Department of Mathematics
University of Utah
Real and Complex Analysis Preliminary Examination

August 18, 2010

Instructions: Do seven problems and list on the front of your blue book the seven problems to be graded. Do at least three problems from each part.

Part A:

Problem 1. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $f : X \to \mathbb{R}$ is a real-valued function on X. Suppose further that $E_r := \{x \mid f(x) > r\}$ is measureable for each rational number r. Either prove the following assertion or find a counter-example: f is measureable.

Problem 2. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and fix p and q finite such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $f_1, f_2, \ldots$ be a sequence of functions in $L^p(X)$ converging (in L^p) to f, and let $g_1, g_2, \ldots$ be a sequence of functions in L^q converging (in L^q) to g. Prove that the sequence $f_1 g_1, f_2 g_2, \ldots$ converges to fg in L^1. Does the same conclusion hold if $p = 1$ and $q = \infty$?

Problem 3. Let H be a Hilbert space and suppose that $\{x_n\}$ is a sequence in H with the following property: for each $y \in H$,

$$\sup_n |\langle x_n, y\rangle| < \infty.$$

Prove that $\sup_n ||x_n|| < \infty$.

Problem 4. Suppose $1 < p < q < r < \infty$. (Here p and q are arbitrary, not necessarily conjugate.) Prove that $L^p(\mathbb{R}) \cap L^r(\mathbb{R}) \subset L^q(\mathbb{R})$.

Problem 5. Let H be a Hilbert space, M a closed subspace of H, and $x \in H$. Prove that there is a unique point $y \in M$ which is closest to x.

Part B:

Problem 6. Let
$$f(z) = 1 - \cos z.$$

(i) Find all zeros of this function;
(ii) Find the multiplicities of these zeros.

Problem 7. Let
$$f(z) = \sin\left(\frac{z}{z+1}\right).$$

(i) Determine all isolated singularities of f and their type;
(ii) Find the Laurent expansions of f at these singularities;
(iii) Find the residues of f at these singularities.

Problem 8. Evaluate the integral
$$\int_{-\infty}^{\infty} \frac{x \cos x}{x^2 - 2x + 10}\, dx$$
using the residue theorem.

Problem 9. Let n be a positive integer. Denote by V_n the linear space of all entire functions f such that there exists $C > 0$ such that $|f(z)| \leq C|z|^n$ for all $z \in \mathbb{C}$.

(i) Describe precisely the functions in V_n;
(ii) Find the dimension of V_n.

Problem 10. Using Rouché's theorem find the number of zeros of the polynomial $2z^5 - z^3 + 3z^2 - z + 8$ in the region $\{z \mid |z| > 1\}$.