DEPARTMENT OF MATHEMATICS

University of Utah
Real and Complex Analysis Preliminary Examination
August 21, 2013

Instructions: Do seven problems with at least three (3) problems from section A and three (3) problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. List on the front of your blue book the seven problems to be graded.

A. Answer at least three and no more than four of the following questions. Each question is worth ten points.

1. Let (X, M, μ) be a measure space, and suppose that $1 \leq p < q < r$. Consider the vector space $L^p \cap L^r$, with norm $\|f\| = \|f\|_p + \|f\|_r$. Show that $L^q \subset L^p \cap L^r$, and that the inclusion map $i : L^q \to L^p \cap L^r$ is continuous.

2. Let V and W be Banach spaces, and $L(V, W)$ the space of bounded linear maps from V to W. Suppose that $T_n \in L(V, W)$, $n = 1, 2, \ldots$, and that $\lim T_n x$ exists for each $x \in V$. Show that $T : V \to W$ defined by $Tx = \lim T_n x$ is in $L(V, W)$.

3. Consider $\mathbb{R}$ with the Lebesgue measure. Show that $L^2(\mathbb{R})$ is a complete Hilbert space (hint: find a countable, complete orthonormal system).

4. Let (X, M, μ) be a measure space, and suppose that $f : X \times [0, 1] \to \mathbb{R}$ such that $f(\cdot, t) : X \to \mathbb{R}$ is integrable for all t. Define $F(t) = \int_X f(x, t)\, d\mu(x)$ for $t \in [0, 1]$. Suppose that $\frac{\partial f}{\partial t}(x, t)$ exists for all x and t, and that for some integrable function g, $|\frac{\partial f}{\partial t}(x, t)| \leq g(x)$ for all x and t. Show that F is differentiable and that $F'(t) = \int_X \frac{\partial f}{\partial t}(x, t)\, d\mu(x)$

5. Let (X, M, μ) be a σ-finite measure space, and consider $X \times X$ with the product measure. Suppose that $K \in L^2(X \times X)$. Show that if $f \in L^2(X)$, then $Tf \in L^2(X)$ and $\|Tf\|_2 \leq \|K\|_2 \|f\|_2$ where $Tf(x) = \int_X K(x, y) f(y)\, d\mu(y)$.

B. Answer at least three and no more than four of the following questions so that the total number of questions you have answered is seven. Each question is worth ten points.

1. Let f and g be analytic functions defined on an open set $U \subset \mathbb{C}$. Assume that g is not identically equal to zero. Prove that $F = f/g$ does not have essential singularities in U.

2. Describe all entire functions f which satisfy the property:

$$\lim_{z\to\infty} \frac{1}{f(z)} = 0$$

3. Determine the numbers of zeroes of the polynomial $z^4 + 2x^2 + 5z + 1$ in the set $|z| \leq 1$.

4. Let $D = \{z \in \mathbb{C} : |z| < 1\}$, and let $f : D \to D$ be an analytic function. Show that

$$|f(z) - f(0)| \leq |z(1 - \overline{f(0)}f(z))|$$

for all $z \in D$.

5. Evaluate

$$\int_\Gamma \frac{\sin z}{z^6}\, dz$$

where Γ is a positively oriented simple unit circle $\{z \in \mathbb{C} : |z| = 1\}$.