Probability Qualifying Examination

January 7, 2011

There are 10 problems, of which you must turn in solutions for **exactly** 6 (your best 6, in your opinion). Each problem is worth 10 points, and 40 points is required for passing. On the outside of your exam book, indicate which 6 you have attempted.

If you think a problem is misstated, interpret it in such a way as to make it nontrivial.

1. Let $X_1, X_2, \ldots$ be independent identically distributed random variables with characteristic function φ. Let N be a random variable with distribution
$$\mathrm{P}\{N = k\} = \frac{1}{2^k}, \quad k = 1, 2, \ldots.$$
It is assumed that $\{X_i, i \geq 1\}$ and N are independent.
(a) Compute the characteristic function of $Y = X_1 + \ldots X_N$.
(b) Can you weaken the condition that $\{X_i, i \geq 1\}$ and N are independent so that the formula obtained in part (a) remains true?

2. (a) Let Φ and ϕ be the standard normal distribution and density functions, respectively. Show that
$$\lim_{x \to \infty} \frac{1 - \Phi(x)}{(1/x)\phi(x)} = 1.$$
(b) Let $X_1, X_2, \ldots$ be independent identically distributed standard normal random variables. Show that
$$\limsup_{n \to \infty} \frac{1}{\sqrt{\log n}} |X_n| = c \quad \text{almost surely}$$
and compute the value of c.

3. Let $X_1, \ldots, X_n$ be independent identically distributed random variables with $\mathrm{E}X_1 = \mu$ and $0 < \mathrm{Var}(X_1) = \sigma^2 < \infty$. Let
$$Y_n = \sum_{1 \leq i < j \leq n} X_i X_j.$$
Find numerical sequences a_n and b_n such that $(Y_n - a_n)/b_n$ has a non-degenerate limit distribution.

4. Let f be a continous and bounded function on $[0, \infty)$. Compute

$$\lim_{n\to\infty} \int_0^\infty \cdots \int_0^\infty f\left(\frac{x_1 + \ldots + x_n}{n}\right) e^{-(x_1+\ldots+x_n)} dx_1 \ldots x_n.$$

5. Let $X_1, \ldots, X_n$ be independent identically distributed Poisson random variables for each n with parameter λ_n.

   (a) Show that $X_1 + \ldots + X_n$ is asymptotically normal if and only if $n\lambda_n \to \infty$.

   (b) Can you weaken the condition that the X's are identically distributed for each n?

6. (a) Let X be a random variable with characteristic function φ. Show that X is symmetric if and only if $\varphi(t)$ is real for all t.

   (b) Give three distinct examples of real characteristic functions.

7. Let X and Y be integrable random variables on $(\Omega, \mathcal{F}, P)$, and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. If $X = Y$ on $G \in \mathcal{G}$, show that $\mathrm{E}[X \mid \mathcal{G}] = \mathrm{E}[Y \mid \mathcal{G}]$ a.s. on G.

8. Let Z_n be a Galton–Watson branching process with offspring distribution $\{p_k,\ k = 0, 1, 2, \ldots\}$ and $Z_0 = x$ (with x a positive integer), and let $f(\theta) = \sum p_k \theta^k$ be the associated pgf. Suppose that $\rho \in (0, 1)$ satisfies $f(\rho) = \rho$. Show that ρ^{Z_n} is a martingale, and use this to conclude that $\mathrm{P}(Z_n = 0 \text{ for some } n \geq 0) = \rho^x$.

9. Let $X \geq 0$, $\mathrm{E}X^2 < \infty$, and $0 \leq a < \mathrm{E}X$. Apply the Cauchy–Schwarz inequality to prove that $\mathrm{P}(X > a) \geq (\mathrm{E}X - a)^2/\mathrm{E}X^2$.

10. By considering the Poisson distribution, show that

$$e^{-n}\left(1 + n + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!}\right) \to \frac{1}{2}$$

as $n \to \infty$.