Prelim in Probability, Spring 2016

Directions. Turn in solutions for no more than 6 of the 10 problems. Each is worth 10 points. 40 points are required to pass. If a problem seems to be misstated, interpret it so as to be nontrivial.

1. If $X \in L^2$, then it is easy to show that μ is its mean if and only if $E[(X-a)^2]$ is uniquely minimized at $a = \mu$. Assuming $X \in L^1$, show that m is a median of X if and only if $E[|X-a|]$ is minimized (not necessarliy uniquely) at $a = m$.

2. (a) Find a simple asymptotic formula for $P(X > x)$, where X has the GAMMA(θ, α) density,

$$f(x) = \frac{1}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\theta}, \quad x > 0.$$

(b) Let $X_1, X_2, \ldots$ be i.i.d. GAMMA(θ, α). Find an increasing sequence a_n such that $\limsup_{n\to\infty} X_n/a_n = 1$ a.s., and prove it.

3. (a) Find the distribution of $\int_0^1 B_t\, dt$, where (B_t) is Brownian motion.

(b) Given an i.i.d. sequence $X_1, X_2, \ldots$ with mean 0 and variance 1, what does Donsker's invariance principle tell us about this sequence in connection with the distribution of $\int_0^1 B_t\, dt$?

4. Let $X_1, X_2, \ldots$ be i.i.d. with common distribution $P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}$, and put $S_0 := 0$ and $S_n := X_1 + \cdots + X_n$ for each $n \geq 1$. Consider the stopping time $N := \min\{n \geq 1 : S_n = 1\}$.

(a) Let $g(u) := E[u^{X_1}]$ for all $u > 1$, and show that the optional stopping theorem applies to the martingale $M_n := u^{S_n} g(u)^{-n}$, where $u > 1$, and the stopping time N.

(b) Use part (a) to show that

$$E[v^N] = \frac{1 - \sqrt{1 - v^2}}{v}, \qquad 0 < v < 1.$$

(c) Use part (b) to show that

$$P(N = 2m+1) = \frac{1}{m+1}\binom{2m}{m}\frac{1}{2^{2m+1}}, \qquad m \geq 0.$$

5. Suppose g and h are continuous on $\mathbf{R}$ with $g > 0$ and $|h(x)|/g(x) \to 0$ as $|x| \to \infty$. If $X_n \Rightarrow X$ and $\sup_n E[g(X_n)] < \infty$, show that $E[h(X_n)] \to E[h(X)]$.

6. Let $U_0, U_1, \ldots$ be independent and identically distributed random variables, uniform on $[0, 1]$. Let N be a Poisson random variable with $E[N] = 1$, independent of $U_0, U_1, \ldots$. Compute $E[Y]$, where $Y = \max_{0 \le k \le N} U_k$.

7. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables uniform on $[0, 1]$ and let $\alpha > 0$. Show that there are numerical sequences a_n and b_n such that

$$Y_n = \frac{\sum_{k=1}^n k^\alpha X_k - a_n}{b_n}$$

converges in distribution to a standard normal random variable.

8. Let $\{e_i, -\infty < i < \infty\}$ be independent and identically distributed random variables with $E[e_1] = 0$ and $E[e_1^2] = \sigma^2$ and ρ be a real number satisfying $|\rho| < 1$. Show that

$$Y = \sum_{i=0}^{\infty} \rho^i e_i$$

is finite with probability one. Compute $E[Y]$ and $E[Y^2]$. (You need to justify your steps when you compute these two expected values.)

9. Compute

$$\lim_{n\to\infty} (b-a)^{-n} \int_a^b \int_a^b \cdots \int_a^b \frac{x_1 + x_2 + \cdots + x_n}{x_1^2 + x_2^2 + \cdots + x_n^2} \, dx_1 \, dx_2 \cdots dx_n$$

if $0 < a < b < \infty$.

10. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with characteristic function φ. Let N be a random variable with

$$P(N = k) = \frac{1}{2^k}, \quad k = 1, 2, \ldots.$$

We assume that $\{X_i, i \ge 1\}$ and N are independent.

(a) Compute the characteristic function of $Y = X_1 + X_2 + \cdots + X_N$.

(b) Can you weaken the condition that $\{X_i, i \ge 1\}$ and N are independent and the formula obtained in (a) remains valid?