Preliminary Exam, Numerical Analysis, August 2010

Instructions: This exam is closed books and notes, and no electronic devices are allowed. The allotted time is three hours and you need to work on any three out of questions 1-4 and any three out of questions 5-8. All questions have equal weight and a score of 75 % is considered a pass. Indicate clearly the work that you wish to be graded.

-1- (The Companion Matrix.) Show that

$$\det\left(\begin{bmatrix} \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_1 & \alpha_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} - \lambda I\right) = (-1)^n \left[\lambda^n - \sum_{j=0}^{n-1} \alpha_j \lambda^j\right]. \tag{1}$$

Describe how this fact can be used to find the roots of a polynomial. Also briefly describe why the reverse process, finding the eigenvalues of a matrix by computing the characteristic polynomial, and finding its roots, is not generally viable.

-2- (Sherman-Morrison-Woodbury Formula.) Suppose A is an invertible $n \times n$ matrix, and U and V are $n \times k$ matrices such that $A + UV^T$ and the $k \times k$ matrix $I + V^T A^{-1} U$ are invertible. Show that

$$(A + UV^T)^{-1} = A^{-1} - A^{-1}U(I + V^T A^{-1} U)^{-1} V^T A^{-1} \tag{2}$$

Explain why this means that a rank k change of A leads to a rank k change of A^{-1}.

-3- (SVD.) Let A be an $m \times n$ matrix. Define what we mean by the singular value decomposition of A. Show that the singular value decomposition exists.

-4- (The interpolant to symmetric data is symmetric.) Suppose you are given symmetric data

$$(x_i, y_i), \quad i = -n, -n+1, \cdots, n-1, n, \tag{3}$$

such that

$$x_{-i} = -x_i, \quad \text{and} \quad y_{-i} = y_i \quad i = 0, 1, \cdots, n. \tag{4}$$

What is the required degree of the interpolating polynomial p? Show that the interpolating polynomial is even, i.e.,

$$p(x) = p(-x) \tag{5}$$

for all real numbers x.

-5- (Explicit Methods.) Define what we mean by Linear Multistep Method and what we mean by its region of absolute stability. Show that a convergent explicit linear multistep method cannot have an unbounded region of absolute stability.

-6- (Crank Nicolson Method.) Describe the Crank-Nicolson Method for the solution of the equation

$$u_t = u_{xx} \tag{6}$$

and use the von Neumann method to examine its stability.

-7- (Local Truncation Error.) What is the local truncation error of a numerical scheme? What does *consistency* mean? Give a specific and precise example.

-8- (Local Truncation Error.) Consider the wave equation

$$u_{tt} = c^2 u_{xx} \tag{7}$$

and its discretization

$$x_m = mh, \qquad t_n = nk, \qquad U_m^n \approx u(x_m, t_n) \tag{8}$$

and

$$U_m^{n+1} - 2U_m^n + U_m^{n-1} = r(U_{m+1}^n - 2U_m^n + U_{m-1}^n) \tag{9}$$

where $r = c^2k^2/h^2$ is the grid constant. For the purposes of this problem, ignore the issues of initial and boundary conditions. Compute the local truncation error of (9) and show that there is a value of r for which the local truncation error is exactly zero. Give a physical interpretation of this fact.