REAL AND COMPLEX ANALYSIS QUALIFYING EXAMINATION
January, 2010

Complete seven of the following ten problems so that at least three of your solutions are for problems 1–5 and at least three are for problems 6–10. Clearly indicate on the front of your blue book which seven problems are to be graded.

1. Let $[0,1]$ be the unit interval equipped, as usual, with Lebesque measure and fix $1 \leq p < \infty$. Suppose $f_1, f_2, \ldots$ is a sequence in $L^p([0,1])$ which is Cauchy with respect to the L^p norm.

(a) Prove that there is a subsequence $g_1, g_2, \ldots$ of $f_1, f_2, \ldots$ such that $\lim_{i\to\infty} g_i(x)$ exists for almost x.

(a) Show (by producing an example of a Cauchy sequence $f_1, f_2, \ldots$ in $L^p([0,1])$) that $\lim_{i\to\infty} f_i(x)$ can fail to exist for almost all x.

2. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and that $L^p(X) \subset L^q(X)$ for some $1 \leq p < q < \infty$.

(a) Show that the inclusion map $L^p(X) \to L^q(X)$ is bounded.

(b) Let $\mathcal{M}'$ denote the subsets in $\mathcal{M}$ of nonzero measure. Use (a) to prove that

$$\inf_{E' \in \mathcal{M}'} \mu(E') > 0.$$

3. Let H be a Hilbert space, and let $V \subsetneq H$ be a nonzero proper closed subspace. Let π_V be the orthogonal projection onto V.

(a) Show that π_V has operator norm 1, is idempotent (i.e. $\pi_V^2 = \pi_V$), and is self-adjoint (i.e. $\pi_V^* = \pi_V$).

(b) Conversely, if $P : H \to H$ is a self-adjoint idempotent continuous linear transformation of operator norm 1, show that $P = \pi_V$ for some closed subspace of H.

4. (a) Let ℓ^1 and ℓ^∞ denote the Banach spaces of sequences of complex numbers $\xi = (\xi_1, \xi_2, \ldots)$ such that

$$\xi \in \ell^1 \text{ iff } \parallel \xi \parallel_1 := \sum_{i=1}^{\infty} |\xi_i| < \infty$$
$$\xi \in \ell^\infty \text{ iff } \parallel \xi \parallel_\infty := \sup_i |\xi_i| < \infty.$$

Let $(\ell^\infty)^*$ denote the Banach space of continuous linear functionals (equipped with the operator norm) on ℓ^∞. Show that the map $\Lambda : \ell^1 \to (\ell^\infty)^*$ defined by

$$\Lambda(\xi)(\eta_1, \eta_2, \ldots) = \sum_i \xi_i \eta_i$$

is a well-defined norm-preserving injection of ℓ^1 into $(\ell^\infty)^*$ which is not onto. (Hint for the failure of surjectivity: show that $(\ell^\infty)^*$ contains nontrivial functionals which vanish on the subspace consisting of those ξ such that $\xi_i \to 0$ as $i \to \infty$.)

(b) Let (X, μ) be a measure space. State (without proof) precise conditions on μ, p, and q guaranteeing that there is a norm-preserving isomorphism of $L^p(X)$ onto $L^q(X)^*$.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) < \infty$. Let $f\colon X \to X$ be a measure-preserving transformation in the sense that $\mu(E) = \mu(f(E))$ for all $E \in \mathcal{M}$. Prove that for any $E \in \mathcal{M}$, the set of those points x of E such that $f^n(x) \notin E$ for all $n > 0$ has zero measure. (Hint: suppose not and derive a contradiction with $\mu(X) < \infty$.)

6. Let
$$f(z) = \frac{6z+2}{(z^2-1)(z+3)}.$$

(a) Use the residue theorem to compute $\int_\gamma f(z)dz$ where γ is the positively oriented circle $|z| = 2$.
(b) Compute the Laurent series for $f(x)$ which converges in the annulus $1 < |z| < 3$. Use this computation to check the value of the integral in (a).

7. Let n be a positive integer. Use a contour integral to compute
$$\int_0^{2\pi} \cos^{2n}(\theta) d\theta$$
in terms of n.

8. Let U be an open connected domain in the complex plane,and let f be an analytic function on U.

(a) Prove or disprove: If f is injective on U, then its complex derivative df/dz is never zero on U.
(b) Prove or disprove: If df/dz is never zero on U, then f is injective on U.

9. Find all conformal transformations $f(z)$ from the upper half disc $\{z \mid |z| < 2 \text{ and } \Re(z) > 0\}$ to the unit disc $\{z \mid |z| < 1\}$ with the property that $f(i) = 0$. It suffices to exhibit your solutions as compositions of more elementary functions.

10. Let $f(z)$ be analytic in the unit disc $\{z \mid |z| < 1\}$ and have bounded modulus in the sense that $|f(z)| \leq M$ for all z in the disc. Let $0 < r < 1$. Find a constant C depending on M and r so that for all points z, w of modulus less than r,
$$|f(z) - f(w)| \leq C|z - w|.$$