Preliminary Exam, Numerical Analysis, January 2012

Instructions: This exam is closed books, no notes and no electronic devices are allowed. You have three hours and you need to work on any three out of questions 1-4, and any three out of questions 5-8. All questions have equal weight and a score of 75% is considered a pass. Indicate clearly the work that you wish to be graded.

Problem 1(**Full rank matrix**).
Given $A \in \mathrm{C}^{m\times n}$ with $m \geq n$, show that $A^\star A$ is nonsingular if and only if A has full rank.

Problem 2(QR **and Cholesky Factorizations**).
Let A be a nonsingular square matrix and let $A = QR$ and $A^\star A = U^\star U$ be the QR factorization of A and the Cholesky factorization of $A^\star A$, respectively. Assume that the usual normalizations $r_{jj}, u_{jj} > 0$ are in effect. Is it true or false that $R = U$? Justify your answer.

Problem 3(**Midpoint Rule**).
Derive error estimate for the midpoint rule in the form:

$$|E_n^M| \leq \frac{h^2(b-a)}{24} \max_{a\leq x\leq b} |f''(x)|$$

Midpoint rule:

$$M_n(f) = h(f(x_1) + f(x_2) + ... + f(x_n))$$

where $h = (b-a)/n$ and

$$x_j = a + (j - \frac{1}{2})h, \quad j = 1, ..., n$$

Problem 4(**Spectral Radius**).
Let $A \in C^{m\times m}$ and $||\cdot||$ denote any operator norm on $m \times m$ matrices. Show that

$$\lim_{k\to\infty} ||A^k||^{1/k} = \rho(A).$$

Problem 5(**Interpolating Polynomial**).

Define $l_{i,n}(x)$ - Lagrange basis functions with $x_0, x_1, ..., x_n$.
Prove that for any $n \geq 1$,

$$\sum_{i=0}^{n} l_{i,n}(x) = 1$$

for all x.

Problem 6(**Linear Multistep Methods**).
Construct an example of:
a) a consistent but not stable (not zero-stable) linear multistep method
b) a stable (zero-stable) but not consistent linear multistep method
What kind of behavior do you expect from the numerical solution produced by the methods in a) and in b)?

Problem 7(**Convergence of Linear Multistep Method**).
Consider the method

$$y_{n+2} - 2y_{n+1} + y_n = \frac{h}{2}\Big(f(t_{n+2}, y_{n+2}) - f(t_n, y_n)\Big)$$

Apply the method to the scalar IVP $y' = y, y(0) = 1$ and solve exactly the resulting difference equation, considering the starting values to be $y_0 = y_1 = 1$. Show theoretically that the numerical solution does not converge as $h \to 0$ and $n \to \infty$.

Problem 8(**Stability of the Scheme**).
Using the von Neumann method investigate the stability of the implicit downwind scheme:

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} - \frac{u_m^{n+1} - u_{m-1}^{n+1}}{h} = f_m^n,$$

$$u_m^0 = g_m, \quad m = 0, \pm 1, \pm 2, ..., \quad n = 0, 1, ..., [T/\Delta t] - 1.$$