Preliminary Exam, Numerical Analysis, January 2009

Instructions: This exam is closed books and notes, and no electronic devices are allowed. The allotted time is three hours and you need to work on any three out of questions 1-4 and any two out of questions 5-7. All questions have equal weight and a score of 75 % is considered a pass. Indicate clearly the work that you wish to be graded.

-1- (Polynomial Interpolation.) Let $I = [a, b]$, let x_i, $i = 0, \dots, n$, be $n+1$ distinct points in I, and let y_i, $i = 0, \dots, n$, be $n+1$ given real numbers. Show that there exists a unique polynomial

$$p(x) = \sum_{i=0}^{n} \alpha_i x^i$$

such that

$$p(x_i) = y_i, \qquad i = 0, \dots, n.$$

-2- (Error Analysis.) Consider the linear system

$$Ax = b \tag{1}$$

where A is an invertible matrix. Let $\hat{x}$ be an approximation of the solution of (1) and let

$$e = x - \hat{x} \quad \text{and} \quad r = b - A\hat{x}.$$

Let $\|\cdot\|$ denote any vector norm or the corresponding induced matrix norm. Show that

$$\frac{\|e\|}{\|x\|} \le \|A\|\|A^{-1}\|\frac{\|r\|}{\|b\|}.$$

Comment on the significance of the condition number $\|A\|\|A^{-1}\|$ and give a lower bound for it in terms of the eigenvalues of A.

-3- (Linear Programming.) Define the phrase "Linear Programming Problem". Let A be a given $m \times n$ matrix with $m > n$, and let $b \in \mathbb{R}^m$ be a given vector. Write the problem

$$\text{Find } x \in \mathbb{R}^n \quad \text{such that} \quad \|Ax - b\|_\infty = \min$$

as a linear programming problem.

-4- (The Gershgorin Theorem.) Let λ be an eigenvalue of $A \in A^{n \times n}$. Show that there exists an

$$i \in \{1, 2, \cdots, n\} \tag{2}$$

such that

$$|a_{ii} - \lambda| \le \sum_{\substack{j=1 \\ i \ne j}}^{n} |a_{ij}| \tag{3}$$

For every eigenvalue λ, the inequality (3) describes a circle in the complex plane called a *Gershgorin circle*. Let S be a set that is the union of $k \leq n$ Gershgorin circles such that the intersection of S with all other Gershgorin circles is empty. Show that S contains precisely k eigenvalues of A (counting multiplicities). Without proof or counterproof state whether it is possible for a Gershgorin Circle not to contain any eigenvalue at all.

-5- (Adaptive Quadrature.) Describe the basic idea of adaptive quadrature, and give a simple example, including formulas.

-6- (Linear Multistep Methods.) Consider the initial value problem

$$y' = f(x, y), \quad y(a) = y_0$$

Let h be some stepsize, $x_n = a + nh$, $y_n \approx y(x_n)$, and $f_n = f(x_n, y_n)$, for $n = 0, 1, 2, 3, \ldots$. Let k be some step number, and ignore the question of obtaining starting values y_1, y_2, $\ldots$, y_{k-1}. Suppose the approximations y_n, $n = k, k+1, \ldots$ are obtained by the *Linear Multistep Method*

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}. \tag{4}$$

Define what is meant by the *local truncation error* and the *order* of the linear multistep method (4). Compute the order and the local truncation error of Euler's Method

$$y_{n+1} - y_n = h f_n. \tag{5}$$

-7- (Numerical PDEs.) Consider the one-dimensional heat equation: Find $u(x, t)$ such that

$$u_t = u_{xx}, \quad t \geq 0, \quad x \in [0, 1], \quad u(x, 0) = f(x), \quad u(0, t) = u(1, t) = 0.$$

Describe how this problem might be solved by applying the Method of Lines and Euler's Method. Give formulas that could be used to write a suitable computer code.