DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN ANALYSIS
August 2019

Instructions: Do at least three (3) problems from section A and three (3) problems from section B. Three completely correct problems in section A and two completely correct problems in section B is a passing exam. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first problems answered will be scored.

A. Answer at least three of the following questions. Each question is worth ten points.

1. Let $f \geq 0$ be an integrable function on a measure space X. Prove that, for every $\epsilon > 0$, there exists a set $E \subset X$ of finite measure such that
$$\int_E f > (\int_X f) - \epsilon.$$

2. Let V be a normed space and U a closed subspace. For every $x \in V$, let $||x+U|| = \inf\{||x+y|| \mid y \in U\}$. Then $||x+U||$ is a norm on V/U. (You do not need to prove that.) Prove that the map $P : V \to V/U$ defined by $P(x) = x+U$ has operator norm 1.

3. Let $T : U \to V$ be a linear map between two Banach spaces such that, for any linear functional f on V, the composite $f \circ T$ is a continuous functional on U. Prove that T is continuous. Hint: use the closed graph theorem.

4. Let V be a Banach space, and $T : V \to V$ a bounded linear map. Assume that for every $v \in V$ there exists a non-negative integer n such that $T^n(v) = 0$. Prove that there exists an integer n such that $T^n(v) = 0$ for all $v \in V$.

5. Let $X = [0,1]$ with the usual measure. For $n = 1, 2, \ldots$, let $f_n = n^{1/p} \cdot \chi_{[0,1/n]}$. Prove that the sequence f_n does not converge weakly to 0 in $L^1([0,1)]$.

B. Answer at least three of the following questions so that the total number of questions you have answered is seven. Each question is worth ten points.

Notation: Let $\mathbb{H}$ be the upper half plane and $\mathbb{D}$ be the (open) unit disk.

6. Prove the following special case of Runge's approximation theorem: If $f : B(0,4) \setminus (B(0,.1) \cup B(1,.1))$ is holomorphic and $\epsilon > 0$ is given then there exists q a rational functions so that $|f(z) - q(z)| < \epsilon$ for all $z \in \overline{B(0,2) \setminus (B(0,.2) \cup B(1,.2))}$.

7. The following statement is false: If $\Omega \subset \mathbb{C}$ is any simply connected set and $x \in \Omega$ then there exists a unique r and biholomorphism $f : \Omega \to B(0,r)$ so that $f(x) = 0$ and $f'(x) = 1$. Fix the statement and provide a proof for the correct statement.
   Note: Simply connected implies the set is open and connected.

8. Prove that if an entire function is not a polynomial then there is a number z so that the cardinality of $f^{-1}(z)$ is infinite.
   Do not use Picard's theorems.

9. Prove that if f is entire and sends the real line to itself then $f(\bar{z}) = \overline{f(z)}$ for all $z \in \mathbb{C}$.

10. Compute $\int_0^{2\pi} \frac{d\theta}{4+\cos(\theta)}$ using the methods of complex analysis.