to the speed of convergence of the CG algorithm?

6) Numerical Methods for ODEs: Consider the Linear Multistep Method

$$y_{n+1} - y_n = \frac{1}{2} h (f_{n+1} + f_n)$$

for solving an initial value problem $y' = f(y, x)$, $y(0) = \eta$. You may assume that f is Lipschitz continuous with respect to y for all x.

a) Analyze the consistency, stability, accuracy, and convergence properties of this method.

b) Would it be more reasonable to use this method or Euler's method for the initial value problem:

$$y' = -10^8[y - cos(x)] - sin(x), \quad y(0) = 1?$$

Justify your answer.

7) Finite-Difference Methods: Consider the initial boundary value problem

$$v_t = \beta v_{xx}, \quad 0 < x < 1, \quad 0 < t, \quad v(x,0) = f(x)$$

with Dirichlet boundary conditions

$$v(0,t) = 0, \quad v(1,t) = 0.$$

Suppose that the interval $[0,1]$ is partitioned by meshpoints $x_j = jh$ for $j = 0, 1, \ldots, N$ and $h = 1/N$. A possible discretization of the above problem uses the Crank-Nicolson scheme for the differential equation:

$$(I - \frac{\beta k}{2} D_+ D_-) u_j^{n+1} = (I + \frac{\beta k}{2} D_+ D_-) u_j^n$$

for $j = 1, \ldots, N-1$, combined with the boundary conditions:

$$u_0^n = 0 \text{ and } \mathrm{u}_\mathrm{N}^\mathrm{n} = 0$$

for each time level n. Initial data is obtained from by setting $u_j^0 = f_j$ for $j = 0, 1, \ldots, N$. (Note that D_+ and D_- are the usual forward and backward difference operators.) Set up the matrix system that must be solved in order to advance the numerical solution from time level n to time level $n+1$. Show that the suggested discretization is stable in the l_2 norm

$$\|u\|_2 = \left(\sum_{j=0}^{N} h |u_j|^2 \right)^{1/2}$$

for *all* choices of timestep k and spacestep h. What other issues should be considered in deciding whether the suggested scheme is a good choice for the problem?

8) Finite-Element Methods: Consider the problem:

$$-u'' + u = f, \qquad \text{on } [0,1]$$

with

$$u(0) = 0 \qquad \text{and} \qquad \mathrm{u}(1) = 1.$$

Derive a variational formulations of this problem (be sure to define the space on which your variational problem is to be solved). Formulate a Galerkin finite-element method for the variational problem using piecewise linear functions. Prove that the linear system which arises in the finite-element method has a solution. In what sense is the solution optimal?

Fact 1: A real symmetric $n \times n$ matrix A can be diagonalized by an orthogonal similarity transformation, and A's eigenvalues are real.

Fact 2: The $(N-1) \times (N-1)$ matrix M defined by

$$m_{ij} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

has eigenvalues $\mu_l = 4\sin^2(\frac{\pi l}{2N})$, $l = 1, 2, \ldots, N-1$.

Fact 3: The $(N+1) \times (N+1)$ matrix:

```
[ -1  1  0  0  0  .  .  .  0  0  0  0 ]
[  1 -2  1  0  0  .  .  .  0  0  0  0 ]
[  0  1 -2  1  0  .  .  .  0  0  0  0 ]
[  0  0  1 -2  1  .  .  .  0  0  0  0 ]
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[  0  0  0  0  0  .  .  .  1 -2  1  0 ]
[  0  0  0  0  0  .  .  .  0  1 -2  1 ]
[  0  0  0  0  0  .  .  .  0  0  1 -1 ]
```

has eigenvalues

$$\mu_l = -4\sin^2\left(\frac{\pi l}{2(N+1)}\right), \quad l = 0, 1, \ldots, N.$$

Fact 4: For a real $n \times n$ matrix A, the Rayleigh quotient of a vector $x \in R^n$ is the scalar

$$r(x) = \frac{x^T A x}{x^T x}.$$

The gradient of $r(x)$ is

$$\nabla r(x) = \frac{2}{x^T x}(Ax - r(x)x).$$

If x is an eigenvector of A then $r(x)$ is the corresponding eigenvalue and $\nabla r(x) = 0$.

Preliminary Examination, Numerical Analysis, Fall 1997

Instructions: This exam is closed books and notes. Time: 3 hours. Answer any 6 of the following 9 questions. Indicate clearly which of your answers you wish to have graded. A score of 70% constitutes a pass. All questions have equal weight. On the last page of the exam, are some facts that you may find useful.

1) Sensitivity of Solution of Linear System: Suppose that A is a real non-singular $n \times n$ matrix. Consider the family of linear systems $Ax(\epsilon) = b + \epsilon\, c$, where b and c are given nonzero vectors in $\mathbf{R}^n$, and ϵ is a real parameter. Let $x = x(0)$ be the solution for $\epsilon = 0$. Let $\|\cdot\|_2$ denote the Euclidean vector norm. Show that

$$\frac{\|x(\epsilon) - x\|_2}{\|x\|_2} \leq \|A\|_2 \|A^{-1}\|_2 \frac{\|\epsilon\, c\|_2}{\|b\|_2}$$

The expression $\|A\|_2 \|A^{-1}\|_2$ is the condition number of A. Interpret the above inequality and explain its significance?

2) Singular Value Decomposition: Suppose that A is a real $m \times n$ matrix with $m \geq n$. What is the *singular value decomposition* (SVD) of A? Does every matrix A of this type have an SVD? If your answer is yes, prove the assertion. If your answer is no, give an example of a matrix A which has no SVD. For a matrix A which does have an SVD, how would having the SVD give easy access to information about the rank of A? If A were $n \times n$ and nonsingular, how would knowing A's SVD help you determine the sensitivity of the solution of a linear system of the following form

$$Ax = b\ ?$$

3) Eigenvalue Problems: Let A be a real $n \times n$ matrix with simple eigenvalue λ, right eigenvector x (*i.e.*, $Ax = \lambda x$), and left eigenvector y (*i.e.*, $y^* A = \lambda y^*$). Assume that $\|x\|_2 = \|y\|_2 = 1$. a) Show that the condition number for λ to changes in A is given by $s(\lambda) = 1/|y^* x|$. b) Give an example of a matrix A which has a badly conditioned eigenvalue, and another matrix A which has only well-conditioned eigenvalues. c) Sketch a reasonably efficient algorithm for finding the largest eigenvalue of a symmetric matrix A with distinct eigenvalues. How fast do you expect this algorithm to converge?

4) Linear Least Squares: The Linear Least Squares problem for an $m \times n$ real matrix A and $b \in \mathbf{R}^m$ is the problem:

$$\text{Find} \quad \mathrm{x} \in \mathbf{R}^{\mathrm{n}} \text{ such that } \|\mathrm{Ax} - \mathrm{b}\|_2 \text{ is minimized.}$$

a) Suppose that you have data $\{(t_j, y_j)\}$, $j = 1, 2, \ldots, m$ that you wish to approximate by an expansion

$$p(t) = \sum_{k=1}^{n} x_k \phi_k(t).$$

Here, the functions $\phi_k(t)$ are given functions. Which norm on the difference between the approximation function p and the data gives rise to a linear least squares problem for the unknown expansion coefficients x_k? What is the matrix A in this case, and what is the vector b?

b) Suppose that A is a real $m \times n$ matrix of full rank and let $b \in \mathbf{R}^m$. What are the 'normal equations' for the Least Squares problem? How can they be used to solve the Least Squares problem? What is the QR factorization of A and how can it be used to solve the Least problem? What is the QR factorization of A and how can it be used to solve the Least Squares problem? Compare and contrast these approaches for numerically solving the Least Square problem.

5) Iterative Methods for Linear Systems: Consider the boundary value problem

$$-u''(x) = f(x), \quad \text{for } 0 \leq x \leq 1$$

with $u(0) = u(1) = 0$, and the following discretization of it:

$$-U_{j-1} + 2\,U_j - U_{j+1} = F_j$$

for $j = 1, 2, \ldots, N-1$ where $Nh = 1$ and $F_j \equiv h^2\, f(jh)$.

Analyze the convergence properties of both the Jacobi and Gauss Seidel iterative methods for this problem. In particular, for each, express the speed of convergence as a function of the discretization stepsize h. How does the number of iterations required to reduce the initial error by a factor δ depend on h for each method? In practice, would you use one of these methods to solve the given problem? If so, explain why this is a good idea? If not, how would you solve it in practice?

6) Newton's Method: Suppose $f(x)$ is a real-valued function defined for all values of the real variable x, and that f' is Lipschitz continuous. State and prove a *local* convergence theorem for Newton's method for finding a simple zero of f. How does this result change if the root is a double root? Is Newton's method globally convergent? If you answered yes, prove your claim. If you answered no, sketch an example which illustrates your claim, and outline a method that has the Newton method's local convergence properties but which is globally convergent.

7) Broyden's Method: Consider the nonlinear system:

$$F(x) = 0 \qquad \text{where } F : \mathbf{R}^n \rightarrow \mathbf{R}^n$$

and the iteration

$$x^{(k+1)} = x^{(k)} - (A^{(k)})^{-1} F(x^{(k)})$$

where $x^{(0)} \in \mathbf{R}^n$ is given.
Newton's Method is defined by the choice $A^{(k)} = F'(x)$ where $F'(x)$ is the Jacobean matrix of F. An alternative choice for $A^{(k+1)}$ is made in Broyden's Method and is motivated as follows:

Let $A^{(0)}$ be a given and suitably chosen matrix. The scalar Secant Method suggests imposing on $A^{(k+1)}$ the condition

$$A^{(k+1)}(x^{(k+1)} - x^{(k)}) = F(x^{(k+1)}) - F(x^{(k)}).$$

This however does not uniquely determine A^{k+1} as it does not give any information on how A^{k+1} acts on vectors perpendicular to $(x^{(k+1)} - x^{(k)})$. Thus require also that:

$$A^{(k+1)}t = A^{(k)}t \qquad \text{for all } \mathrm{t} \in \mathbf{R}^{\mathrm{n}} \text{ such that } \mathrm{t}^{\mathrm{T}}(\mathrm{x}^{(\mathrm{k}+1)} - \mathrm{x}^{(\mathrm{k})}) = 0$$

Use these conditions to derive an explicit expression for $A^{(k+1)}$ in terms of $A^{(k)}$, $x^{(k)}$, and $x^{(k+1)}$.

8) Numerical Methods for ODEs: Consider the Linear Multistep Method

$$y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2}{3}hf_{n+2}$$

for solving an initial value problem $y' = f(y,x)$, $y(0) = \eta$. You may assume that f is Lipschitz continuous with respect to y for all x.

a) Analyze the consistency, stability, accuracy, and convergence properties of this method.

b) Would it be more reasonable to use this method or Euler's method for the initial value problem:

$$y' = -10^8[y - cos(x)] - sin(x), \quad y(0) = 1?$$

Justify your answer.

9) Finite-Difference Methods: Consider the initial boundary value problem

$$v_t = \beta v_{xx}, \quad 0 < x < 1, \quad 0 < t, \quad v(x,0) = f(x)$$

with Neumann boundary conditions

$$v_x(0,t) = 0, \quad v_x(1,t) = 0.$$

Suppose that the interval $[0,1]$ is partitioned by meshpoints $x_j = jh$ for $j = 0, 1, \ldots, N$ and $h = 1/N$. A possible discretization of the above problem uses the Crank-Nicolson scheme for the differential equation:

$$(I - \frac{\beta k}{2}D_+D_-)u_j^{n+1} = (I + \frac{\beta k}{2}D_+D_-)u_j^n$$

for $j = 0, 1, \ldots, N$, combined with the boundary condition discretizations:

$$\frac{u_1^n - u_{-1}^n}{h} = 0 \text{ and } \frac{\mathrm{u}_{\mathrm{N+1}}^{\mathrm{n}} - \mathrm{u}_{\mathrm{N-1}}^{\mathrm{n}}}{\mathrm{h}} = 0$$

for each time level n. Initial data is obtained from by setting $u_j^0 = f_j$ for $j = 0, 1, \ldots, N$. (Note that D_+ and D_- are the usual forward and backward difference operators.) Set up the matrix system that must be solved in order to advance the numerical solution from time level n to time level $n + 1$. Show that the suggested discretization is stable in the l_2 norm

$$||u||_2 = \left(\sum_{j=0}^{N} h|u_j|^2 \right)^{1/2}$$

for *all* choices of timestep k and spacestep h. What other issues should be considered in deciding whether the suggested scheme is a good choice for the problem?

Fact 1: A real symmetric $n \times n$ matrix A can be diagonalized by an orthogonal similarity transformation, and A's eigenvalues are real.

Fact 2: The $(N-1) \times (N-1)$ matrix M defined by

$$m_{ij} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

has eigenvalues $\mu_l = 4\sin^2(\frac{\pi l}{2N})$, $l = 1, 2, \ldots, N-1$.

Fact 3: The $(N+1) \times (N+1)$ matrix:

```
[ -1  1  0  0  0  .  .  .  0  0  0  0 ]
[  1 -2  1  0  0  .  .  .  0  0  0  0 ]
[  0  1 -2  1  0  .  .  .  0  0  0  0 ]
[  0  0  1 -2  1  .  .  .  0  0  0  0 ]
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[  0  0  0  0  0  .  .  .  1 -2  1  0 ]
[  0  0  0  0  0  .  .  .  0  1 -2  1 ]
[  0  0  0  0  0  .  .  .  0  0  1 -1 ]
```

has eigenvalues

$$\mu_l = -4\sin^2\left(\frac{\pi l}{2(N+1)}\right), \quad l = 0, 1, \ldots, N.$$

Preliminary Exam, Numerical Analysis, September 1996[-1-]

Notes:

1. Please answer any **eight** of the following **twelve** questions. Clearly indicate which questions you want scored.
2. A score of 70% or more constitutes a pass. Do not use books or calculators. Duration of the exam: 3 hours.
3. To avoid disruption and confusion I will not be able to answer questions during the exam. If you believe there's something wrong with one of the questions, indicate it and if you are right you will receive generous credit. TO be on the safe side, do not include that question among those you want scored.
4. We want to know if you understand the answers to these questions to the point that you can work with the associated concepts. So go beyond just giving the name of something, but do not bother to give trivial details. Use your judgment!

Math 661: Linear Algebra

-1- **(Philosophy.)** Why are matrices important? Why do we multiply a matrix with a vector the way we do?

-2- **(The QR Factorization.)** Let A be an $m \times n$ matrix, where $m \geq n$. What is the QR factorization of A? How do you compute it? What do you use it for?

-3- **(Gaussian Elimination is LU Factorization.)** What is the LU factorization of a square matrix? How do you solve a linear system $Ax = b$ by Gaussian Elimination? Show that for a general 4×4 system the matrix part of Gaussian Elimination is equivalent to computing the LU factorization of A.

-4- **(Condition Number.)** What is the condition number of a square matrix A? Suppose you solve a linear system $Ax = b$. How do you use the condition number to gauge the accuracy of your numerical solution? Give a formula.

Math 662: Interpolation, Quadrature, Approximation, Nonlinear Systems, Optimization.

-5- **(Gaussian Quadrature.)** What is Gaussian quadrature? How do you pick the weights and knots? Why?

-6- **(Least Squares.)** Describe in general the problem of approximating a given function in the Least Squares sense by a linear combination of certain basis functions. Discuss the

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significance of orthogonal basis functions. If you have a basis that is not orthogonal. how can you obtain an orthogonal basis that spans the same space?

-7- (Interpolation.) Suppose you are given an arbitrarily smooth function f of a single variable x, and points

$$x_0 < x_1 < \ldots < x_n.$$

Let p be the interpolating polynomial of degree n satisfying

$$p(x_i) = f(x_i), \quad i = 0, 1, \ldots, n.$$

Derive an expression for the error

$$f(x) - p(x)$$

for a general value of x, in terms of the derivatives of f.

-8- (Discrete Approximation.) Let A be an $m \times n$ matrix and $b \in \mathbb{R}^m$ a given vector. Consider the problem of finding $x \in \mathbb{R}^n$ such that

$$\|Ax - b\|_p = \min$$

Depending on the value of p you obtain different kinds of problems. Describe and derive these problems for $p = 1$, $p = 2$, and $p = 3$.

Math 663: Solution of Differential Equations.

-9- (Discretization and Convergence.) What is a discretization? What does it mean for a discretization to be convergent? Give a precise definition for a differential equation problem and a discretization of your choice.

-10- (Local Truncation Error.) What is the local truncation error of a numerical scheme? What does *consistency* mean? Give a specific and precise example.

-11- (Stability.) What does stability of a numerical scheme mean? Give a specific example.

-12- (Convergence is equivalent to stability and convergence.) Describe and prove a specific example for this fundamental result.

Preliminary Examination, Numerical Analysis, Fall 1995

Instructions: This exam is closed books and notes. Time: 3 hours. Answer any 5 of the following 8 questions. Indicate clearly which of your answers you wish to have graded. A score of 70% constitutes a pass. All questions have equal weight.

1) Sensitivity of Solution of Linear System: Suppose that A is a real non-singular $n \times n$ matrix. Consider the family of linear systems $Ax(\epsilon) = b + \epsilon\, c$, where b and c are given nonzero vectors in $\mathbf{R}^n$, and ϵ is a real parameter. Let $x = x(0)$ be the solution for $\epsilon = 0$. Let $\|\cdot\|_2$ denote the Euclidean vector norm.

a) Show that

$$\frac{\|x(\epsilon) - x\|_2}{\|x\|_2} \leq \|A\|_2 \|A^{-1}\|_2 \frac{\|\epsilon\, c\|_2}{\|b\|_2}$$

The expression $\|A\|_2\|A^{-1}\|_2$ is the condition number of A. Interpret the above inequality and explain its significance?

b) What is the Singular Value Decomposition (SVD) of A? Express the condition number above in terms of A's singular values.

c) When the condition number is very large, the matrix is called ill-conditioned. Sometimes people say that an ill-conditioned matrix is 'nearly-singular'. Make this precise by expressing the (relative) distance between A and the nearest singular matrix to A in terms of the condition number of A.

2) Linear Least Squares: Let A be a real $m \times n$ matrix of full rank and let $b \in \mathbf{R}^m$ and consider the Least Squares problem:

$$\text{Find} \quad \mathrm{x} \in \mathbf{R}^{\mathrm{n}} \text{ such that } \|\mathrm{Ax} - \mathrm{b}\|_2 \text{ is minimized.}$$

What are the 'normal equations' for the Least Squares problem? How can they be used to solve the Least Squares problem? What is the QR factorization of A and how can it be used to solve the Least Squares problem? What is the Singular Value Decomposition (SVD) of A and how can it be used to solve the Least Squares problem? Compare and contrast these approaches for numerically solving the Least Square problem.

3) Iterative Methods for Linear Systems: Consider the boundary value problem

$$-u''(x) = f(x), \quad \text{for } 0 \leq x \leq 1$$

with $u(0) = u(1) = 0$, and the following discretization of it:

$$-U_{j-1} + 2\, U_j - U_{j+1} = F_j$$

for $j = 1, 2, \ldots, N-1$ where $Nh = 1$ and $F_j \equiv h^2\, f(jh)$. Imagine applying the Jacobi iterative method to solve this system of linear equations. Analyze the convergence properties of the Jacobi method for this problem. In particular, express the speed of convergence as a function of the discretization stepsize h. You may find it helpful to recall that the $(N-1) \times (N-1)$ matrix M defined by

$$m_{ij} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

has eigenvalues $\mu_l = 4\sin^2(\frac{\pi l}{2N})$, $l = 1, 2, \ldots, N-1$. (NOTE: This is a model problem designed to give you a simple linear system on which to analyze the Jacobi method; one would certainly not use an iterative method to solve a tri-diagonal system in practice!)

4) Eigenvalue Problems: Let A be a real $n \times n$ matrix with simple eigenvalue λ, right eigenvector x (*i.e.*, $Ax = \lambda x$), and left eigenvector y (*i.e.*, $y^*A = \lambda y^*$). Assume that $\|x\|_2 = \|y\|_2 = 1$. a) Show that the condition number for λ to changes in A is given by $s(\lambda) = 1/|y^*x|$. b) Give an example of a matrix A which has a badly conditioned eigenvalue, and another matrix A which has only well-conditioned eigenvalues. c) Sketch a reasonably efficient algorithm for finding the largest eigenvalue of a symmetric matrix A with distinct eigenvalues. How fast do you expect this algorithm to converge?

5) Newton Type Methods: Suppose $f(x)$ is a real-valued function defined for all values of the real variable x, and that f' is Lipschitz continuous. Consider the discrete Newton method: Guess x_0 and for $k = 0, 1, 2, \ldots$, do:

$$a_k = (f(x_k + h_k) - f(x_k))/h_k$$

$$x_{k+1} = x_k - f(x_k)/a_k$$

Analyze the local convergence properties of this method. How does the speed of convergence depend on the choice of h_k (ignore any effect of round-off errors in this analysis)? Now taking roundoff into account what would be a reasonable size for h_k in practice?

6) Numerical Methods for ODEs: Consider the Linear Multistep Method

$$y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2}{3}hf_{n+2}$$

for solving an initial value problem $y' = f(y, x)$, $y(0) = \eta$. You may assume that f is Lipschitz continuous with respect to y for all x. a) Analyze the consistency, stability, accuracy, and convergence properties of this method. b) Would this method be a reasonable one to use for the initial value problem:

$$y' = -10^8[y - cos(x)] - sin(x), \quad y(0) = 1?$$

Justify your answer.

7) Finite-Difference Methods: Consider the spatially-periodic initial value problem for $v(x, t)$:

$$v_t = c\, v_x, \quad \text{for } 0 \le x \le 1$$

with

$$v(x,0) = f(x), \quad \text{for } 0 \le x \le 1$$

Here the function f has period 1, and c is a positive constant. Note that the exact solution of the problem is $v(x,t) = f(x+ct)$.

Consider the following difference approximations to the PDE:

i) $u_j^{n+1} = u_j^n + \frac{ck}{h}(u_{j+1}^n - u_j^n)$

ii) $u_j^{n+1} = u_j^n + \frac{ck}{h}(u_j^n - u_{j-1}^n)$

iii) $u_j^{n+1} = u_j^n + \frac{ck}{2h}(u_{j+1}^n - u_{j-1}^n)$

In each case, $Nh = 1$, and the difference scheme is supplemented with the discrete initial conditions $u_j^0 = f(jh)$ for $j = 0, 1, \ldots, N$ and boundary conditions $u_N^n = u_0^n$, $u_{-1}^n = u_{N-1}^n$, and $u_{N+1}^n = u_1^n$ for all time levels $n \ge 0$.
For each scheme, tell whether it is a reasonable scheme to use on the above problem, and justify your answers.

8) Finite-Element Methods: Consider the problem:

$$-u'' + u = f, \qquad \text{on } [0,1]$$

with

$$u(0) = 0 \qquad \text{and} \qquad u(1) = 1.$$

Derive a variational formulations of this problem (be sure to define the space on which your variational problem is to be solved). Formulate a Galerkin finite-element method for the variational problem using piecewise linear functions. Prove that the linear system which arises in the finite-element method has a solution. In what sense is the solution optimal?

Preliminary Examination, Numerical Analysis, Fall 1994

Instructions: This exam is closed books and notes. Time: 3 hours.

Answer any 5 of the following 9 questions (equal weight). A score of 70% constitutes a pass.

1. a) Derive a multistep method to approximate $y' = f(t,y)$ of the form

$$y_{n+1} = y_n + h(b_{-1}y'_{n+1} + b_0 y'_n + b_1 y'_{n-1})$$

with $y'_j \equiv f(t_j, y_j)$ $\quad t_j = t_o + jh$ by interpolating $f(s, y(s))$ at t_{n-1}, t_n, t_{n+1}.

b) Analyze the local truncation error of this method and the consequences regarding convergence. What would be an appropriate predictor for this method?

c) Discuss the region of absolute stability of this method, and the predictor-corrector method described in part b).

---

2. a) Show that the best uniform ($L^\infty[-1,1]$) approximation of an even polynomial $p_n(x)$ of (even) degree n by polynomials of degree $\leq n-2$ on $[-1,1]$, is given by $p_n(x) - c_n T_n(x)$, where T_n is the Tchebyshev polynomial of degree n, and c_n is an appropriately chosen constant. Hint: First find the best approximation by polynomials of degree $\leq n-1$.

b) What are the zeroes of T_n and what are their significance in interpolation. In particular, estimate the error of interpolation at Tchebyshev nodes.

3. Compare the stability of the Euler Predictor-Trapezoidal Corrector and Backwards Euler timestepping methods for solution of the heat equation

$$u_t = u_{xx} \quad u(x,0) = u_o(x)$$

using the centered second difference operator

$$u_j \to (u_{j+1} - 2u_j + u_{j-1})/h^2 \qquad j = 1, \ldots, J$$

$u_j \approx u(jh)$ on the interval $[0, 2\pi]$ with periodic boundary conditions, and mesh-size $h = \frac{2\pi}{J}$. Discuss the accuracy of these methods?

4. a) What set of four nonlinear equations are satisfied by the nodes and weights for Gaussian quadrature with degree of precision 3, on the interval $[0, 1]$.

b) Describe Newton's method as applied to solving this system of equations.

c) Show how and why the weights and nodes for Gaussian quadrature may be modified from the standard interval to an arbitrary interval.

5. a) What is the best least squares approximation to $f(x) = \cos 3x$ by functions of the form

$$a_0 + a_1 \cos x + a_2 \cos^2 x + b_1 \sin x + b_2 \sin^2 x$$

with the inner product given by

$$(v, w)_C \equiv \frac{1}{2\pi} \int_0^{2\pi} v(x)\bar{w}(x)dx$$

b) What if the inner product is

$$(v, w)_D \equiv \frac{1}{J} \sum_{j=0}^{J-1} v(x_j)\bar{w}(x_j)$$

with $x_j \equiv jh, \quad h = \frac{2\pi}{J}$

c) Briefly discuss the normal equations (do they form a well- or ill-conditioned system?), the Gram-Schmidt process, QR factorization in the context of parts a) and b). Observe the connection with problem #2.

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6. a) Show that Jacobi's method for solving $Ax = b$ iteratively is convergent when A is strictly diagonally dominant. What does this imply about the spectral radius of another matrix obtained from A.

b) Describe the successive over-relaxation (SOR) method with parameter ω and it's rate of convergence. Give a sufficient condition upon A and ω for convergence.

c) Briefly compare the complexity of solving Poisson's problem in three dimensions by direct and iterative methods. Assume the simplest methods grid, differencing scheme, boundary conditions, etc.

7. a) Give an example to demonstrate the role of pivoting in Gaussian elimination in solving $Ax = b$. Define and explain the significance of the condition number of A. Compare the condition numbers, with respect to the norm of your choice, of A and PA, where P is a permutation matrix.

b) Explain the LU decomposition of a matrix A, in particular why the entries of L are multipliers used in Gaussian elimination, and how to solve $Ax = b$ using L and U. How much work does this take?

8. a) Give an identity that is the basis of one stage of the Fast Fourier Transform algorithm applied to a vector of length 2^J. Explain the approximate operation count for the complete algorithm.

b) Describe the application of the FFT to the solution of the heat equation with periodic boundary conditions, and how this may be viewed as performing a fast convolution.

9. a) Find the order/rate of convergence of Newton's method to a multiple root r of multiplicity $m > 1$ of the scalar equation $f(x) = 0$.

b) What method is obtained by applying Aitken's acceleration in this situation? Show that it is at least quadratically convergent.

c) How would you extrapolate an integration formula whose error behaved asymptotically as

$$I - I_n \; c_3 h^3 + c_6 h^6 + \dots$$

where $I \equiv \int_a^b f(x)dx$ is the exact value, I_n is the approximation with n subintervals, and $h \equiv \frac{b-a}{n}$.