Real and Complex Analysis Exam

Aug 18, 2006

Do 7 of the following 10 problems – at least 3 from each part of the exam.

A. Real Variables

Do at least 3 of these.

In all of the following problems, unless stated otherwise, all functions are measurable on a measure space (X, μ).

1. Assume that f is non-negative and
$$\int_X f d\mu < \infty.$$
Show that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if A is a measurable subset of X such that $\mu(A) \leq \delta$ then
$$\int_A f d\mu \leq \epsilon.$$

2. We say that $f_n \to f$ *in measure* if for all $\epsilon > 0$ there exists an $N > 0$ such that if $n \geq N$ then $\mu(\{x \| f_n(x) - f(x)| > \epsilon\}) < \epsilon$. Show that if $f_n \to f$ pointwise and $\mu(X) < \infty$ then $f_n \to f$ in measure.

3. Let α be an irrational number. Let f be a real valued measurable function on $\mathbb{R}$ such that $f(x) = f(x+1) = f(x+\alpha)$. Use Fourier series to show that f is constant almost everywhere.

4. Let $E_1, \ldots, E_n$ by a finite collection of disjoint measurable subsets of X and let M be a subspace of $L^2(X)$ spanned by the functions $\{u_1, \ldots, u_n\}$ where each u_i is in $L^2(X)$ and has support contained in E_i. Let $P : L^2(X) \longrightarrow \mathbb{R}$ be the orthogonal projection. Find a function $h : X \times X \longrightarrow \mathbb{R}$ such that

$$Pf(x) = \int_X h(x, y) f(y) d\mu(y)$$

for all $f \in L^2(X)$.

5. The space $C[0, 1]$ of continuous functions on $[0, 1]$ is a subspace of all the L^p spaces, $L^p[0, 1]$ for $1 \leq p \leq \infty$. Define a linear operator $T : C[0, 1] \longrightarrow \mathbb{R}$ by $Tf = f(1/2)$. Show that T is a bounded operator for the L^p-norm if and only if $p = \infty$. Show that T extends to a bounded operator on all of $L^\infty[0, 1]$.

B. Complex Variables

Do at least 3 of these.

In what follows, $D_R(z_0)$ and $\overline{D}_R(z_0)$ denote the open and closed discs, respectively, centered at z_0, with radius R.

6. Let f be a holomorphic function on an open set containing $\overline{D}_1(0)$. Prove that if z is in $D_1(0)$, then

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{\sin(w-z)}\,dw$$

where $z \in D_1(0)$ and γ traverses the unit circle once in the positive direction.

7. Prove that if an entire function f satisfies an inequality of the form

$$|f(z)| \leq A + B\log|f(z)|$$

for some positive constants A and B and for all $z \in C$, then f is a constant.

8. Suppose f is an entire function with $f^{(n)}(0) \geq 0$ for all non-negative integers n. Find a point at which $|f(z)|$ assumes its maximum on $\overline{D}_R(0)$ and tell what that maximum is. Justify your answer.

9. Suppose f is holomorphic in an open set Ω and $\overline{D}_R(z_0) \subset \Omega$. If f has a zero of order k at z_0 and no other zeroes in $\overline{D}_R(z_0)$, then prove that there is an $\epsilon > 0$ such that, if $|\lambda| < \epsilon$, then f takes on λ as a value k times (counting multiplicity) in the disc $D_R(z_0)$.

10. Suppose Ω and Λ are open subsets of the complex plane. Suppose Ω is connected and Λ is a simply connected, proper subset of $\mathbb{C}$. If $\{f_n\}$ is a sequence of holomorphic functions from Ω to Λ, prove that either $\{f_n\}$ has a subsequence which converges uniformly on compacta to a holomorphic function on Ω or $\{f_n\}$ itself converges to ∞ at every point of Ω.

(REAL AND COMPLEX)
ANALYSIS PRELIMINARY EXAMINATION
August, 2005

Complete seven of the following ten problems. Of the seven, at least three must be from Part A, and at least three must be from Part B. Clearly indicate on the front of your examination booklet which seven you wish to be graded.

Part A

1. Consider a measurable subset A of the circle $\{z \in \mathbb{C} \mid |z| = 1\}$. Fix a real number θ so that θ/π is irrational. Define

$$A_\theta = \{e^{i\theta}z \mid z \in A\},$$

the rotation of A through θ radians. Prove that if

$$A = A_\theta$$

then either A has measure zero or else A is the complement of a set of measure zero. (Hint: consider the Fourier series of $\chi_A - \chi_{A_\theta}$; here χ_S denotes the characteristic function of S.)

2. Let $(\mathcal{M}, \mu, X)$ be a positive measure space. Let $\mathcal{M}'$ denote the collection of subsets (in $\mathcal{M}$) whose measures are finite. Prove that

$$L^2(X) \subset L^1(X)$$

if and only if

$$\sup_{E \in \mathcal{M}'} \mu(E) < \infty.$$

3. Let H be a Hilbert space with inner product $(\cdot, \cdot)$. Suppose $\{x_n\}$ is a sequence of elements in H with the property that the sequence $\{(x_n, y)\}$ converges for each $y \in H$. Prove that there is an element $x \in H$ such that $\{(x_n, y)\}$ converges to (x, y) for each $y \in H$.

4. Construct a sequence of continuous real-valued functions f_n on $[0, 1]$ such that $0 \le f_n \le 1$ and

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0,$$

but such that the sequence $\{f_n(x)\}$ converges for no $x \in [0, 1]$.

5. Suppose f and g are two complex measurable functions on a measure space X. Let

$$S = \{x \mid f(x) \neq g(x)\}$$

Prove or disprove: S is a measurable subset of X.

Part B

6. Provide a rigorous computation of the sum of the following series:

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

(Hint: one possible approach is to consider the integral of $f(z) = \frac{\pi}{z^4}\cot(\pi z)$ over an appropriate sequence of contours.)

7. Determine the number of zeros of

$$p(z) = 3z^3 - 2z^2 + 2iz - 8$$

which lie in the annulus $1 < |z| < 2$. Prove your answer is correct.

8. Fix positive integers k and l. Describe the space of meromorphic functions on the Riemann sphere that have a have a pole of order at most k at ∞ and a zero of order at least l at the origin.

9. The series

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n^2} z^n$$

defines a holomorphic function near the origin. Describe a maximal open subset of the complex plane on which f can be analytically continued.

10. Determine the region of values of z for which

$$f(z) = \int_0^{\infty} \frac{e^{tz}}{1+t^2}\,dt$$

defines a holomorphic function. (Hint: one approach is to use the holomorphicity criterion of Morera's Theorem. If you pursue this approach, make certain to justify any application of Fubini's theorem.)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF UTAH
REAL AND COMPLEX ANALYSIS PRELIMINARY EXAMINATION

January 3, 2005

Instructions: Do seven problems and list on the front of your blue book the seven problems to be graded. Do at least three problems from each part.

Part A:

1. Let f and g be two real measurable functions on a measurable space X. Show that

$$\{x \in X \mid \sin(f(x)) \geq \exp(g(x))\}$$

is measurable.

2. Let $f_1, f_2, \ldots$ be a sequence of real-valued convex functions on $\mathbb{R}$. Define their upper-limit via

$$\overline{f}(x) = \limsup_{n\to\infty} f_n(x);$$

and the lower-limit via

$$\underline{f}(x) = \liminf_{n\to\infty} f_n(x).$$

Prove or find a counterexample to each of the following two statements:

(i) $\overline{f}$ is convex.
(ii) $\underline{f}$ is convex.

3. Let f be a bounded linear functional on a linear subspace M of a Hilbert space H.

(i) Prove that f has a unique extension to a bounded linear functional F on H so that the norm of f coincides with the norm of F.
(ii) Prove that the extension F vanishes on $M^{\perp}$.

4. Let H be an infinite dimensional Hilbert space. Describe a closed subset of H which contains no element of minimal norm.

5. Suppose X consists of two points a and b. Define $\mu(\{a\}) = 1$, $\mu(\{b\}) = \mu(X) = \infty$, and $\mu(\emptyset) = 0$. For this measure μ, prove or disprove: $L^{\infty}(\mu)$ is the dual space of $L^1(\mu)$.

Part B:

1. Determine if there exists an entire function such that
$$f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^3}$$
for all $n \in \mathbb{N}$.

2. Let
$$f(z) = \frac{e^{\frac{1}{z-1}}}{e^z - 1}.$$
Determine all isolated singularities of f and their type. Find the residues of f at its poles.

3. Evaluate the integral
$$\int_0^{2\pi} \frac{\cos^2 3\phi}{1 - 2a\cos\phi + a^2}\, d\phi$$
where a is a complex number such that $|a| < 1$.

4. Let f be a holomorphic function with isolated singularity at point a. Assume that $g = \frac{1}{f}$ also has an isolated singularity at a, and that a is not a removable singularity for either f or g. Determine what type of isolated singularity a is for f and g.
Do you know an example of such a function f?

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5. Let f be a function holomorphic in $\mathbb{C} - \{0, 2\}$. Assume that:
(i) 0 and 2 are first order poles of f;
(ii) that f is bounded for $|z| \geq R$ for sufficienty large R;
(iii) the integral $\int_\gamma f(z)dz$ is equal to:
a) $2\pi i$ if γ is a positively oriented circle of radius 1 centered at 0;
b) 0 if γ is a positively oriented circle of radius 3 centered at 0.

Determine the function f.

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF UTAH

REAL AND COMPLEX PRELIMINARY EXAMINATION

AUGUST 17 2004

Instructions: Do seven problems and list on the front of your blue book the seven problems to be graded. Do at least three problems from each part.

Part A

1. Let μ be a positive measure on a measure space (X, Σ), and let $f \geq 0$ be a function which is integrable with respect to μ. Let $f_1 \leq f_2 \leq \cdots \leq f_n \leq \cdots$ be a sequence of real valued, measurable functions on X, such that $f_n \leq f$ for each n and $\int_X f_n d\mu \to \int_X f d\mu$. Show that $f_n \to f$ almost everywhere

2. Let μ be a positive measure on a measure space (X, Σ) such that $\mu(X) = 1$. Let f be a bounded measurable function on X. Show that $f \in L^p(X)$ for $1 \leq p \leq \infty$ and $||f||_p \leq ||f||_q$ for $1 \leq p \leq q \leq \infty$

3. Let $u_1, u_2, \cdots u_n, \cdots$ be an orthonormal set of vectors in a Hilbert space H. Show that for every $x \in H$
$$\lim_{n \to \infty} (x, u_n) = 0$$

4. Let $f_1, f_2, \cdots f_n, \cdots$ be a sequence of functions in $L^1(\mathbb{R})$ such that
$$\lim_{n \to \infty} \int_{\mathbb{R}} f_n(x) g(x) dx = 0$$
for every $g \in L^\infty(\mathbb{R})$. Show that $f_1, f_2, \cdots f_n, \cdots$ is a bounded sequence in $L^1(\mathbb{R})$

5. Let $f, g \in L^1(\mathbb{R})$. Let $A = f^{-1}(\mathbb{C} - \{0\})$ and $B = g^{-1}(\mathbb{C} - \{0\})$. Show that
$$(f \star g)^{-1}(\mathbb{C} - \{0\}) \subset A + B$$
where $A + B = \{x + y : x \in A, y \in B\}$

Part B

1. Let f be an entire function such that $f' = f$. Show that

$$f(z) = f(0)e^z = f(0)\sum_{n=0}^{\infty}\frac{z^n}{n!}$$

for any $z \in \mathbb{C}$.

2. Let

$$f(z) = e^{\frac{z}{z-1}}.$$

Determine all isolated singularities of f and their type. Find the residues of f at these isolated singularities.

3. Evaluate the integral

$$\int_0^{\infty} \frac{\sin ax}{x(x^2+b^2)}\,dx$$

where a and b are real numbers and $b \neq 0$.

4. Let f be a meromorphic function on the Riemann sphere. Show that f is a rational function.

5. Let f be a holomorphic automorphism of $\mathbb{C}^*$. Show that either $f(z) = \alpha z$ or $f(z) = \frac{\alpha}{z}$ for some complex number $\alpha \neq 0$.

Department of Mathematics
University of Utah

Ph.D. Preliminary Examination in Real/Complex Analysis

January 7, 2004

INSTRUCTIONS

Work seven problems and list on the front of your blue book the problems to be graded. Your list must include at least **three** problems from each part.

PART A

1. Let $(X, \mathfrak{M}, \mu)$ be a positive measure space, and let $f \in L^1(X)$. Let $X_n = \{x \in X : |f(x)| > n\}$. Show that

$$\lim_{n \to \infty} n\mu(X_n) = 0$$

2. Let $\{f_n\}$ be a sequence of complex valued measurable functions defined on a finite interval $(a, b) \subset \mathbb{R}$ (" measurable" means with respect to the Lebesque measure). Suppose that there exists a constant $M < \infty$ such that $\int_a^b |f_n(x)|^2 dx < M$ for all n, and $f_n(x) \to 0$ a. e. Prove that

$$\lim_{n \to \infty} \int_a^b f_n(x) dx = 0$$

Show also that this assertion fails in general if one replaces the interval (a, b) by $\mathbb{R}$ (give a counterexample).

3. Let $X = [0, 1]$, and $C(X)$ the space of complex valued continuous functions on X, equipped with the sup norm: $||f||_\infty = \sup_{x \in X} |f(x)|$. Let f_n be a sequence in $C(X)$. Prove that the following statements are equivalent:

(a) For each bounded linear functional λ on $C(X)$, $\lambda(f_n) \to 0$;

(b) $f_n(x) \to 0$ for each $x \in X$, and $\sup_n ||f_n||_\infty$ is finite.

4. Let $(X, \mathfrak{M}, \mu)$ be a positive measure space, and let $f \geq 0$ be a measurable function on X. Define $g(t) = \mu(\{x : f(x) \geq t\})$. Show that

$$\int_X f(x) d\mu(x) = \int_0^\infty g(t) dt$$

5. Let λ be a bounded linear functional on a linear subspace V of a Hilbert space H. Show that there exists a *unique* bounded linear functional Λ on H such that: (a) $||\Lambda|| = ||\lambda||$, and (b) $\Lambda(x) = \lambda(x)$ for $x \in V$.

PART B

Notation: $D = \{z \in \mathbb{C} : |z| < 1\}$, $H(\Omega)$ is the space of holomorphic functions defined on an open sunset $\Omega \subset \mathbb{C}$

1. Let $f : D \to D$ be holomorphic. Show that

$$|f(z) - f(0)| \leq |z(1 - \overline{f(0)}f(z))|$$

for all $z \in D$

2. Let f be an entire function such that $|\Re f(z)| + |\Im f(z)| \geq 1$ for each $z \in \mathbb{C}$. Prove that f is a constant function

3. Characterize all entire functions f which satisfy the property:

$$\lim_{z \to \infty} \frac{1}{f(z)} = 0$$

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4. Let Ω open in $\mathbb{C}$, $b \in \Omega$, $f \in H(\Omega)$, and assume that $f'(b) \neq 0$. Show that

$$\frac{2\pi i}{f'(b)} = \int_C \frac{1}{f(z) - f(b)} dz$$

for sufficiently small positively oriented simple circles centered at b.

5. Determine the poles and their orders of the function

$$\frac{1}{e^z - 1} - \frac{1}{z}.$$

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF UTAH
REAL AND COMPLEX PRELIMINARY EXAM, 2003

Instructions: Do seven problems and list on the front of your blue book the seven problems to be graded. Do at least three problems from each part.

Part A

1. Prove that a non-decreasing function from $\mathbb{R}$ to $\mathbb{R}$ is measurable with respect to the Borel σ-algebra.

2. Let μ be a positive measure on a measure space (X, Σ) and let f be a complex valued function on X which is integrable with respect to μ. For each n, set
$$E_n = \{x \in X : 1/n \leq |f(x)| \leq n\}.$$
Then each E_n is a measurable set on which $|f|$ is bounded. Prove that
   (a) $\mu(E_n)$ is finite for each n; and
   (b) $\lim_{n\to\infty} \int_{E_n} f(x)\, d\mu(x) = \int_X f(x)\, d\mu(x)$.

3. Prove that if X is a Banach space with a closed unit ball which is compact, then X must be finite dimensional. Hint: consider the family of sets of the form $\{x \in X : ||x|| = 1, \text{ and } \ f(x) = 0\}$ where f is a bounded nonzero linear functional on X.

4. Consider $L^1(T)$ to be the space of periodic functions on $\mathbb{R}$ of period 2π which are integrable on $[0, 2\pi]$. Let M be a proper closed translation invariant subspace of $L^1(T)$.
   (a) Prove that there exists a non-zero $h \in L^\infty(T)$ such that $f * h(x) \equiv 0$ for all $f \in M$, where $f * h(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x-y)h(y)\, dy$;
   (b) use part (a) to prove there must be an $n \in \mathbb{Z}$ such that $\hat{f}(n) = 0$ for every $f \in M$, where $\hat{f}(n)$ is the n th Fourier coeficient of f.

5. Let A be a closed subspace of $C([0,1])$ which is also closed in $L^2([0,1])$. Prove that the orthogonal projection P which projects $L^2([0,1])$ onto A is continuous as a linear transformation from $L^2([0,1])$ into $C([0,1])$. Note that $C([0,1])$ has the topology determined by the norm $||\cdot||_\infty$, while $L^2([0,1])$ has the topology determined by the norm $||\cdot||_2$

Part B

6. Let f be holomorphic on the unit disc U. Let $h(z,w)$ be the function on $U \times U$ defined by

$$h(z,w) = \left\{ \begin{array}{ll} \dfrac{f(z)-f(w)}{z-w} & \text{if } z \neq w \\ f'(z) & \text{if } z = w \end{array} \right\}$$

Prove that f is continuous as a function of two variables on $U \times U$. You need only check the continuity at points of the diagonal. At other points it is obvious.

7. Let Ω be an unbounded domain in the plane with the property that there exists a sequence $\{h_n\}$ of functions holomorphic on a neighborhood of $\overline{\Omega}$ such that each h_n vanishes at infinity on $\overline{\Omega}$, $|h_n| \leq 1$ on $\overline{U}$ for each n, and $\{h_n\}$ converges uniformly to 1 on each compact subset of $\overline{\Omega}$. Prove that if f is a function which is bounded and holomorphic in a neighborhood of $\overline{\Omega}$ and if $|f| \leq 1$ on $\partial\Omega$, then $|f| \leq 1$ on all of $\overline{\Omega}$.

8. Suppose f is continuous on $U = \{z \in \mathbb{C} : |z| < 1\}$ and holomorphic on the complement in U of the interval $(-1,1)$. Prove that f is actually holomorphic on U.

9. Determine how many zeroes, counting multiplicity, the polynomial

$$p(x) = z^5 - 6z^4 + z^3 + 2z - 1$$

has in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Prove that your answer is correct.

10. Suppose a is a real number greater than 1. Compute

$$\int_0^{2\pi} \frac{d\theta}{a + \sin\theta}$$

using residues. Hint: $\sin\theta = \dfrac{z - z^{-1}}{2i}$ where $z = e^{i\theta}$.

Qualifying Examination, August 14, 2002

Do at least seven problems, making sure to do at least three from each set.

Real Analysis

1) Let $\alpha(x,y)$ be a continuous function on $[0,1]\times[0,1]$. Show that

$$A(f)(x) = \int_0^1 \alpha(x,y)f(y)\,dy$$

defines a bounded operator from $L^2([0,1])$ to $C([0,1])$, considered as a Banach space with respect to $||f|| = \sup_x |f(x)|$.

2) Let P be a self-adjoint operator on a Hilbert space H such that $P^2 = P$. Show that P is a projection on a closed subspace. (Warning: we do not assume, apriori, that P is bounded).

3) Consider

$$HC = \{(x_n) \in \mathbf{C}^{\mathbf{Z}} \mid |x_n| \le \frac{1}{n} \text{ for all n} \in \mathbf{Z}\}.$$

Note that HC is simply a product of (infinitely many) closed discs. As such, it is compact with respect to the product topology. The product topology on HC is defined so that the sets

$$U_x(m,\delta) = \{(y_n) \in HC \mid |x_n - y_n| < \delta \text{ for all } 1 \le \text{n} \le \text{m}\}$$

provide a fundamental system of neighborhoods of x. Next, note that HC can be considered a subset of the Hilbert space $\ell^2(\mathbf{Z})$. Let $B(x,\epsilon)$ be, as usual, the open ball of radius ϵ in $\ell^2(\mathbf{Z})$. Show that for every $\epsilon > 0$ there exists a positive integer m and $\delta > 0$ such that

$$U_x(m,\delta) \subseteq B(x,\epsilon) \cap HC$$

and, for every positive integer m and $\delta > 0$ there exists $\epsilon > 0$ such that

$$U_x(m,\delta) \supseteq B(x,\epsilon) \cap HC$$

In other words, show that the restricted topology on HC coincides with the product topology.

4) Let X be a measure space with $\mu(X) = 1$. Let f be a measurable function in $L^\infty(X)$. Note that $||f||_p \le ||f||_\infty$ for all $p \ge 1$. In particular, f can be considered an element of $L^p(X)$ for all $p \ge 1$. Show that

$$\lim_{p\to\infty} ||f||_p = ||f||_\infty.$$

5) Show that the parallelogram law fails for $L^1([0,1])$ so it is not a Hilbert space. Hint: Use $f = \chi_{[0,1/2]}$ and $g = \chi_{[1/2,1]}$.

Complex Analysis

1) Evaluate the integral:

$$\int_0^{\infty} \frac{dx}{1+x^3}$$

2) Let B and C be two discs of the same radius which are tangent at a point in $\mathbf{C}$. Let A be another disc of the same radius whose boundary goes through the point of tangency of B and C. Find a Riemann mapping of $A - (B \cup C)$ onto the unit disc.

3) Let f be a complex valued function defined on the domain $D \subset \mathbf{C}$. Suppose, that as a mapping of $\mathbf{R}^2$ to $\mathbf{R}^2$, $f = u + iv$ is twice differentiable, and that the jacobian

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

is everywhere, as a linear transformation, a composition of a dilation and a rotation. Show that f is a holomorphic function.

4) Let D be a domain in the plane, and $\{f_n\}$ a sequence of functions holomorphic on D. Suppose the sequence is uniformly bounded; that is, there is a positive number M such that

$$|f_n(z)| < M \quad \text{for all} \quad n \quad \text{and} \quad z \in D\ .$$

Suppose, in addition, that there is an infinite compact set K in D such that the sequence $f_n|K$ converges uniformly on K. Show that the sequence f_n converges uniformly on all compact subsets of D.

5) Suppose that f is meromorphic on the plane, is never zero on the plane, and that

$$\lim_{z \to \infty} |f(z)| = 0\ .$$

Show that f is the inverse of a polynomial.

Real and Complex Variables Preliminary Exam - 2001

Do any seven problems, including at least 3 problems from Part A and at least 3 problems from Part B. Specify on the front of your bluebook which problems are to be graded.

Part A

1) Let X be a Banach space and let T be a function which assigns to each real number $s \in \mathbb{R}$ a bounded linear operator $T(s) : X \to X$. The derivative $T'(s)$ of T at s is the operator defined by

$$T'(s)x = \lim_{h \to 0} h^{-1}(T(s+h) - T(s))x$$

where the domain of $T'(s)$ is the set of all $x \in X$ for which this limit exists. Prove that if the domain of $T'(s)$ is all of X for a given s, then $T'(s)$ is a bounded linear operator on X.

2) Prove that if $f, g \in L^2(R)$ and the convolution product $f * g$ is defined by

$$f * g(x) = \int f(y)g(x-y)\,dy,$$

then $f * g(x)$ is a continuous function which vanishes at infinity.

3) Let X be a compact Hausdorff space and denote by $C(X)$ the Banach space of continuous complex valued functions on X and by $P(X)$ the cone consisting of all $f \in C(X)$ such that $\mathrm{Re}(f) \geq 0$. If H is a linear subspace of $C(X)$ such that $H \cap P(X) = (0)$, prove that

(a) $||\lambda + h||_\infty \geq |\lambda|$ for all $\lambda \in \mathbb{C}$, $h \in H$ (hint: show this is equivalent to $||1-h||_\infty \geq 1$ for all $h \in H$);

(b) there exists a positive Borel measure μ on X such that $\mu(X) = 1$ and

$$\int h\,d\mu = 0 \quad \text{for all} \quad h \in H$$

(recall that a measure μ on X is positive if $||\mu|| = \mu(X)$).

4) Let H_n be the linear span in $L^2([-\pi, \pi])$ of the set $\{e^{ikt}\}_{k=-n}^{n}$. The orthogonal projection P_n of $L^2([-\pi, \pi])$ onto H_n may be written in the form

$$P_n f(x) = h_n * f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h_n(x-t)f(t)\,dt.$$

Find the function h_n and prove that it has this property.

5) Prove that

(a) $\log(1+\mathrm{e}^t) \leq t + \log(2)$ for all $t \geq 0$;

(b) if f is a real valued function in $L^1([0.1])$ then

$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 \log(1+\mathrm{e}^{nf(x)})\, dx = \int_0^1 f^+(x)\, dx,$$

where $f^+(x) = f(x)$ if $f(x) \geq 0$ and $f^+(x) = 0$ if $f(x) < 0$.

Part B

1) Suppose that f is holomorphic in the open disc of radius r, centered at z_0, but does not have an analytic extension to any neighborhood of the closure of this disc. Show that the radius of convergence of the power series expansion for f centered at z_0 is r.

2) Let f be a holomorphic function defined on a neighbourhood of the closed disc $|z| \leq 1$. Asumme that $f(0) = 1$, and $|f(z)| > 1$ if $|z| = 1$. Must f have a zero in the open disc $|z| < 1$? Justify your answer.

3) Determine the number of zeroes of the polynomial

$$z^5 + 4z^2 + 2z + 1$$

in the region $|z| \leq 1$.

4) Suppose f is holomorphic in $r < |z| < R$, and suppose that for some $\rho, r < \rho < R$,

$$\int_{|z|=\rho} f(z) z^n dz = 0$$

for all negative integers n. Show that f has a holomorphic extension to the set $|z| > r$.

5) Assume that f is a one-to-one conformal mapping of the open disc $|z| < 1$ onto a regular n-gon with center at 0, and $f(0) = 0$. Prove that $f(\rho z) = \rho f(z)$ where ρ is any n-th root of 1. Remember that conformal maps preserve angles.

Department of Mathematics
University of Utah

Ph.D. Preliminary Examination in Real/Complex Analysis

August 2000

INSTRUCTIONS

Work seven problems and list on the front of your blue book the problems to be graded. Your list must include at least **three** problems from each part.

PART A

1. Let f and g be two real valued measurable functions on a measurable space X. Show that

$$\{x \in X : \sin f(x) \geq \cos g(x)\}$$

is a measurable subset of X.

2. Let $\mathbb{C}^n = \{z : z = (z_1, \cdots, z_n)\}$ be the n-dimensional complex space with norm $||z|| = \Sigma_{k=1}^{n} |z_k|$. Prove that there is no scalar product on C^n with the property that for each $z \in \mathbb{C}^n$, $||z||^2 = (z, z)$.

3. Let $X = [0, 1]$, and $C(X)$ the space of complex valued continuous functions on X, equipped with the sup norm: $||f||_\infty = \sup_{x \in X} |f(x)|$. Let f_n be a sequence in $C(X)$. Prove that the following statements are equivalent:
(a) For each bounded linear functional λ on $C(X)$, $\lambda(f_n) \to 0$;
(b) $f_n(x) \to 0$ for each $x \in X$, and $\sup_n ||f_n||_\infty$ is finite.

4. Let X be a compact Hausdorff space. Let $C(X)$ be the space of complex valued continuous functions on X. We assume that $C(X)$ is equipped with a norm $||.||$, which turns it into a Banach space. Assume moreover, that, for each $x \in X$ the linear functional λ_x, defined by: $\lambda_x(f) = f(x)$ is bounded on this Banach space.
Show that there exist positive constants A and B, such that for every $f \in C(X)$

$$A||f||_\infty \leq ||f|| \leq B||f||_\infty$$

Here $||f||_\infty$ denotes, as usual, the sup norm.

5. Let f be a continuous positive valued function on $\mathbb{R}$, and let g be the characteristic function of a finite nonempty open interval in $\mathbb{R}$. Show that the function $h = fg$ is in $L^1(\mathbb{R})$, but its Fourier transform $\hat{h}$ is not.

PART B

1. Let f be an automorphism of the unit disc $|z| < 1$. Asumme that f fixes two points. Show that $f(z) = z$ for all z in the unit disc.

2. Determine the number of zeroes of the polynomial

$$z^5 + z^3/3 + z^2/4 + z/3$$

in the region $1/2 < |z| < 1$.

3. Let γ be a closed path in $\mathbb{C} \setminus [0, 1]$. Show that

$$\int_\gamma \frac{1}{z(1-z)} dz = 0.$$

4. Let f be a holomorphic function on the unit disc. For every $0 \leq r < 1$ define

$$M(r) = \max_{|z|=r} |f(z)|.$$

Show that $M(r)$ is an increasing, continuous function on $[0, 1)$.

5. Determine the poles and their orders of the function

$$\frac{1}{e^z - 1} - \frac{1}{z}.$$

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF UTAH
REAL AND COMPLEX PRELIMINARY EXAM, 1999

Instructions: Do seven problems and list on the front of your blue book the seven problems to be graded. Do at least three problems from each part.

Part A

1. Let f be a non-negative measurable function and μ a finite positive measure on a measure space (X, Σ). Prove that

$$\lim_{n\to\infty} \int \frac{f^n(x)}{1+f^n(x)}\, d\mu(x) = \mu(E) + \frac{1}{2}\mu(F),$$

where $E = \{x \in X; f(x) > 1\}$ and $F = \{x \in X; f(x) = 1\}$.

2. Let μ be a measure on a measure space X and let $f \geq 0$ be a measurable function on X. Prove that

$$\int_X f(x)\, d\mu(x) = \int_0^\infty \phi(y)\, dy$$

where $\phi(y) = \mu(\{x : f(x) \geq y\})$.

3. Let D be the closed unit disc in $\mathbb{C}$ and T the unit circle. Prove that for every μ positive measure of total mass one on D there is a positive measure ν of total mass one on T such that

$$\int f\, d\mu = \int f\, d\nu$$

for every f which is continuous on D and holomorphic on the interior of D.

4. A bounded linear transformation A from a Banach space X to a Banach space Y is said to be "compact" if the image under A of the unit ball in X has compact closure in Y. Prove that if a compact linear transformation is also surjective, then Y is finite dimensional (you may assume that a locally compact Banach space must be finite dimensional).

5. Let H be a Hilbert space with inner product $(\cdot, \cdot)$ and $\{x_n\}$ a sequence of elements of H with the property that the sequence $\{(x_n, y)\}$ converges for each $y \in H$. Prove that there is an element $x \in H$ so that $\{(x_n, y)\}$ converges to (x, y) for each $y \in H$.

Part B

6. Let U be the open unit disc in $\mathbb{C}$. Let U^+ be the intersection of U with the upper half plane.
   (a) Let $f(z) = \frac{1}{2}(z + \frac{1}{z})$. Describe $f(U^+)$.
   (b) Give a conformal equivalence of U^+ and U.

7. Let $\lambda > 1$. Use Rouche's Theorem to show that the equation $\lambda - z - e^{-z} = 0$ has one solution for $Re(z) > 0$.

8. Let f be a non-vanishing holomorphic function defined in a neighborhood of the closed unit disc $\bar{U}$ such that $|f(z)| = 1$ if $|z| = 1$. Show that f is constant.

9. Let f be an injective entire function. Put $g(z) = f(1/z)$. Show that g does not have an essential singularity at 0. In particular, it follows that $f(z) = az + b$. Why?

10. Compute
$$\int_0^\infty \frac{x^2}{x^4+1} dx.$$

Department of Mathematics
University of Utah

Ph.D. Preliminary Examination in Real/Complex Analysis

August 1998

INSTRUCTIONS

Work seven problems and list on the front of your blue book the problems to be graded. Your list must include at least **three** problems from each part.

PART A

1. Let f and g be two complex measurable functions on a measurable space X. Show that

$$\{x \in X : f(x) \neq g(x)\}$$

is a measurable subset of X.

2. Let μ be a measure on X. Let $f_1 \geq f_2 \geq \cdots f_n \geq \cdots \geq 0$ be a sequence of measurable functions. Assume $\lim_{n\to\infty} \int_X f_n d\mu = 0$. Show that $f_n(x) \to 0$ a.e.

3. Let T be a bounded linear functional on a (linear) subspace M of a Hilbert space H. Prove that T has a unique norm preserving extension to a bounded linear functional on H, and that this extension vanishes on $M^{\perp}$.

4. Let f_n be a sequence of functions in $L^\infty(\mathbf{R})$ such that

$$\sup_n \left| \int_{\mathbf{R}} f(x)g(x)dx \right| < \infty$$

for all $g \in L^1(\mathbf{R})$. Show that $\sup_n \|f_n\| < \infty$.

5. (a) State the definition of a Fourier transform of $f \in L^1(\mathbf{R})$;
(b) State the Fourier Inversion Theorem;
(c) Show that the Fourier transform is one-to-one on $L^1(\mathbf{R})$.

PART B

In what follows $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

1. Let f be holomorphic in $\Delta - \{0\}$. Assume that f' is bounded in $\Delta - \{0\}$. Show that f extends to a holomorphic function on Δ.

2. Let f be a nonconstant entire function. Show that for every real constant k there exists $z \in \mathbf{C}$ such that $\Im f(z) = k \cdot \Re f(z)$.

3. Using methods of complex analysis, evaluate

$$\int_0^\infty \frac{x^2 dx}{x^4 + x^2 + 1}$$

4. Show that all zeroes of $z^4 - 6z - 3$ lie inside the circle $|z| = 2$.

5. Find a biholomorphism φ of $\Delta - \mathbf{R}^+$ onto Δ. Here $\mathbf{R}^+ = \{x \in \mathbf{R} : x \geq 0\}$. (It will suffice to give explicitly finitely many maps whose composition is equal to φ).

Department of Mathematics
University of Utah

Written Qualifying Exam in Real and Complex Analysis

Fall 1997

Instructions

Work seven problems and list on the front of your exam book the problems to be graded. Your list **must** include at least **three** problems from each part.

Part A

1. Let $f(x) = x^{-1/2}$ if $0 < x < 1$, $f(x) = 0$ if $x \leq 0$ or $x \geq 1$, and $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rationals. Show that if

$$g(x) = \sum_{n=1}^{\infty} \frac{f(x - r_n)}{2^n}$$

then $g \in L^1(\mathbb{R})$ and hence $g(x) < \infty$ m–a.e.. Show however, that even though $g^2(x) < \infty$ m–a.e., that $g \notin L^2(\mathbb{R})$.

2. Let $0 < p < r < q$. Show that if $(X, \mathfrak{M}, \mu)$ is a (positive) measure space, then

$$L^r(\mu) \subset L^p(\mu) + L^q(\mu);$$

more precisely, for every $f \in L^r(\mu)$ there exists $g \in L^p(\mu)$ and $h \in L^q(\mu)$ such that $f = g + h$.

3. Suppose that $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$ ($\hat{f}$ denoting the Fourier transform of f). Then show that

$$f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

4. Let H be a Hilbert space, M a closed subspace, and N a finite dimensional subspace of H. Show that $M + N$ is a closed subspace of H.

5. Let X_1, X_2 be Banach spaces, T a linear map of X_1 into X_2, and $\{\lambda_\alpha\}_{\alpha \in A} \subset X_2^*$ be a subset which seperates points of X_2; that is, if $v_1, v_2 \in X_2$, $v_1 \neq v_2$, then there exists $\alpha \in A$ such that $\lambda_\alpha(v_1) \neq \lambda_\alpha(v_2)$. Suppose that $\lambda_\alpha \circ T \in X_1^*$ for all $\alpha \in A$. Then show that T is continuous.

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Part B

1. Let $f(z)$ be an entire function and let $F_n(z) = \sum_{k=1}^{n} \frac{f^k(z)}{k!}$. Show that for every $R > 0$ there exists N such that all the zeros of F_n, $n \geq N$, are outside the circle of radius R.

2. Let $W = \{z \mid z \in \mathbb{C}, z = x + iy \text{ and } y^2 - x^2 < 1\}$. Let f be an entire function such that $f(\mathbb{C}) \subset W$. Show that f is a constant function.

3.
a. Let Ω be an open set containing the closed unit disk, f holomorphic on Ω, and suppose that f has an analytic logarithm on Ω. Show that if $z = re^{i\theta}$ with $0 \leq r < 1$ then

$$\ln|f(z)| = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) \ln|f(e^{it})|\, dt;$$

$P_r(\theta)$ denoting the Poisson kernel.

b. Let g be holomorphic on Ω and assume that $g(z) \neq 0$ for all $|z| = 1$. If $a_1, \ldots, a_n$ are the zeros of g in the open unit disk, repeated according to their multiplicities, $z = re^{i\theta}$, $0 \leq r < 1$, $g(z) \neq 0$, then show that

$$\ln|g(z)| = \sum_{j=1}^{n} \ln\left|\frac{z - a_j}{1 - \bar{a}_j z}\right| + \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) \ln|g(e^{it})|\, dt.$$

4. Let

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

where for $a > 0$, a^z is defined to be $e^{z \ln a}$, and $R = \{z \mid z \in \mathbb{C} \text{ and } \Re z > 1\}$. Prove that f is holomorphic on R.

5.
a. Let $\Re\alpha < 0$, and $g(z) = \frac{\alpha - z}{\bar{\alpha} + z}$. Show that g is a conformal map of $\{z \mid z \in \mathbb{C} \text{ and } \Re z < 0\}$ onto the open unit disk.(Hint: the ordered triple $\{\alpha, \frac{\alpha - \bar{\alpha}}{2}, -\bar{\alpha}\}$ maps to $\{0, 1, \infty\}$ under g.)

b. Let f be holomorphic on the closed unit disk and suppose that $\Re f(z) < 0$ for all z in the closed disk. Then show that

$$|f'(0)| \leq 2|\Re f(0)|.$$

Department of Mathematics
University of Utah

Ph.D. Preliminary Examination in Real/Complex Analysis

Fall 1996

INSTRUCTIONS

Work seven problems and list on the front of your exam book the problems to be graded. Your list must include at least **three** problems from each part.

PART A

1. Let $(X, \mathfrak{M}, \mu)$ be a measure space, and let $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$ be a sequence of sets in $\mathfrak{M}$.

Is it always true that $\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$? If so, give a proof. Otherwise give a counterexample.

2. Let H be a Hilbert space and let u and v be two distinct elements of H, both of norm 1.

Show that $\frac{u+v}{2}$ has norm strictly less than 1. Conclude that the unit sphere $S = \{w \in H : \| w \| = 1\}$ cannot contain a nontrivial interval (i.e. an interval with distinct endpoints).

3. Let ℓ^∞ be the normed space consisting of all complex valued, bounded sequences

$$\mathbf{a} = (a_1, a_2, \cdots a_n \cdots)$$

with the norm $\| \mathbf{a} \| = \{\sup |a_n| : n \in \mathbb{N}\}$. Let $W \subset \ell^\infty$ be the subspace consisting of all $\mathbf{a} \in \ell^\infty$ such that $\{n \in \mathbb{N} : a_n \neq 0\}$ is a finite set.

Show that there exists a nontrivial bounded linear functional on ℓ^∞ which vanishes on W.

4. Let $f \in L^1(\mathbb{R})$. Define a function φ on $\mathbb{R}$ by the formula $\varphi(x) = \int_{-\infty}^{x} f(t)dt$. Show that φ is a uniformly continuous function on $\mathbb{R}$.

5. Using the definitions of convolution and of Fourier transform show that, for $f, g \in L^1(\mathbb{R})$,

$$(f * g)\hat{}(t) = f\hat{}(t)g\hat{}(t)$$

for $t \in \mathbb{R}$

PART B

1. (a) Let f be an entire function. Assume that there exists a polynomial g such that for each $z \in \mathbb{C}$, $|f(z)| \leq |g(z)|$. Show that f is also a polynomial.

(b) Can we draw the same conclusion if the function g is rational? In other words, let f be an entire function and g a rational function such that for each $z \in \mathbb{C}$, which is not a pole of g, $|f(z)| \leq |g(z)|$. Does it follow that f is a polynomial?

2. Let f be an entire function such that $f(\mathbb{C}) \subset \mathbb{C} - S^1$, where S^1 is the unit circle. Show that f is a constant function.

3. Let D be an open set in $\mathbb{C}$. Let $\mathcal{F}$ be a collection of holomorphic functions defined on D

(a) Assume $\mathcal{F}$ is uniformly bounded on the boundary $\partial\triangle$ of each triangle $\triangle$ contained in D. Show that $\mathcal{F}$ forms a normal family.

(b) Is $\mathcal{F}$ a normal family if we assume it is uniformly bounded when restricted to any interval contained in D?

4. Evaluate

$$\int_\Gamma \frac{\cos z}{z^5} dz$$

where Γ is a positively oriented simple unit circle $\{z : |z| = 1\}$.

5. Let $P(z) = kz + z^k$ where k is a positive integer. Show that P is one-to-one on $\{z : |z| < 1\}$

Department of Mathematics
University of Utah

Written Qualifying Exam in Real and Complex Analysis

Fall 1995

Instructions

Work seven problems and list on the front of your exam book the problems to be graded. Your list **must** include at least **three** problems from each part.

Part A

1. Let μ be a positive finite measure on $(X, \mathfrak{M})$ and $g \in L^1(\mu)$. Suppose there exists $0 < M < \infty$ such that

$$|\int gs\, d\mu| \leq M||s||_p \quad (1 \leq p < \infty)$$

for all simple measurable functions on X. Then show that $g \in L^q(\mu)$ where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

2. Let $(X, \mathfrak{M}, \mu)$ be a positive measure space such that $\mu(X) = 1$. If V is a closed subspace of $L^2(\mu)$ which is contained in $L^\infty(\mu)$ then show that $\dim(V) < \infty$ as follows:

(1) Show that there exists $C > 0$ such that $||f||_\infty \leq C||f||_2$ for all $f \in V$.

(2) Let $\{f_1, \ldots, f_n\}$ be any orthonormal set in V and show that for all $c_1, \ldots, c_n \in \mathbb{C}$, and almost all $x \in X$,

$$|\sum_{i=1}^n c_i f_i(x)| \leq C(\sum_{i=1}^n |c_i|^2)^{1/2}.$$

(3) Choose $c_i = \overline{f_i(x)}$ in (2) and deduce that

$$\sum_{i=1}^n |f_i(x)|^2 \leq C^2 \quad (\text{for almost all } x \in X).$$

(4) Integrate the inequality in (3).

3. Let B be a Banach space and $A \subset B$ a closed subspace. If $x \in B$ but $x \notin A$, prove that there is a bounded linear functional f on B such that $f(x) = 1$, $f(y) = 0$ for every $y \in A$ and $||f|| = (\inf\{||x + y|| : y \in A\})^{-1}$.

4. Recall the k^{th} Fourier coefficient of $f \in L^1([-\pi,\pi])$ is $\hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\,dt$.

(1) If $f(t) = t$ for $-\pi \le t < \pi$, show that $\hat{f}(k) = (-1)^{k+1}\frac{1}{ik}$ for $k \neq 0$ and $\hat{f}(0) = 0$;

(2) Show that $\sum_{k=1}^{\infty} k^{-2} = \frac{\pi^2}{6}$.

5. Let f be an increasing function on $[a,b] \subset \mathbb{R}$. Use the fact that there is a Borel measure μ on $[a,b]$ such that $\mu([a,x]) = f(x) - f(a)$ for all $a \le x \le b$ to prove that

$$\int_a^b f'\,dm \le f(b) - f(a),$$

where m denotes Lebesgue measure on $[a,b]$.

Part B

6. If f is holomorphic in $\Delta(0;1) = \{z \in \mathbb{C} : |z| < 1\}$ and $r < 1$ prove that there exists a constant M so that

$$|f(z) - f(w)| \le M|z-w|$$

for all $z, w \in \Delta(0;r) = \{z \in \mathbb{C} : |z| < r\}$.

7. Suppose $P(z) = a_n z^n + \cdots + a_0$. If $M(\rho) = \sup_\theta |P(\rho e^{i\theta})|$ then show that $\frac{M(r)}{r^n} \ge \frac{M(R)}{R^n}$ if $0 < r < R$.

8. Let f be entire, $|f(t)| = 2$ for all $t \in \mathbb{R}$ and $f(i) = 1$. Then find $f(-i)$.

9. Find $\int_{\gamma_R} \frac{dz}{e^z - 1}$ where γ_R is the curve $|z| = R$ traversed counterclockwise and R is a positive number such that γ_R does not pass through a zero of $e^z - 1$.

10. How many zeroes (counting multiplicity) does the function $f(z) = z^4 + 6z^3 + 2z^2 - z + 1$ have inside the region $|z| < 1$.

Department of Mathematics
University of Utah

Written Qualifying Exam in Real and Complex Analysis

Fall 1994

Instructions

Work seven problems and list on the front of your exam book the problems to be graded. Your list **must** include at least **three** problems from each part.

Part A

1. Let $\mathcal{F} = \{f_\alpha\}_{\alpha \in A}$ be a family of continuous complex valued functions on a compact Hausdorff space. Let $\{c_\alpha\}_{\alpha \in A}$ be a family of complex numbers. Suppose that for each $F \subset A$ with F finite, there exists a complex measure μ_F of total variation one on X such that
$$\int_X f_\alpha \, d\mu_F = c_\alpha \qquad (\forall \alpha \in F).$$
Show that there exists a complex measure μ on X of total variation at most one such that
$$\int_X f_\alpha \, d\mu = c_\alpha \qquad (\forall \alpha \in A).$$

2. Let $(X, \mathfrak{M}, \mu)$ be a positive measure space. Let $\psi \in L^2(\mu) \cap L^\infty(\mu)$. Show that $\psi \in L^p(\mu)$ for all $p \geq 2$.

3. Suppose that $(X, \mathfrak{M}, \mu)$ is a measure space with μ a finite positive measure. Let I be an open interval in $\mathbb{R}$ and $f : X \times I \to \mathbb{C}$ a bounded function such that $x \to f(x,t)$ is $\mathfrak{M}$-measurable for each $t \in I$ and $t \to f(x,t)$ is differentiable for each $x \in X$. If $\dfrac{d}{dt} f(x,t)$ is also bounded on $X \times I$ prove that $h(t) = \int_X f(x,t)\, d\mu(x)$ is differentiable on I and in fact
$$h'(t) = \int_X \frac{d}{dt} f(x,t)\, d\mu(x).$$

4. Let A, B, and C be Banach spaces and let $f : A \to C$ and $g : B \to C$ be bounded linear maps. Suppose that g is one to one and $f(A) \subset g(B)$. Prove that there is a bounded linear map $h : A \to B$ such that $b = h(a)$ if and only if $f(a) = g(b)$.

5. Prove that if A and B are closed subspaces of a Hilbert space H, $A \perp B$, then $A + B$ is also closed in H.

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Part B

1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Assume that $a_0 = 0$ and $a_1 = 1$. Prove that f is one to one on the unit disk $U = \{z \in \mathbb{C} \,|\, |z| < 1\}$ if $\sum_{l=2}^{\infty} l|a_l| \le 1$.(Hint: Show that for every $z_0 \in U$ the function $z \to f(z) - f(z_0) = z - z_0 + \sum_{n=2}^{\infty} a_n(z^n - z_0^n)$ has a simple zero in U.)

2. Let $P(z) = a_n z^n + \cdots + a_0$ and suppose that P has zeroes at $z_1, \ldots, z_r$ each with multiplicity $m_1, \ldots, m_r$ repsectively. Let Ω be some domain **not** containing $z_1, \ldots, z_r$, and γ a cycle in Ω. Suppose further that there exists g holomorphic on Ω such that $g^p = P$. Show that there exists $k \in \mathbb{Z}$ such that

$$m_1 \mathrm{Ind}(\gamma, z_1) + \cdots + m_r \mathrm{Ind}(\gamma, z_r) = kp.$$

3. Let f be an entire function with the property that $f^{-1}(B)$ is bounded or empty for every $B \subset \mathbb{C}$ bounded. Prove that $f(\mathbb{C}) = \mathbb{C}$.

4. Suppose that f and g are one to one analytic functions of the open unit disk $U = \{z \in \mathbb{C} \,|\, |z| < 1\}$ onto some domain Ω. Suppose that $f(0) = g(0)$, and $f'(0) = g'(0) \neq 0$. Show that $f(z) = g(z)$.

5.a. Let f be holomorphic in an open set containing the disk $D = \{z \in \mathbb{C} \,|\, |z| \le r\}$ and suppose that the only zero of $f \in D$ is a simple zero at a belonging to the interior of D. If C_r is the positively oriented contour $|z| = r$ show that

$$\int_{C_r} \frac{1}{f(z)}\, dz = \frac{2\pi i}{f'(a)}.$$

5.b. Use methods of complex analysis to find

$$\int_{-\infty}^{\infty} \frac{dx}{x^4 + 1}.$$

Department of Mathematics
University of Utah

Ph.D. Preliminary Examination in Real/Complex Analysis

Fall 1993

INSTRUCTIONS

Work seven problems and list on the front of your exam book the problems to be graded. Your list must include at least **three** problems from each part.

PART A

1. Let $H = L^2(\mathbb{R})$ be the Hilbert space with the inner product $\int_{\mathbb{R}} f(x)\bar{g}(x)dx$, where dx is the Lebesque measure. Let $T : H \to M$ be the orthogonal projection of H onto a finite dimensional subspace M of H.

Show that there exists a function $h : \mathbb{R}^2 \to \mathbb{C}$ such that

$$Tf(x) = \int_{\mathbb{R}} h(x,y)f(y)dy$$

for all $f \in L^2(I)$.

(Hint: Let $u_1, \cdots u_n$ be an orthonormal basis of M. You may express h in terms of $u_1, \cdots u_n$).

2. Let $X = [a,b] \subset \mathbb{R}$ be a closed interval. Assume $a < b$. Regard X as a measure space with respect to the usual Lebesque measure.

(a) Show directly that the space $C(X)$ of continuous functions on X is *not dense* in $L^\infty(X)$.

(b) Using (a), if necessary, show that there exists a bounded linear functional on $L^\infty(X)$ which vanishes on $C(X)$.

3. Let X be as in Problem 2. Give an example of a sequence of measurable, nonnegative functions $f_n : X \to \mathbb{R}$, $f_1 \geq f_2 \geq \cdots \geq f_n \geq \cdots$, converging pointwisely to zero, such that $\int_X f_n(x)dx$ does not converge to zero.

Can any of the integrals $\int_X f_n(x)dx$ be finite? Justify your answer.

4. For $f : \mathbb{R} \to \mathbb{C}$, and $t \in \mathbb{R}$ define

$$f_t(x) = f(x-t).$$

Suppose f is continuous with compact support. Prove directly that $t \to f_t$ is a continuous map of $\mathbb{R}$ into $L^1(\mathbb{R})$.

5. (a) State the definition of convolution $*$ of two functions in $L^1(\mathbb{R})$.

(b) Prove directly that
$$\| f * g \|_1 \leq \| f \|_1 \| g \|_1$$
Here, as usual, "$\| h \|_1$" denotes the L^1 norm of function h.

PART B

1. (a) Let D be an open connected set in $\mathbb{C}$, and f a holomorphic function on D. Show that if either the real or imaginary part of f is constant then f is constant.

(b) Let f be an entire function. Show that if either the real or imaginary part of f is a bounded function then f is a constant.

2. Find a conformal transformation of the disc $\Delta = \{z : |z| < 1\}$ onto itself, which maps 0 to $-\frac{1}{2}$.

3. Let D be an open connected set in $\mathbb{C}$. Let $\mathcal{F}$ be a collection of holomorphic functions defined on D, which is uniformly bounded on each circle contained in D. Show that $\mathcal{F}$ forms a normal family.

4. Evaluate
$$\int_\Gamma \frac{e^z}{z^3} dz$$
where Γ is a positively oriented simple unit circle $\{z : |z| = 1\}$.

5. Suppose f is an entire function and there exist $M > 0$, $R > 0$ and a number $0 < k < 1$ such that $|f(z)| \leq M|z|^k$ for $|z \geq R$. Show f is a constant.

Department of Mathematics
University of Utah Ph.D Written Qualifying Examination in

Real/complex Analysis
September, 1992

Instructions

Work three (3) problems from one part and four (4) from the other. List on the front of your blue book the seven (7) problems you want graded.

Part A

Problem 1. Let $f \in L^\infty(\mathbf{R})$ define a bounded linear operator M_f from $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$ by $M_f(g) = fg$.

(a) Prove that $\|M_f\| = \|f\|_\infty$.

(b) Prove that if M_f is one to one and onto then there exists $\varepsilon > 0$ and a set E of measure zero such that $|f(x)| \geq \varepsilon$ for $x \in \mathbf{R} - E$.

Problem 2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose μ is finite and positive, and it satisfies the following property:

(*) for each $A \in \mathcal{M}$, such that $\mu(A) > 0$, there exists $B \in \mathcal{M}$ such that $B \subset A$ and $0 < \mu(B) < \mu(A)$.

Prove that for each $r > 0$ there exists $C \in \mathcal{M}$ such that

$$0 < \mu(C) < r.$$

Problem 3. Let $(X, \mathcal{M}, \mu)$ be a measure space, where the measure μ is positive. Suppose $f, f_n \in L^1(x, \mu)$, $f_n \geq 0$ for each n, $f_n \to f$ a.e. and

$$\int_X f_n d\mu \to \int_X f d\mu$$

Prove that $f_n \to f$ in $L^1(X, \mu)$ norm.

Hint: Show first that $(f - f_n)^+ \leq f$ for each n (recall that for real valued function g, $g^+ = \max(g, 0)$).

Problem 4. Let X be a compact Hausdorff space.

(a) Prove that a complex measure μ on X is positive if and only if $\mu(X) = |\mu|(X)$.

(b) Let $A \subset C(X)$ be a linear subspace containing the function 1. Let $S \subset X$ be a compact subset with the property that for each $f \in A$,

$$\sup\{|f(x)| : x \in X\} = \sup\{|f(x)| : x \in S\}$$

Prove that for each $x \in X$ there exists a positive measure μ_x with support contained in S such that $f(x) = \int_S f d\mu_x$ for $F \in A$.

Problem 5. In $L^2([0, 2\pi])$ consider the orthogonal projection onto the subspace spanned by $\{e^{-it}, 1, e^{it}\}$. Express this projection as an integral operator (i.e. an operator of the form $f \to Pf$, where $Pf(y) = \int_0^{2\pi} f(x)h(x, y)dx$).

Part B

Problem 1. Find $\int_{|z|=7} \tan(z)dz$

Problem 2. Show that every continuous function on $\overline{\Delta} = \{z : |z| \leq 1\}$ which is holomorphic on $\Delta = \{z : |z| < 1\}$ can be uniformly approximated on $\overline{\Delta}$ by holomorphic polynomials.

Problem 3. Show that if f is a meromorphic function in $\Delta(0; 1+h) = \{z : |z| < 1+h\}$, where $h > 0$, and $|f(z)| = 1$ for $|z| = 1$, then f is a rational function.

Problem 4. Suppose f is holomorphic in $\{z : |z| < 1\}$ and satisfies $\left|f(\frac{1}{n})\right| \leq \left(\frac{1}{n}\right)^n$ for $n = 2, 3 \ldots$. Can you identify f? Justify your answer.

Problem 5. Let $H(D)$ be the space of holomorphic functions on an open connected set $D \subset \mathbb{C}$. Suppose f_n is a sequence in $H(D)$ converging to a function f uniformly on compact subsets. Suppose also that f is a nonconstant function. Let $a \in D$, $\alpha = f(a)$.

Show there is a sequence $a_n \in D$ which converges to a, such that $f_n(a_n) = \alpha$ for each n.