Probability Prelim Exam

January 2018

Instructions (Read before you begin)

- You may attempt all of 10 problems in this exam. However, you can turn in solutions for **at most** 6 problems. On the outside of your exam booklet, indicate which problem you are turning in.
- Each problem is worth 10 points; 40 points or higher will result in a pass.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 6040 text, then you need to carefully state and prove that result.

Exam Problems:

1. Let (X_n) be a sequence of real-valued random variables and define the tail σ-algebra $\mathcal{T} = \cap_{n\geqslant 1}\sigma(X_n, X_{n+1}, \dots)$. Let $S_n = X_1 + \cdots + X_n$. Prove or disprove the following statements.

   (a) $\{\lim_{n\to\infty} S_n \text{ exists}\} \in \mathcal{T}$.

   (b) $\{\overline{\lim}_{n\to\infty} S_n > 0\} \in \mathcal{T}$.

   (c) $\{\overline{\lim}_{n\to\infty} S_n/c_n > 1\} \in \mathcal{T}$ if $c_n \to \infty$ as $n \to \infty$.

2. Let μ be a translation invariant σ-finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Assume that there exist $\alpha < \beta$ such that $\mu((\alpha, \beta]) < \infty$. Prove that there exists a finite positive constant c such that $c\mu$ is Lebesgue measure.

3. Let (X_n) be i.i.d. random variables with mean μ.

   (a) Prove that

   $$\lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{X_i}{i} = \mu \quad \text{almost surely.}$$

   (b) If the random variables also have a finite variance $\sigma^2 < \infty$, then find a_n such that

   $$a_n\Big(\sum_{i=1}^{n} \frac{X_i}{i} - \mu \log n\Big)$$

   has a weak limit. Identify this weak limit. (You need to prove the weak convergence, once you have identified a_n and the limiting distribution.)

4. (a) Prove the Borel-Cantelli lemma. That is, prove that if $\{A_n\}_{n\geqslant 1}$ is any sequence of events satisfying $\sum_{n\geqslant 1} P(A_n) < \infty$, then $P(A_n \text{ occurs for infinitely many } n) = 0$.

   (b) Prove that if (X_n) is any sequence of real-valued random variables, then there are constants c_n so that $X_n/c_n \to 0$ almost surely as $n \to \infty$.

5. Prove or disprove the following statement. If X_n converges weakly to X as $n \to \infty$ and $Z_n - X_n$ converges weakly to 0 as $n \to \infty$, then Z_n converges weakly to X as $n \to \infty$.

6. Let X and Y be two standard normal random variables (defined on a common probability space and having positive variances).

   (a) Prove that they are independent if and only if they are jointly normal (i.e. (X, Y) is a two-dimensional normal random variable) and uncorrelated.

   (b) Give an example of two standard normal random variables that are uncorrelated but not jointly normal.

7. Let (X_n) be i.i.d. Cauchy distributed random variables. That is, they have probability density function $\frac{1}{\pi(1+x^2)}$. Abbreviate $S_n = X_1 + \cdots + X_n$.

   (a) Prove that
   $$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |X_i| = \infty \quad \text{almost surely.}$$

   (b) Using the fact that the characteristic function of a Cauchy distributed random variable is given by $e^{-|t|}$, prove that S_n/n is again a Cauchy distributed random variable.

   (c) Deduce that
   $$\varlimsup_{n\to\infty} S_n/n = \infty \quad \text{and} \quad \varliminf_{n\to\infty} S_n/n = -\infty, \quad \text{almost surely.}$$

8. Let (X_n) be i.i.d. uniform random variables on $[0, 1]$. For each $n \geqslant 1$, define
   $$Y_n = n \min\{X_k : 1 \leqslant k \leqslant n\}.$$
   Prove that as $n \to \infty$, Y_n converges weakly to an exponential random variable with mean one.

9. Suppose (M_n) is a martingale with $M_0 = 0$ and $|M_{n+1} - M_n| \leqslant 1$ for all $n \geqslant 0$.

   (a) Prove that there exists a finite positive constant C such that $e^x \leqslant 1 + x + Cx^2$ for all $x \in [-1, 1]$.

   (b) Prove that $E[e^{\lambda M_n}] \leqslant (1 + C\lambda^2)^n \leqslant e^{C\lambda^2 n}$ for all $n \geqslant 1$ and $\lambda \in (0, 1)$.

   (c) Prove that $M_n/n^\delta \to 0$ almost surely for every $\delta > 1/2$.

10. Give an example of a sequence of random variables X_n with densities $f_n(x)$ so that X_n converges weakly as $n \to \infty$ to a uniform random variable on $[0, 1]$, but $f_n(x)$ does not converge to 1 for any $x \in [0, 1]$.