Preliminary Examination, Numerical Analysis, August 2007

Instructions: This exam is closed books and notes. The time allowed is three hours and you need to work on any three out of questions 1-5 and any two out of questions 6-8. All questions have equal weights and the passing score will be determined after all the exams are graded. Indicate clearly the work that you wish to be graded.

Note: In problems 6-8, the notations $k = \Delta t$ and $h = \Delta x$ are used. Note also that at the end of the exam there is a list of Facts some of which may be useful to you.

1. Matrix Factorizations:

(a) Prove any two of the following statements:

(i) Schur Decomposition: Any matrix $A \in \mathbb{C}^{m\times m}$ can be factored as $A = Q^*TQ$, where Q is unitary and T is upper triangular.

(ii) Singular Value Decomposition: Any matrix $A \in \mathbb{C}^{m\times n}$ can be factored as $A = U\Sigma V^*$, where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary and $\Sigma \in \mathbb{R}^{m\times n}$ is a rectangular matrix whose only nonzero entries are non-negative entries on its diagonal.

(iii) QR Factorization: Any full-rank matrix $A \in \mathbb{R}^{m\times n}$ for $m \geq n$ can be factored $A = QR$ where $Q \in \mathbb{R}^{m\times m}$ is orthogonal and $R \in \mathbb{R}^{m\times n}$ is upper triangular with positive diagonal entries.

b) Discuss situations in which each of these factorizations is useful in numerical analysis and explain why the factorizations are useful in those situations.

2. Least Squares:

Suppose $A \in \mathbb{R}^{m \times n}$ has linearly independent columns, and $C \in \mathbb{R}^{l \times n}$ has linearly independent rows. Let $d \in \mathbb{R}^l$ and $b \in \mathbb{R}^m$. Show how two full-rank QR factorizations can be used to solve the *linearly constrained* least squares problem:

$$\text{Find } \ \mathrm{x}^* \in \{\mathrm{x} \in \mathbb{R}^{\mathrm{n}} : \mathrm{Cx} = \mathrm{d}\} \text{ which minimizes } \ \|\mathrm{b} - \mathrm{Ax}\|_2.$$

3. Interpolation and Integration:

a) Consider equally spaced points $x_j = a + jh$, $j = 0, \ldots, J+1$ on the interval $[a, b]$, where $(J+1)h = b - a$. Let $f(x)$ be a function defined on $[a, b]$. Consider the problem of finding a cubic spline approximation $s(x)$ to $f(x)$ that interpolates f at the points x_j, is twice continuously differentiable, and satisfies $s''(a) = s''(b) = 0$. Does this problem always have a solution? If your answer is yes, derive formulas by which to determine the spline. If your answer is no, explain your reasoning.

b) Let $I_n(f)$ denote the result of using the composite Trapezoidal rule to approximate $I(f) \equiv \int_a^b f(x)dx$ using n equally sized subintervals of length $h = (b-a)/n$. It can be shown that the integration error $E_n(f) \equiv I(f) - I_n(f)$ satisfies

$$E_n(f) = d_2h^2 + d_4h^4 + d_6h^6 + \ldots$$

where $d_2, d_4, d_6, \ldots$ are numbers that depend only on the values of f and its derivatives at a and b. Suppose you have a black-box program that, given f, a, b, and n, calculates $I_n(f)$. Show how to use this program to obtain an $O(h^4)$ approximation and an $O(h^6)$ approximation to $I(f)$.

4. Eigenvalue Problems:

a) Let A be a real $n \times n$ matrix with simple eigenvalue λ, right eigenvector x (*i.e.*, $Ax = \lambda x$), and left eigenvector y (*i.e.*, $y^* A = \lambda y^*$). Assume that $\|x\|_2 = \|y\|_2 = 1$. Consider the perturbed matrix $A + \delta A$, where $\|\delta A\|_2 / \|A\|_2 = \epsilon << 1$. Show that the matrix $A + \delta A$ has an eigenvalue μ such that $|\lambda - \mu|$ is approximately $\epsilon / |y^* x|$. (Note that this implies that $|\lambda - \mu| \leq (2/|y^* x|)\epsilon$ for ϵ sufficiently small.)

b) Consider a matrix A with double eigenvalue λ and a perturbation δA of A as above. Must there be a constant C such that $A + \delta A$ has an eigenvalue μ for which $|\lambda - \mu| \leq C\epsilon$ for all sufficiently small ϵ?. If yes, prove it. If no, construct and analyze an example that shows such a C need not exist.

c) Consider the problem of computing an eigenvalue/eigenvector pair of a real symmetric $m \times m$ matrix A with eigenvalues that satisfy $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_m|$. One algorithm for trying to do this is Power Iteration with Rayleigh Quotient approximation of the eigenvalue:

Guess $v^{(0)}$ with $\|v^{(0)}\|_2 = 1$.
Set $\lambda^{(0)} = (v^{(0)})^T A v^{(0)}$.
For $k = 1, 2, 3, \ldots$
  Let $w = Av^{(k-1)}$
  $v^{(k)} = w / \|w\|_2$
  $\lambda^{(k)} = (v^{(k)})^T A v^{(k)}$
End

Analyze the convergence of the approximate eigenvector $v^{(k)}$ and the approximate eigenvalue $\lambda^{(k)}$. To what do these quantities converge, and how fast do they converge?

5. Iterative Methods:
Consider the fixed-point iteration

$$u^{(k+1)} = Tu^{(k)} + c$$

for finding a solution of the problem

$$u = Tu + c$$

where T is an $m \times m$ real matrix and c is a real m-vector. It is a fact that the fixed-point iteration converges for any choice of starting point $u^{(0)}$ if and only if the spectral radius of T, $\rho(T)$, satisfies $\rho(T) < 1$.

a) Consider a linear system $Ax = b$ where A is strictly diagonally dominant (that is, $a_{ii} > \sum_{j \neq i} |a_{ij}|$ for all i). Show that the Gauss-Seidel iterative scheme converges for this problem. Describe a situation in numerical analysis where a strictly diagonally-dominant matrix system arises.

b) For convergence of fixed-point iteration from any starting point, is it sufficient that $\|T\| < 1$ for some operator norm $\|\cdot\|$? Justify your answer.

6. ODEs

Consider the initial value problem for a system of ODEs

$$y' = Ay$$

where A has eigenvalues with real parts ranging from -10^6 to -1. Which of the following schemes would be the best choice for solving this problem? Justify your answer in terms of stability, accuracy, and efficiency. Does your answer change depending on the size and condition number of A? Explain your answer.

$$y^{n+1} = y^n + kAy^n, \tag{1}$$

$$y^{n+1} = y^n + kAy^{n+1}, \tag{2}$$

$$y^{n+1} = y^n + \frac{k}{2}(Ay^n + Ay^{n+1}). \tag{3}$$

7. Heat Equation Stability:

a) Consider the initial value problem for the constant-coefficient diffusion equation

$$v_t = \beta v_{xx}, \ t > 0$$

with initial data $v(x,0) = f(x)$. The Crank-Nicolson scheme for this problem is:

$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{\beta}{2h^2} \left\{ u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1} + u_{j-1}^n - 2u_j^n + u_{j+1}^n \right\}.$$

Analyze the 2-norm stability of this scheme and show that the scheme is stable for any choice of $k > 0$ and $h > 0$.

b) Consider the variable coefficient diffusion equation

$$v_t = (\beta v_x)_x, \qquad 0 < x < 1, \ t > 0$$

with Dirichlet boundary conditions

$$v(0,t) \ = \ 0, \qquad v(1,t) \ = \ 0$$

and initial data $v(x,0) = f(x)$. Assume that $\beta(x) \geq \beta_0 > 0$, and that $\beta(x)$ is smooth. Let $\beta_{j+1/2} = \beta(x_{j+1/2})$. The Crank-Nicolson scheme for this problem is:

$$\begin{aligned}\frac{u_j^{n+1} - u_j^n}{k} = \frac{1}{2h^2} \ & \left\{ \beta_{j-1/2} u_{j-1}^{n+1} - (\beta_{j-1/2} + \beta_{j+1/2}) u_j^{n+1} + \beta_{j+1/2} u_{j+1}^{n+1} \right. \\ & \left. + \beta_{j-1/2} u_{j-1}^n - (\beta_{j-1/2} + \beta_{j+1/2}) u_j^n + \beta_{j+1/2} u_{j+1}^n \right\}.\end{aligned}$$

Analyze the 2-norm stability of this scheme for solving this initial boundary value problem. Do not neglect the fact that there are boundary conditions!

8. Elliptic Problems:

Consider the standard five-point difference approximation (centered difference for both the gradient and divergence operators) for the variable coefficient Poisson equation

$$-\nabla \cdot (a\nabla v) = f$$

with Dirichlet boundary conditions, in a two-dimensional rectangular region. We assume that $a(x, y) \geq a_0 > 0$. The approximate solution $\{u_{i,j}\}$ satisfies a linear system $Au = b$.

1. State and prove the maximum principle for the numerical solution $u_{i,j}$.

2. Derive the matrix A in the one-dimensional case and show that it is symmetric and positive definite.

3. For the one-dimensional and *constant-coefficient* case, show that the global error $e_j = v(x_j) - u_j$ satisfies $\|e\|_2 = O(h^2)$ as the space step $h \to 0$.

4. Discuss the advantages and disadvantages of trying to solve the system for the two-dimensional problem using (i) the SOR (Successive Over Relaxation) method and (ii) the (preconditioned) Conjugate Gradient method.

Fact 1: A real symmetric $n \times n$ matrix A can be diagonalized by an orthogonal similarity transformation, and A's eigenvalues are real.

Fact 2: The $(N-1) \times (N-1)$ matrix M defined by

```
[ -2  1  0  0  0  .  .  .  0  0  0  0 ]
[  1 -2  1  0  0  .  .  .  0  0  0  0 ]
[  0  1 -2  1  0  .  .  .  0  0  0  0 ]
[  0  0  1 -2  1  .  .  .  0  0  0  0 ]
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[  0  0  0  0  0  .  .  .  1 -2  1  0 ]
[  0  0  0  0  0  .  .  .  0  1 -2  1 ]
[  0  0  0  0  0  .  .  .  0  0  1 -2 ]
```

has eigenvalues $\mu_l = -4\sin^2(\frac{\pi l}{2N})$, $l = 1, 2, \ldots, N-1$.

Fact 3: The $(N+1) \times (N+1)$ matrix:

```
[ -1  1  0  0  0  .  .  .  0  0  0  0 ]
[  1 -2  1  0  0  .  .  .  0  0  0  0 ]
[  0  1 -2  1  0  .  .  .  0  0  0  0 ]
[  0  0  1 -2  1  .  .  .  0  0  0  0 ]
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[  0  0  0  0  0  .  .  .  1 -2  1  0 ]
[  0  0  0  0  0  .  .  .  0  1 -2  1 ]
[  0  0  0  0  0  .  .  .  0  0  1 -1 ]
```

has eigenvalues $\mu_l = -4\sin^2\left(\frac{\pi l}{2(N+1)}\right)$, $\quad l = 0, 1, \ldots, N$.

Fact 4: For a real $n \times n$ matrix A, the Rayleigh quotient of a vector $x \in R^n$ is the scalar

$$r(x) = \frac{x^T A x}{x^T x}.$$

The gradient of $r(x)$ is

$$\nabla r(x) = \frac{2}{x^T x}(Ax - r(x)x).$$

If x is an eigenvector of A then $r(x)$ is the corresponding eigenvalue and $\nabla r(x) = 0$.