DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN ANALYSIS

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Instructions: Do seven problems with at least three (3) problems from section A and three (3) problems from section B. You need at least two problems completely correct from each section to pass. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first problems answered will be scored.

A. Answer at least three and no more than four of the following questions. Each question is worth ten points. Let λ denote Lebesgue measure.

1. If $f_i : \mathbb{R} \to \mathbb{R}$ are measurable functions then is

$$\{x : 2 \text{ is a limit point of } \{f_i(x)\}_{i=1}^{\infty}\}$$

measurable?

Recall that p is a limit point of a sequence $\{a_i\}_{i=1}^{\infty}$ if there exists an increasing sequence of natural numbers n_i so that $p = \lim_{i\to\infty} a_{n_i}$.

2. Show that if $f, g \in L^1(\lambda, \mathbb{R})$ then $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$. Recall

$$(f * g)(x) = \int f(t)g(x-t)d\lambda(t).$$

3. Recall that if A is a linear operator on Hilbert space with bounded operator norm then it has an adjoint A^* which is the unique operator so that for all v, w

$$< Av, w >=< v, A^* w > .$$

   (a) Let $P : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $P\bar{v} = \bar{w}$ where $w_j = 0$ if j is not prime and v_j otherwise. What is P^*?

   (b) Let $P : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $P\bar{v} = \bar{w}$ where $w_j = v_{p_j}$ where p_j is the j^{th} prime. What is P^*?

4. If $f : \mathbb{R} \to \mathbb{C}$ is in $L^1(\lambda)$ show that the sequence of $L^1(\lambda)$ functions $g_n(x) = f(x + \frac{1}{n})$ converges to f in $L^1(\lambda)$.

5. (a) Let $A \subset \mathbb{R}$. Prove that if $\lambda(A) > 0$ then there exists $\delta_0 > 0$ so that for all $0 \leq \delta \leq \delta_0$ there exists a and $a + \delta$ so that both of them are in A. Note, a is allowed to depend on δ.

   (b) Show that if $A \subset \mathbb{R}$ that is closed under addition and subtraction then $\lambda(A) = 0$ or $A = \mathbb{R}$.

B. Answer at least three and no more than four of the following questions so that the total number of questions you have answered is seven. Each question is worth ten points.

6. Let f be an automorphism of the unit disk $|z| < 1$. Assume that f fixes two points. Prove that $f(z) = z$ for all z in the unit disk

7. Let $f(z) = 1 + 3z + z^n$.

   (a) Show that f has exactly one root in the unit disk $\{z : [z] < 1\}$.

   (b) Show that all roots of f lie in the disk $\{z : |z| \leq \frac{3+\sqrt{13}}{2}\}$

8. Let f be an entire function. Suppose that there exist $A > 0, R > 0$ and a positive integer k such that

   $$|f(z)| \leq A|z|^k \ln |z|$$

   for $|z| \geq R$.

   Show that f is a polynomial of degree at most k.

9. Let f be a holomorphic function defined on a domain Ω in $\mathbb{C}$. Show that f is either a polynomial, or there exists $z \in \Omega$ such that

   $$f^{(n)}(z) \neq 0$$

   for all integers $n \geq 0$.

10. Evaluate

    $$\int_C \frac{1}{z^2 - z} dz$$

    where C is a closed path in $\mathbb{C} - [0, 1]$.