DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN ANALYSIS
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Instructions: Do at least three (3) problems from section A and three (3) problems from section B. Three completely correct problems in section A and two completely correct problems in section B is a passing exam. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first problems answered will be scored.

A. Answer at least three of the following questions. Each question is worth ten points.

1. Let $\mathcal{M}$ be a σ-algebra of subsets of X and μ a measure on $\mathcal{M}$. Let $A_1 \subseteq A_2 \subseteq \ldots$ be a sequence of sets in $\mathcal{M}$. Let $A = \cup_{i=1}^{\infty} A_i$. Prove that $\lim_{i \mapsto \infty} \mu(A_i) = \mu(A)$. Now assume that $\mu(X) < \infty$, and let $A_1 \supseteq A_2 \supseteq \ldots$ be a sequence of sets in $\mathcal{M}$. Let $A = \cap_{i=1}^{\infty} A_i$. Prove that $\lim_{i \mapsto \infty} \mu(A_i) = \mu(A)$. Give an example where this fails if $\mu(X) = \infty$.

2. Let $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = e^{-x}$. Explain why f is Lebesgue integrable and compute its integral.

3. Let X be a normed space and X^* be the space of bounded functionals. Every $x \in X$ defines a functional $\hat{x}$ on X^* by $\hat{x}(f) = f(x)$ for all $f \in X^*$. Prove that $||\hat{x}|| = ||x||$.

4. Let H be a Hilbert space and e_1, $e_2, \ldots$ and an orthonormal basis. Prove that 0 is a weak limit of the sequence e_n. Prove that the unit ball B, $||x|| \leq 1$, is closed in the weak topology. Prove that the unit sphere S, $||x|| = 1$, is dense in B in the weak topology.

5. Prove the following special case of the Baire category theorem, without directly using the general theorem: $\mathbb{R}^2$ is not a countable union of lines.

B. Answer at least three of the following questions so that the total number of questions you have answered is seven. Each question is worth ten points.

Notation: Let $\mathbb{H}$ be the upper half plane and $\mathbb{D}$ be the (open) unit disk.

6. State and prove Rouché's theorem.

7. Give an example of a sequence of numbers $a_1, ...$ so that there is no holomorphic function $f : \mathbb{D} \to \mathbb{C}$ so that $a_1 = f'(0), a_2 = f''(0), a_3 = f'''(0), ...$ Justify your answer.

8. Show that if f is an entire function and f restricted to the unit disk is a bijection, (that is, $f|_{\mathbb{D}} : \mathbb{D} \to \mathbb{D}$ is a bijection) then f is a rotation. This is, there exists $c \in \mathbb{C}$ with $|c| = 1$ so that $f(z) = cz$ for all z.

9. Using the methods of complex analysis compute $\int_0^\infty \frac{1}{x^4+16} dx$.

10. Prove or disprove $\{f : \mathbb{H} \to \mathbb{H} | f \text{ is holomorphic}\}$ is a normal family.